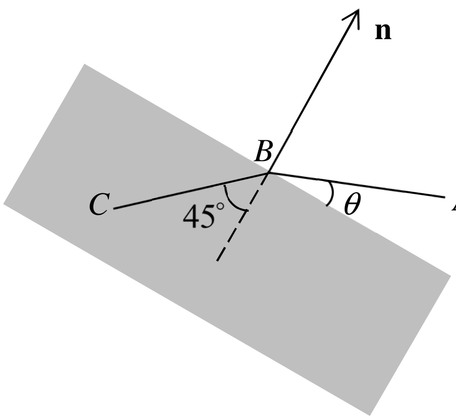
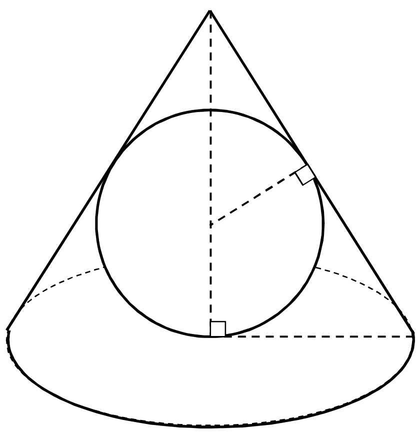
1	Without using a graphic calculator, solve the inequality $\frac{4x^2+7x+1}{3x+1} \leq x+2$. [3] Hence solve the inequality $\frac{4x+7\sqrt{x}+1}{3\sqrt{x}+1} \leq \sqrt{x}+2$. [2]
2	**(i)** Find $\int n\cos^{-1}(nx)\,\mathrm{d}x$, where n is a positive constant. [3] **(ii)** Hence find the exact value of $\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,\mathrm{d}x$. [2]
3	The vectors **p** and **q** are given by **p** = 2**i** + **j** +a**k** and **q** = b**i** + **j**, where a and b are non-zero constants. **(i)** Find $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})$ in terms of a and b. [2] Given that the **i**- and **j**- components of the answer to part **(i)** are equal, find the value of b. [1] Use the value of b you have found to solve parts **(ii)** and **(iii)**. **(ii)** Given that the magnitude of $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})$ is 80, find the possible exact values of a. [2] **(iii)** Given instead that $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular, find the exact value of $\lvert\mathbf{p}\rvert$. [3]
4	A graphic calculator is **not** to be used in answering this question. **(a)** The equation $w^3+pw^2+qw+30=0$, where p and q are real constants, has a root $w=2-\mathrm{i}$. Find the values of p and q, showing your working. [3] **(b)** The equation $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=0$ has a root $z=3+u\mathrm{i}$, where u is real constant. Find the value of u and hence find the second root of the equation in cartesian form, $a+b\mathrm{i}$, showing your working. [5]

5	A sequence $u_1,\ u_2,\ u_3,...$ is such that $$u_n = \frac{1}{2n^2(n-1)^2} \quad \text{and} \quad u_{n+1} = u_n - \frac{2}{n(n-1)^2(n+1)^2}, \text{ for all } n \geq 2 .$$ **(i)** Find $\sum_{n=2}^{N} \frac{2}{n(n-1)^2(n+1)^2}$. [3] **(ii)** Explain why $\sum_{n=2}^{\infty} \frac{2}{n(n-1)^2(n+1)^2}$ is a convergent series, and state the value of the sum to infinity. [2] **(iii)** Using your answer in part **(i)**, find $\sum_{n=1}^{N} \frac{2N}{(n+1)n^2(n+2)^2}$. [2]
6	**(i)** The variables x and y are related by $$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} + ky = 2 \quad \text{and} \quad y = 1 \text{ at } x = 0,$$ where k is a constant. Show that $(x+y)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+k)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 0$. [1] By further differentiation of this result, find the Maclaurin series for y, up to and including the term in x^3, giving the coefficients in terms of k. [4] **(ii)** Given that x is small, find the series expansion of $\mathrm{g}(x) = \frac{1}{\sin^2\left(2x + \frac{\pi}{2}\right)}$ in ascending powers of x, up to and including the term in x^2. If the coefficient of x^2 in the expansion of $\mathrm{g}(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for y found in part **(i)**, find the value of k. [4]

7	A population of a certain organism grows from an initial size of 5. After 5 days, the size of the population is 20, and after *t* days, the size of the population is *M*. The rate of growth of the population is modelled as being proportional to $\left(100^2 - M^2\right)$. **(i)** Write down a differential equation modelling the population growth and find *M* in terms of *t*. [6] **(ii)** Find the size of the population after 15 days, giving your answer correct to the nearest whole number. [2] **(iii)** Find the least number of days required for the population to exceed 80. [2]
8	It is given that $\mathrm{f}(x) = \begin{cases} 2x-1 & 0 \le x \le 2, \\ 2-(x-3)^3 & 2 < x \le 4, \\ 1 & \text{otherwise.} \end{cases}$ Sketch, on separate diagrams, for $0 \le x \le 8$, the graphs of **(i)** $y = \mathrm{f}(x)$ and state the range of f, [5] **(ii)** $y = \dfrac{1}{\mathrm{f}(x)}$. [4] In each graph, indicate clearly the coordinates of the end points, points of intersection with the axes and stationary point, if any. State clearly the equation of any asymptote. **(iii)** Deduce the value of $\int_{-6}^{-4} \mathrm{f}(-x)\,\mathrm{d}x$. [1]
9	Given that $\mathrm{f}(x) = \sin 2x + \cos 2x$, express $\mathrm{f}(x)$ as $R\sin(2x+\alpha)$, where $R > 0$, $0 < \alpha < \dfrac{\pi}{2}$ and *R* and α are constants to be found. [2] **(i)** Describe a sequence of transformations involved that transformed $y = \sin x$ to $y = \mathrm{f}(x)$. [3]

	(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $0 \le x \le \frac{3\pi}{8}$, indicating clearly the exact coordinates of the maximum point and the end points of the graph. [3] **(iii)** The region bounded by the curve $y = \mathrm{f}(x)$, the line $x = \frac{\pi}{8}$ and both axes is rotated about the y-axis through 2π radians. Find the volume of the solid of revolution correct to 4 decimal places. [4]
10	When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point $A(1, 2, 2)$ and enters a glass object at point $B(0, 0, 2)$. The surface of the glass object is a plane with normal vector **n**. The diagram shows a cross-section of the glass object in the plane of the light ray and **n**.  **(i)** Find a vector equation of the line AB. [1] The surface of the glass object is a plane with equation $x + z = 2$. AB makes an acute angle θ with the plane. **(ii)** Calculate the value of θ, giving your answer in degrees. [2] The line BC makes an angle of 45° with the normal to the plane, and BC is parallel to the unit vector $\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix}$.

	(iii) By considering a vector perpendicular to the plane containing the light ray and **n**, or otherwise, find the values of p and q. [4] The light ray leaves the glass object through a plane with equation $3x+3z=-4$. **(iv)** Find the exact thickness of the glass object, taking one unit as one cm. [2] **(v)** Find the exact coordinates of the point at which the light ray leaves the glass object. [3]
11	[It is given that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$ and the volume and surface area of a sphere of radius r are $\frac{4}{3}\pi r^3$ and $4\pi r^2$ respectively.] In a distant Northern kingdom of Drivenbell, Elsanna builds a spherical snowball with radius 3 m. The snowball is inscribed in a right conical container of base radius r m and height h m. The container is specially designed to allow the snowball to remain intact with fixed radius 3 m (see diagram).  **(i)** By considering the slant height of the cone, show that $r=\frac{3h}{\sqrt{\left(h^2-6h\right)}}$. [3] **(ii)** Use differentiation to find the values of h and r that give a minimum volume for the container. Find the value of the minimum volume. [6]

The snowball is being removed from the container and it starts to melt under room temperature.

(iii) Assuming that the snowball remains spherical as it melts, find the rate of decrease of its volume at the instant when the radius of the sphere is 2.5 m, given that the surface area is decreasing at 0.75 m^2 per minute at this instant. [5]

ANNEX B

IJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	$x \le -1$ or $-\frac{1}{3} < x \le 1$; $0 \le x \le 1$
2	Integration techniques	(i) $(nx)\cos^{-1}(nx) - \sqrt{(1-n^2x^2)} + C$ (ii) $\frac{\pi}{6} - \frac{\sqrt{3}}{2} + 1$
3	Vectors	(i) $20\begin{pmatrix} -a \\ ab \\ 2-b \end{pmatrix}$ $b = -1$ (ii) $\pm\frac{\sqrt{14}}{2}$ (iii) $\frac{5\sqrt{2}}{2}$
4	Complex numbers	(a) $p=2$, $q=-19$ (b) $u = -5$, $z = 2 + 3\mathrm{i}$
5	Sigma Notation and Method of Difference	(i) $\frac{1}{8} - \frac{1}{2N^2(N+1)^2}$ (ii) $\frac{1}{8}$ (iii) $\frac{N}{8}\left[1 - \frac{4}{(N+1)^2(N+2)^2}\right]$
6	Maclaurin series	(i) $y = 1 + (2-k)x + \left(\frac{3k-6}{2}\right)x^2 + \left(k^2 - 6k + 8\right)x^3 + \ldots$ (ii) $1 + 4x^2 + \ldots$; $k = \frac{10}{3}$
7	Differential Equations	(i) $\frac{\mathrm{d}M}{\mathrm{d}t} = k\left(100^2 - M^2\right)$, $k > 0$

		$M=\dfrac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1}$ (ii) $M\approx 47$ (iii) 35
8	Graphs and Transformation	(i) $R_f=[-1,3]$ (iii) 2
9	Application of Integration	$\mathrm{f}(x)=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)$ (i) A: A translation of $\frac{\pi}{4}$ units in the negative x-direction B: A scaling/stretch with scale factor $\frac{1}{2}$ parallel to the x-axis. C: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the y-axis. (iii) 0.6506
10	Vectors	(i) $\underset{\sim}{r}=\begin{pmatrix}1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}$ (ii) $\theta=18.4^{\circ}$ (iii) $p=-\frac{2}{3}$; $q=-\frac{1}{3}$ (iv) $\frac{5\sqrt{2}}{3}$ cm (v) $\left(-\frac{20}{9},-\frac{20}{9},\frac{8}{9}\right)$
11	Differentiation & Applications	(ii) $h=12$; $r=\frac{6}{\sqrt{2}}$; $V=72\pi\ \mathrm{m}^3$ (iii) 0.9375 m^3 per minute

1	$\dfrac{4x^2+7x+1}{3x+1} \le x+2$ $\dfrac{4x^2+7x+1-(x+2)(3x+1)}{3x+1} \le 0$ $\dfrac{4x^2+7x+1-(3x^2+x+6x+2)}{3x+1} \le 0$ $\dfrac{x^2-1}{3x+1} \le 0$ $\dfrac{(x-1)(x+1)}{3x+1} \le 0$ $-$ $+$ $-$ $+$ -1 $-\dfrac{1}{3}$ 1 $\therefore \; x \le -1$ or $-\dfrac{1}{3} < x \le 1$ $\dfrac{4x+7\sqrt{x}+1}{3\sqrt{x}+1} \le \sqrt{x}+2$ Replace x with $\sqrt{x}$, $\therefore \; \sqrt{x} \le -1$ or $-\dfrac{1}{3} < \sqrt{x} \le 1$ (rejected as $\sqrt{x} \ge 0$) Since $\sqrt{x} \ge 0$, $-\dfrac{1}{3} < \sqrt{x} \le 1 \quad \Rightarrow \quad 0 \le \sqrt{x} \le 1$ $0 \le x \le 1$
2	(i) $\int n\cos^{-1}(nx)\,\mathrm{d}x$ $=(nx)\cos^{-1}(nx)-\int (nx)\left(-\dfrac{n}{\sqrt{1-(nx)^2}}\right)\mathrm{d}x$ $=(nx)\cos^{-1}(nx)-\dfrac{1}{2}\int \left(-2n^2x\right)\left(1-n^2x^2\right)^{-1/2}\,\mathrm{d}x$ $=(nx)\cos^{-1}(nx)-\dfrac{1}{2}\times\dfrac{\left(1-n^2x^2\right)^{1/2}}{\frac{1}{2}}+C$ $=(nx)\cos^{-1}(nx)-\sqrt{\left(1-n^2x^2\right)}+C$

	(ii) $\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,\mathrm{d}x$ $=\left[(nx)\cos^{-1}(nx)-\sqrt{(1-n^2x^2)}\ \right]_0^{\frac{1}{2n}}$ $=\left[\frac{1}{2}\cos^{-1}\frac{1}{2}-\sqrt{1-\frac{1}{4}}\right]-(0-1)$ $=\frac{\pi}{6}-\frac{\sqrt{3}}{2}+1 \quad or \quad \frac{\pi}{6}+\frac{2-\sqrt{3}}{2}$
3	(i) $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})=4\mathbf{p}\times\mathbf{p}+10\mathbf{p}\times\mathbf{q}-10\mathbf{q}\times\mathbf{p}-25\mathbf{q}\times\mathbf{q}$ $=20\mathbf{p}\times\mathbf{q}$ $=20\begin{pmatrix}2\\1\\a\end{pmatrix}\times\begin{pmatrix}b\\1\\0\end{pmatrix}$ $=20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ Alternative: $(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})=\left(2\begin{pmatrix}2\\1\\a\end{pmatrix}-5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)\times\left(2\begin{pmatrix}2\\1\\a\end{pmatrix}+5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)$ $=\begin{pmatrix}4-5b\\-3\\2a\end{pmatrix}\times\begin{pmatrix}4+5b\\7\\2a\end{pmatrix}$ $=\begin{pmatrix}-6a-14a\\-(8a-10ab-8a-10ab)\\28-35b+12+15b\end{pmatrix}$ $=\begin{pmatrix}-20a\\20ab\\40-20b\end{pmatrix}=20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ Given that the **i**- and **j**- components of the vector $20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ are equal, $-a=ab$ $ab+a=0$ $a(b+1)=0$ Since $a\neq 0$, thus $b=-1$

	(ii) $\lvert(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})\rvert = 80$ $\left\lvert 20\begin{pmatrix}-a\\ ab\\ 2-b\end{pmatrix}\right\rvert = 80$ $\left\lvert\begin{pmatrix}-a\\ -a\\ 2+1\end{pmatrix}\right\rvert = 4$ $\sqrt{2a^2+9} = 4$ $2a^2+9 = 16$ $a^2 = \frac{7}{2}$ $a = \pm\sqrt{\frac{7}{2}}\ \ or\ \ \pm\frac{\sqrt{14}}{2}$ (iii) Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular, $(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q}) = 0$ $4\lvert\mathbf{p}\rvert^2 - 25\lvert\mathbf{q}\rvert^2 = 0$ $\lvert\mathbf{p}\rvert^2 = \frac{25}{4}\lvert\mathbf{q}\rvert^2$ $= \frac{25}{4}\left((-1)^2+1^2\right)$ $= \frac{25}{2}$ $\lvert\mathbf{p}\rvert = \frac{5\sqrt{2}}{2}$ Alternative: $(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q}) = \begin{pmatrix}4+5\\ -3\\ 2a\end{pmatrix}\begin{pmatrix}4-5\\ 7\\ 2a\end{pmatrix}$ $= 16-25-21+4a^2$ $= 4a^2-30$ Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular, $(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q}) = 0$ $4a^2-30 = 0$ $a^2 = \frac{15}{2}$ $\lvert\mathbf{p}\rvert = \sqrt{2^2+1+a^2} = \sqrt{5+\frac{15}{2}} = \sqrt{\frac{25}{2}} = \frac{5\sqrt{2}}{2}$

4	(a) **Method 1** Since the coefficients are real, $w=2+\mathrm{i}$ is another root of the equation. $(w-2+\mathrm{i})(w-2-\mathrm{i})=(w-2)^2-(\mathrm{i})^2$ $=w^2-4w+4+1$ $=w^2-4w+5$ $w^3+pw^2+qw+30=0$ $(w^2-4w+5)(w+6)=0$ (By inspection) Comparing coefficients of w^2, $p=6-4=2$ Comparing coefficients of w, $q=-24+5=-19$ **Method 2** Substitute $w=2-\mathrm{i}$ (or $w=2+\mathrm{i}$) into the given eqn, $(2-\mathrm{i})^3+p(2-\mathrm{i})^2+q(2-\mathrm{i})+30=0$ $(3-4\mathrm{i})(2-\mathrm{i})+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$ $(6-3\mathrm{i}-8\mathrm{i}-4)+p(3-4\mathrm{i})+q(2-\mathrm{i})+30=0$ $(32+3p+2q)+(-11-4p-q)\mathrm{i}=0$ Comparing the real parts, $3p+2q=-32$ --- (1) Comparing the imaginary parts, $4p+q=-11$ ---- (2) (1) − (2) × 2: $3p-8p=-32+11\times 2$ $-5p=-10$ $p=2$ From (2): $q=-11-4\times 2=-19$ $\therefore p=2$, $q=-19$ (b) Substitute $z=3+u\mathrm{i}$ into the given eqn, $(3+u\mathrm{i})^2+(-5+2\mathrm{i})(3+u\mathrm{i})+(21-\mathrm{i})=0$ $9+6u\mathrm{i}-u^2-15-5u\mathrm{i}+6\mathrm{i}-2u+21-\mathrm{i}=0$ $(15-2u-u^2)+(u+5)\mathrm{i}=0$ Compare imaginary coefficient: $u+5=0$ $u=-5$ $\therefore z=3-5\mathrm{i}$ [Note: if using $15-2u-u^2=0$, need to reject $u=3$] **Method 1** Let the other root be w. $z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})=(z-3+5\mathrm{i})(z-w)$ Comparing coefficients of z, $-5+2\mathrm{i}=-w-3+5\mathrm{i}$ $w=2+3\mathrm{i}$

Method 2

Let the other solution be $a+b\mathrm{i}$,

$$z^2+(-5+2\mathrm{i})z+(21-\mathrm{i})$$
$$=(z-(3-5\mathrm{i}))(z-(a+b\mathrm{i}))$$
$$=z^2-(a+b\mathrm{i})z-(3-5\mathrm{i})z+(3-5\mathrm{i})(a+b\mathrm{i})$$
$$=z^2-\left[a+3+(b-5)\mathrm{i}\right]z+(3-5\mathrm{i})(a+b\mathrm{i})$$

Compare the z term: $-(a+3)=-5 \quad => a=2$

$-(b-5)=2 \quad => b=3$

$\therefore z=2+3\mathrm{i}$ is another root.

5

(i)

$$\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$$
$$=\sum_{n=2}^{N}\left[u_n-u_{n+1}\right]$$
$$=\begin{bmatrix}(u_2-u_3)\\+(u_3-u_4)\\+(u_4-u_5)\\ \dots\\ \dots\\+(u_{N-1}-u_N)\\+(u_N-u_{N+1})\end{bmatrix}$$
$$=u_2-u_{N+1}$$
$$=\frac{1}{2(2^2)(2-1)^2}-\frac{1}{2(N+1)^2((N-1)+1)^2}$$
$$=\frac{1}{8}-\frac{1}{2N^2(N+1)^2}$$

(ii)

As $N\to\infty$, $\dfrac{1}{2N^2(N+1)^2}\to 0$

$\displaystyle\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}\to\frac{1}{8}$ which is a constant, hence it is a convergent series.

$$\sum_{n=2}^{\infty}\frac{2}{n(n-1)^2(n+1)^2}=\frac{1}{8}-0$$
$$=\frac{1}{8}$$

	(iii) **Method 1** $\sum_{n=1}^{N}\frac{2N}{(n+1)n^2\left(n+2\right)^2}=N\sum_{n=1}^{N}\frac{2}{(n+1)n^2\left(n+2\right)^2}$ $=N\sum_{n=2}^{N+1}\frac{2}{(n)\left(n-1\right)^2\left(n+1\right)^2}$ $=N\left[\frac{1}{8}-\frac{1}{2\left(N+1\right)^2\left(N+2\right)^2}\right]$ $=\frac{N}{8}\left[1-\frac{4}{\left(N+1\right)^2\left(N+2\right)^2}\right]$ **Method 2** By listing the terms $\sum_{n=2}^{N}\frac{2}{n\left(n-1\right)^2\left(n+1\right)^2}$ $=\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{N(N-1)^2(N+1)^2}$ $\sum_{n=1}^{N}\frac{2N}{(n+1)n^2\left(n+2\right)^2}$ $=N\left[\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{(N+1)(N)^2(N+2)^2}\right]$ $=N\sum_{n=2}^{N+1}\frac{2}{n\left(n-1\right)^2\left(n+1\right)^2}$ $=N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$ $=\frac{N}{8}\left[1-\frac{4}{\left(N+1\right)^2\left(N+2\right)^2}\right]$
6	(i) $(x+y)\frac{dy}{dx}+ky=2 \qquad \cdots(1)$ Differentiating (1) w.r.t. x: $(x+y)\frac{d^2y}{dx^2}+\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}+k\frac{dy}{dx}=0$ $(x+y)\frac{d^2y}{dx^2}+(1+k)\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2=0 \quad \cdots(2)$

Differentiating (2) w.r.t. x:

$$(x+y)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=0$$

$$(x+y)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(2+3\frac{\mathrm{d}y}{\mathrm{d}x}+k\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$$

$x=0,\quad y=1:\ \frac{\mathrm{d}y}{\mathrm{d}x}=2-k$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3k-6$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=6k^2-36k+48=6\left(k^2-6k+8\right)$$

$$\therefore y\ =1+(2-k)x+\left(\frac{3k-6}{2!}\right)x^2+\left(\frac{6\left(k^2-6k+8\right)}{3!}\right)x^3+\ldots$$

$$=1+(2-k)x+\left(\frac{3k-6}{2}\right)x^2+\left(k^2-6k+8\right)x^3+\ldots$$

(ii)

$$\sin\left(2x+\frac{\pi}{2}\right)=\sin 2x\cos\frac{\pi}{2}+\cos 2x\sin\frac{\pi}{2}=\cos 2x$$

$$\frac{1}{\sin^2\left(x+\frac{\pi}{2}\right)}=\frac{1}{\cos^2 2x}$$

$$\approx\left(1-\frac{(2x)^2}{2}\right)^{-2}$$

$$=\left(1-2x^2\right)^{-2}$$

$$=1+4x^2+\ldots$$

$$4=2\left(\frac{3k-6}{2}\right)$$

$$k=\frac{10}{3}$$

7

(i) $\frac{\mathrm{d}M}{\mathrm{d}t}=k\left(100^2-M^2\right)\ ,\quad k>0$

Since $\frac{\mathrm{d}M}{\mathrm{d}t}>0$ and $M>0,\Rightarrow\left(100^2-M^2\right)>0$ and $0<M<100$

$$\int\frac{1}{\left(100^2-M^2\right)}\,\mathrm{d}M=\int k\ \mathrm{d}t$$

$$\frac{1}{200}\ln\left(\frac{100+M}{100-M}\right)=kt+C$$

$$\ln\left(\frac{100+M}{100-M}\right)=200kt+C'$$

$$\frac{100+M}{100-M}=Ae^{200kt}\quad,\text{ where } A=e^{C'}$$

When $t=0,\quad M=5\quad\Rightarrow\quad A=\frac{105}{95}=\frac{21}{19}$

When $t=5,\quad M=20\quad\Rightarrow\quad \frac{3}{2}=\frac{21}{19}e^{1000k}$

$$e^{1000k}=\frac{19}{14}\quad or\quad 200k=\frac{1}{5}\ln\left(\frac{19}{14}\right)$$

Thus $\frac{100+M}{100-M}=\frac{21}{19}\left(e^{1000k}\right)^{\frac{t}{5}}=\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}$

$$100+M=\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}(100-M)$$

$$M\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1\right]=100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]$$

$$M=\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1}\quad OR\quad \frac{100\left[21\left(\frac{19}{14}\right)^{\frac{t}{5}}-19\right]}{21\left(\frac{19}{14}\right)^{\frac{t}{5}}+19}\quad OR\quad \frac{100\left[\left(\frac{19}{14}\right)^{\frac{t}{5}}-\frac{19}{21}\right]}{\left(\frac{19}{14}\right)^{\frac{t}{5}}+\frac{19}{21}}$$

(ii)

When $t=15$, $M=\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{3}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{3}+1}=46.847$

$M\approx 47$ (nearest whole number)

(iii)

<u>Method 1:</u> Graphical Method

Sketch the graphs of $M=f(t)$ and $M=80$

From the graph, when $t>34.336397,\quad M>80$

Least number of days required is 35.

<u>Method 2:</u> Use GC table

When $t=34,\ M=79.627<80$

When $t=35,\ M=80.718>80$

When $t=36,\ M=81.756>80$

$\Rightarrow t\geq 35$

Thus least number of days required is 35.

Method 3:

$$\frac{100\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]}{\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1}>80$$

$$\frac{5}{4}\left[\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}-1\right]>\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}+1$$

$$\frac{1}{4}\cdot\frac{21}{19}\left(\frac{19}{14}\right)^{\frac{t}{5}}>\frac{9}{4}$$

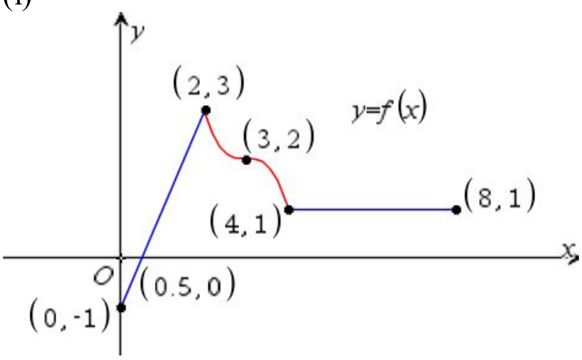
$$\left(\frac{19}{14}\right)^{\frac{t}{5}}>\frac{57}{7}$$

$$t>\frac{5\ln\left(\frac{57}{7}\right)}{\ln\left(\frac{19}{14}\right)}=34.336397$$

Least number of days required is 35.

8

(i)

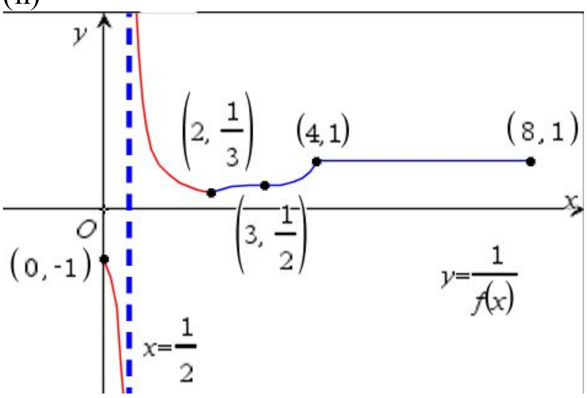


Range of f is $[-1,3]$

or $R_f=[-1,3]$

or $R_f=\{y:-1\le y\le 3\}$

(ii)



(iii)

$$\int_{-6}^{-4} \mathrm{f}(-x)\,\mathrm{d}x = \int_{4}^{6} \mathrm{f}(x)\,\mathrm{d}x$$
$$= \text{area of rectangle}$$
$$= 2$$

9

$\mathrm{f}(x) = \sin 2x + \cos 2x$

$R = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan\alpha = 1 \quad \Rightarrow \quad \alpha = \dfrac{\pi}{4}$

$\mathrm{f}(x) = \sin 2x + \cos 2x = \sqrt{2}\sin\left(2x + \dfrac{\pi}{4}\right)$

(i)

Transforming $y = \sin x$ to $y = \sqrt{2}\sin\left(2x + \dfrac{\pi}{4}\right)$

Sequence of Transformation:

Either

A: A translation of $\dfrac{\pi}{4}$ units in the negative x-direction

B: A scaling/stretch with scale factor $\dfrac{1}{2}$ parallel to the x-axis.

C: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the y-axis.

Acceptable sequence: ABC, ACB, CAB.

OR $y = \sqrt{2}\sin\left[2\left(x + \dfrac{\pi}{8}\right)\right]$

D: A scaling/stretch with scale factor $\dfrac{1}{2}$ parallel to the x-axis.

E: A translation of $\dfrac{\pi}{8}$ units in the negative x-direction.

F: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the y-axis.

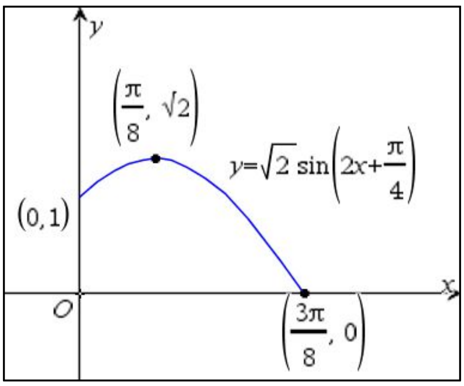
Acceptable sequence: DEF, DFE, FDE

(ii)

Max point occurs when $\sin\left(2x + \dfrac{\pi}{4}\right) = 1$

$\Rightarrow \quad \left(2x + \dfrac{\pi}{4}\right) = \dfrac{\pi}{2}$

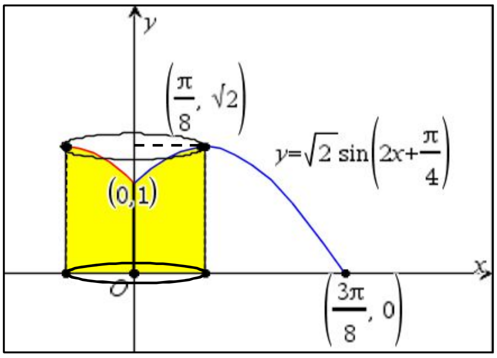
$\Rightarrow \quad x = \dfrac{\pi}{8}, y = \sqrt{2}$



(iii)

$y=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)$

The curve is one-one for $0\le x\le\frac{\pi}{8}$, thus inverse function exists.



$\sin\left(2x+\frac{\pi}{4}\right)=\frac{y}{\sqrt{2}}$

$2x+\frac{\pi}{4}=\sin^{-1}\frac{y}{\sqrt{2}}$

$x=\frac{1}{2}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)-\frac{\pi}{4}\right]$

Volume = Volume of cylinder - $\pi\int_{1}^{\sqrt{2}} x^2 \ \mathrm{d}y$

$=\pi\left(\frac{\pi}{8}\right)^2\sqrt{2}-\pi\int_{1}^{\sqrt{2}}\frac{1}{4}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)-\frac{\pi}{4}\right]^2 \ \mathrm{d}y$

$=0.6506458$

$\approx 0.6506 \ \ (4 \ \text{d.p.})$

10

(i)

$\overrightarrow{AB}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}$

$l_{AB}:\underset{\sim}{r}=\begin{pmatrix}1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}$ *or* $\underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\lambda\in$ □ or equivalent

(ii)

$\sin\theta=\frac{\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\square\begin{pmatrix}1\\0\\1\end{pmatrix}\right|}{\sqrt{5}\sqrt{2}}=\frac{1}{\sqrt{10}}$

$\theta=18.4^\circ$

(iii)

Let $\underset{\sim}{m}$ be a vector perpendicular to the plane containing the light ray and $\underset{\sim}{n}$.

$\underset{\sim}{m}=\underset{\sim}{n}\times\overrightarrow{AB}=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ or $\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\2\end{pmatrix}$

$$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{2}} \Rightarrow \frac{2}{3} - q = 1$$

$$q = -\frac{1}{3}$$

$$\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix} \perp \underset{\sim}{m} \Rightarrow \begin{pmatrix} -\frac{2}{3} \\ p \\ -\frac{1}{3} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 0$$

$$-\frac{4}{3} - p + \frac{2}{3} = 0 \Rightarrow p = -\frac{2}{3}$$

(iv)
Glass upper surface is $x + z = 2$

Glass bottom surface is $3x + 3z = -4 \Rightarrow x + z = -\frac{4}{3}$

Distance between two planes $= \frac{\left|2 - \left(-\frac{4}{3}\right)\right|}{\sqrt{2}} = \frac{10}{3\sqrt{2}} = \frac{5\sqrt{2}}{3}$

Thickness of the glass object is $\frac{5\sqrt{2}}{3}$ cm

(v)
Let the point at which the light ray leaves the glass object be F.

Method 1:

$$l_{BF} : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad or \quad \underset{\sim}{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

At F,

$$\left[\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = -4$$

$$6 + \mu(6+3) = -4$$

$$\mu = -\frac{10}{9}$$

OR

$$\left[\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = -4$$

$$6 + \mu(-2-1) = -4$$

$$\mu = \frac{10}{3}$$

The coordinates of F are

$$\left(-\frac{20}{9}, -\frac{20}{9}, \frac{8}{9}\right)$$

Method 2:

$$\cos 45^\circ = \frac{\frac{5\sqrt{2}}{3}}{BF} \Rightarrow \left|\overrightarrow{BF}\right| = \frac{5\sqrt{2}}{3} \times \sqrt{2} = \frac{10}{3}$$

(or using Pythagoras' theorem)

	$\vec{BF} = \frac{10}{3}\begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} = -\frac{10}{9}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ $\vec{OF} = -\frac{10}{9}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} -20 \\ -20 \\ 8 \end{pmatrix}$ The coordinates of F are $\left(-\frac{20}{9}, -\frac{20}{9}, \frac{8}{9}\right)$
11	(i) Let l be the slant height of the cone. $l^2 = h^2 + r^2 \quad ----(1)$ Using similar triangles, $\frac{h-3}{l} = \frac{3}{r}$ $l = \frac{rh-3r}{3} \quad ----(2)$ Equating (1) and (2), $\left(\frac{rh-3r}{3}\right)^2 = h^2 + r^2 \quad ----(*)$ $r^2h^2 - 6r^2h + 9r^2 = 9h^2 + 9r^2$ $r^2(h^2 - 6h) = 9h^2$ $\therefore r = \frac{3h}{\sqrt{h^2 - 6h}} \qquad (\text{Since } r > 0)$ (ii) Volume of cone, $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3}\pi\left(\frac{3h}{\sqrt{h^2-6h}}\right)^2 h$ $= \frac{3\pi h^3}{h^2 - 6h}$ $= \frac{3\pi h^2}{h-6}$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{6\pi h(h-6) - 3\pi h^2}{(h-6)^2}$ $= \frac{3\pi h^2 - 36\pi h}{(h-6)^2}$

$\frac{dV}{dh} = 0 \quad \Rightarrow \quad 3\pi h^2 - 36\pi h = 0$

$h(h-12) = 0$

$h = 12$ or $h = 0$ (reject $\because h > 0$)

h	12^-	12	12^+
Sign of $\frac{dV}{dh}$	– ve	0	+ ve
Tangent	\	—	/

Thus, V is a minimum when $h = 12$

When $h = 12$,

$$r = \frac{3(12)}{\sqrt{(12)^2 - 6(12)}} = \frac{6}{\sqrt{2}} \qquad (\approx 4.2426)$$

$$V = \frac{3\pi(12)^2}{12-6} = 72\pi \qquad (\approx 226.195)$$

(iii)

Let R be the radius of the snowball

$$S = 4\pi R^2 \quad \Rightarrow \quad \frac{dS}{dt} = 8\pi R \frac{dR}{dt}$$

$$V = \frac{4}{3}\pi R^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}$$

When $R = 2.5$, $\frac{dS}{dt} = -0.75 \quad \Rightarrow \quad 8\pi(2.5)\frac{dR}{dt} = -0.75$

$$\frac{dR}{dt} = -\frac{3}{80\pi} \quad or \quad -\frac{0.0375}{\pi} \quad or \quad -0.0119366$$

$$\frac{dV}{dt} = 4\pi(2.5)^2\left(-\frac{3}{80\pi}\right) = -\frac{15}{16} \quad or \quad -0.9375$$

At the instant when $R = 2.5$ m, the rate of decrease of volume is 0.9375 m^3 per minute.

1 The complex number z is such that $|z| = 1$ and $\arg z = \theta$, where $0 < \theta < \dfrac{\pi}{4}$.

(i) Mark a possible point A representing z on an Argand diagram. Hence, mark the points B and C representing z^2 and $z + z^2$ respectively on the same Argand diagram corresponding to point A. [2]

(ii) State the geometrical shape of $OACB$. [1]

(iii) Express $z + z^2$ in polar form, $p\cos(q\theta)\left[\cos(k\theta) + \mathrm{i}\sin(k\theta)\right]$, where p, q and k are constants to be determined. [2]

2 The function f is given by $\mathrm{f} : x \mapsto 3 + \dfrac{1}{x-2}$ for $x \in \mapsto,\ x > 2$.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(ii) Explain why the composite function f^2 exists. [1]

(iii) Find the value of x for which $\mathrm{f}^2(x) = x$. Explain why this value of x satisfies the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [3]

3 It is given that a curve C has parametric equations

$$x = t^2 - t, \quad y = \frac{1}{t^2 + 1} \quad \text{for } -2 \le t < 2.$$

(i) Sketch C, indicating clearly the coordinates of the end points and the points where C cuts the y-axis. [4]

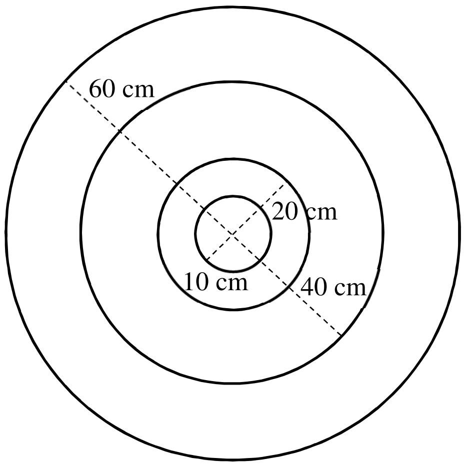
(ii) Find the equation of the tangent to C that is parallel to the y-axis. [4]

(iii) Express the area of the region bounded by C, the tangent found in part **(ii)** and both axes, in the form

$$\int_a^b \mathrm{f}(t)\ \mathrm{d}t,$$

where the function f and the constants a and b are to be determined. Hence find this area, leaving your answer in exact form. [5]

4	A farmer owns a plot of farmland. To prepare for wheat planting, the farmer has to plough the farmland before sowing wheat seeds. At the start of the first week, 300 m^2 of the farmland is ploughed. The farmer ploughs another 100 m^2 of the farmland at the beginning of each subsequent week. To sow wheat seeds, the farmer is considering two different options. **(a)** In the first option, the farmer sows wheat seeds on 60% of the **unsown** ploughed land at the end of each week. **(i)** Find the area of **unsown** ploughed land at the end of the second week. [1] **(ii)** Show that the area of **unsown** ploughed land at the end of the nth week is given by $$\left[0.4^n(300)+k\left(1-0.4^{n-1}\right)\right]\text{ m}^2,$$ where k is an exact constant to be determined. [3] **(iii)** Find the number of complete weeks required for the area of **unsown** ploughed land to first fall below 70 m^2. [3] **(b)** In the second option, the farmer sows 80 m^2 of the **unsown** ploughed land at the end of the first week. At the end of each subsequent week, he sows 20 m^2 of the **unsown** ploughed land more than in the previous week. This means that the area of sown ploughed land is 100 m^2 in the second week, 120 m^2 in the third week, and so on. **(i)** Find, in terms of n, the area of **unsown** ploughed land at the end of the nth week. [4] **(ii)** Find the number of complete weeks required for the farmer to finish sowing all the ploughed farmland in this option. Deduce the area of ploughed land to be sown in the final week. [4]
5	A group of twelve people consists of six married couples. Each couple consists of a husband and a wife. **(i)** The twelve people are to stand in a straight line. Find the number of different arrangements if each husband must stand next to his wife. [2] **(ii)** The group of twelve people finds a round table with ten chairs. Assuming only ten people are to be seated, find the probability that five married couples are seated such that each husband sits next to his wife and husbands and wives alternate. [3]

6	Seven red counters and two blue counters are placed in a bag. All the counters are indistinguishable except for their colours. Clark and Kara take turns to draw a counter from the bag at random with replacement. The first player to draw a blue counter wins the game and the game ends immediately. If Clark draws first, find the probability that **(i)** Clark wins the game at his third draw, [2] **(ii)** Kara wins the game. [3]
7	An archer shoots an arrow into a circular target board that has a radius of 60 cm. The target board further consists of three inner concentric circular sections, with radii 40 cm, 20 cm and 10 cm respectively as shown in the diagram.  The archer scores • 50 points if the arrow lands in the centre circle of radius 10 cm, • 20 points if the arrow lands in the ring with outer radius 20 cm, • 10 points if the arrow lands in the ring with outer radius 40 cm, • 0 point otherwise. Assume that the arrow will definitely hit the target board and is equally likely to hit any portion of the target board. **(i)** Let X be the number of points scored for one arrow shot. Find the expectation of X, leaving your answer in 4 significant figures. [3] **(ii)** Interpret, in this context, the value obtained in part **(i)**. [1] **(iii)** The archer shot at the target board forty times. Find the probability that the average score obtained by the archer is between 10 and 20 points (inclusive). [4]

8 At a hospital, records show that 84.5% of patients turn up for their appointments. It is known that on any day, the doctor has time to see 20 patients.

On one particular day, there are 20 patients who make appointments to see the doctor.

(i) State, in this context, one condition that must be met for the number of patients who turn up for their appointments to be well modelled by a binomial distribution. [1]

For the remainder of this question, assume that the condition stated in part **(i)** is met.

(ii) Find the probability that more than 15 patients turn up for their appointments. [2]

(iii) Given that at least 12 patients turn up for their appointments, find the probability that more than 2 patients fail to turn up for their appointments. [3]

(iv) To improve efficiency, the hospital decides to increase the number of appointments that can be made on each day. Given that there will still be enough time for the doctor to see 20 patients, find the greatest number of appointments that can be made so that there is a probability of at least 0.85 of the doctor having time to see all patients who turn up. [2]

9 In order to recruit the best possible employees, a large corporation has designed an entrance test that consists of three components, namely Logical Reasoning, Personality and Communication. The scores obtained by candidates in each of the three components are independent random variables L, P and C which are normally distributed with means and standard deviations as shown in the table.

	Mean	Standard deviation
Logical Reasoning, L	35.2	5.2
Personality, P	24.6	3.8
Communication, C	29.3	4.3

(i) For a particular role in the corporation, the Logical Reasoning and the Personality scores of a candidate is valued and hence a special score of $3L+2P$ is computed.

(a) Find the special score that is exceeded by only 1% of candidates taking the test. Leave your answer in 1 decimal place. [4]

(b) Five candidates are selected randomly. Find the probability that three of them obtained a special score of more than 150, and the other two obtained less than 140. [3]

(ii) For another role in the corporation, a candidate must achieve a result such that his special score of $3L+2P$ differs from $5C$ by less than 25. Find the percentage of candidates who will be able to achieve this. [4]

10 The following table shows the mass (m) of a foetus, in grams, taken at various weeks (t).

t	12	16	20	24	28	32	36
m	14	100	300	600	1005	1702	2622

(i) Draw a scatter diagram to illustrate the data, labelling the axes clearly. [1]

(ii) Calculate the product moment correlation coefficient between t and m, giving your answer correct to 5 decimal places. Explain why this value does not necessarily mean that the linear model is the best model for the relationship between t and m. [2]

It is proposed that the mass of the foetus at week t can be modelled by

$$m = at^b,$$

where a and b are positive constants.

(iii) By using logarithm to transform $m = at^b$ into a linear equation, calculate the value of the product moment correlation coefficient and give two reasons why this model may be a better model. [4]

(iv) Calculate the values of a and b. [2]

(v) Using the equation of a suitable regression line, estimate the mass of the foetus at 26 weeks, giving your answer to the nearest grams. Comment on the reliability of the estimate. [2]

11 The mass of strawberry jam in a randomly chosen jar follows a normal distribution and has a mean mass of 200 grams. A retailer suspects that the mean mass of the strawberry jam is being overstated. He takes a random sample of 30 jars of strawberry jam and weighs the content, x grams, in each jar. The results are summarized as follows.

$$\sum(x-200) = -66 \quad \text{and} \quad \sum(x-200)^2 = 958$$

(i) Test at 2% significance level, whether the retailer's suspicion is justifiable. [6]

(ii) Explain, in this context, the meaning of 'at 2% significance level'. [1]

(iii) Suppose the retailer now decides to test whether the mean mass differs from 200 grams at 2% significance level. Without carrying out the test, explain whether the conclusion would change in part **(i)**. [1]

The manufacturing process has now been improved and the population standard deviation is 3.5 grams. The retailer selects a new random sample of 20 jars of strawberry jam and the sample mean is found to be k grams. Find the range of possible values of k so that the retailer's suspicion that the mean mass differs from 200 grams is not justified at the 2% significance level. Give your answer correct to one decimal place. [4]

ANNEX B

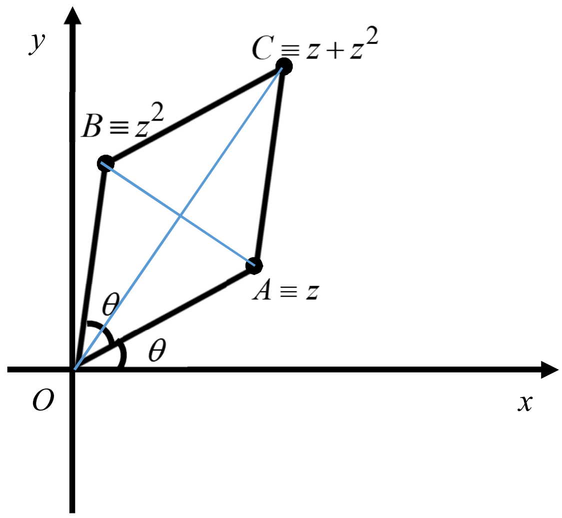
IJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Complex numbers	(ii) rhombus (iii) $2\cos\frac{\theta}{2}\left[\cos\frac{3\theta}{2}+\mathrm{i}\sin\frac{3\theta}{2}\right]$
2	Functions	(i) $\mathrm{f}^{-1}(x)=2+\frac{1}{x-3},\ x\in\mathbb{R},\ x>3$ (iii) $x=3.62$
3	Differentiation & Applications	(ii) $x=-\frac{1}{4}$ (iii) $\ln\frac{8}{5}-\frac{\pi}{4}+\tan^{-1}\frac{1}{2}$
4	AP and GP	(a)(i) $88\ \mathrm{m}^2$ (a)(ii) k is $\frac{200}{3}$ (a)(iii) 5 (b)(i) $-10n^2+30n+200$ (b)(ii) number of complete weeks is 7; $120\,\mathrm{m}^2$
5	P&C, Probability	(i) 46080 (ii) 0.0000120
6	P&C, Probability	(i) 0.0813 (ii) 0.4375
7	DRV	(i) 6.389 (iii) 0.00965
8	Binomial Distribution	(i) Whether a randomly chosen patient turns up for an appointment is independent of any other patient. (ii) 0.812 (iii) 0.618 (iv) 22
9	Normal Distribution	(i)(a) $a=195.2$ (i)(b) 0.0875 (ii) 61.3%
10	Correlation & Linear Regression	(ii) $r=0.94597$ (iii) $r=0.990$ (iv) $a=2.30\times10^{-4}$, $b=4.59$ (v) $m=728$ Since the value of 26 is within the range of values of t and the value of r is close to 1, this estimate is reliable.

11	Hypothesis Testing	(ii) At 2% significance level means that there is a probability of 0.02 that **<u>the test will indicate</u>** that the mean mass of the strawberry jam in the jar is less than 200 g when in fact it is 200 g. (iii) This will result in a different conclusion; $198.2 < k < 201.8$

1

(i)



(ii)

Since $OACB$ is a parallelogram with 4 equal sides, it is a **rhombus**.

(iii)

$$\begin{aligned}
&z+z^2 \\
&=\cos\theta+\mathrm{i}\sin\theta+\left(\cos\theta+\mathrm{i}\sin\theta\right)^2 \\
&=\cos\theta+\mathrm{i}\sin\theta+\cos^2\theta+2\mathrm{i}\cos\theta\sin\theta-\sin^2\theta \\
&=\left(\cos\theta+\cos 2\theta\right)+\mathrm{i}\left(\sin\theta+\sin 2\theta\right) \\
&=2\cos\frac{3\theta}{2}\cos\frac{\theta}{2}+2\mathrm{i}\sin\frac{3\theta}{2}\cos\frac{\theta}{2} \\
&=2\cos\frac{\theta}{2}\left[\cos\frac{3\theta}{2}+\mathrm{i}\sin\frac{3\theta}{2}\right]
\end{aligned}$$

<u>Alternative</u>

$$\arg\left(z+z^2\right)=\theta+\frac{\theta}{2}=\frac{3}{2}\theta$$

$$\left|z+z^2\right|=2OM=2\cos\left(\frac{\theta}{2}\right)$$

$$z+z^2=2\cos\left(\frac{\theta}{2}\right)\left[\cos\left(\frac{3}{2}\theta\right)+\mathrm{i}\sin\left(\frac{3}{2}\theta\right)\right]$$

$$\therefore p=2,\ q=\frac{1}{2},\ k=\frac{3}{2}$$

2

(i)

$$\mathrm{f}: x \mapsto 3+\frac{1}{x-2},\ x\in\mathbb{R},\ x>2$$

Let $y=\mathrm{f}(x)$.

$$y = 3 + \frac{1}{x-2}$$

$$x - 2 = \frac{1}{y-3}$$

$$x = 2 + \frac{1}{y-3}$$

$$\therefore \ \mathrm{f}^{-1}(x) = 2 + \frac{1}{x-3}, \ x \in \mathbb{R}, \ x > 3$$

(ii)

$\mathrm{D_f} = (2, \infty)$

$\mathrm{R_f} = (3, \infty)$

Since $\mathrm{R_f} \subseteq \mathrm{D_f}$, the composite function f^2 exists.

(iii)

$$\mathrm{f}^2(x) = x$$

$$\mathrm{f}\left(3 + \frac{1}{x-2}\right) = x$$

$$3 + \frac{1}{3 + \frac{1}{x-2} - 2} = x$$

$$3 + \frac{1}{\left(\frac{x-1}{x-2}\right)} = x$$

$$\frac{3(x-1) + (x-2)}{x-1} = x$$

$$4x - 5 = x(x-1)$$

$$x^2 - 5x + 5 = 0$$

Using GC, $x = 1.38$ (rej $\because 1.38 \notin \mathrm{D_f}$) or $x = 3.62$

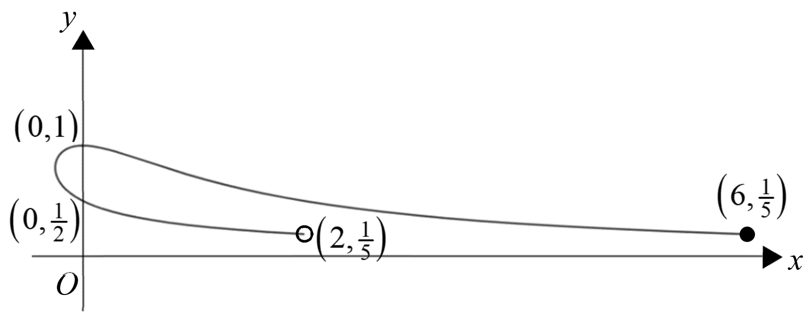
$$\mathrm{ff}(x) = x$$

$$\mathrm{f}^{-1}\mathrm{ff}(x) = \mathrm{f}^{-1}(x)$$

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$$

Therefore $x = 3.62$ satisfies $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$.

3



When $x = 0$, $t(t-1) = 0 \quad \Rightarrow t = 0$ or $t = 1$

$\Rightarrow y = 1$ or $y = \frac{1}{2}$

Coordinates are $(0,1)$ and $\left(0, \frac{1}{2}\right)$.

(ii)

$\frac{dx}{dt} = 2t - 1, \quad \frac{dy}{dt} = \frac{-2t}{\left(t^2+1\right)^2}$

$$\therefore \frac{dy}{dx} = \frac{-2t}{\left(t^2+1\right)^2} \times \frac{1}{2t-1}$$

$$= \frac{-2t}{\left(t^2+1\right)^2 (2t-1)}$$

When tangent is parallel to y-axis,

$$\left(t^2+1\right)^2 (2t-1) = 0 \Rightarrow t = \frac{1}{2} \qquad \left(\because \left(t^2+1\right)^2 > 0\right)$$

Equation of tangent: $x = -\frac{1}{4}$

(iii)

Area of the required region

$$= \int_{-1/4}^{0} y \quad dx$$

$$= \int_{1/2}^{1} \frac{1}{t^2+1}(2t-1)\, dt$$

When $x = -\frac{1}{4}$, $t = \frac{1}{2}$ When $x = 0$, $t = 1$

$$= \int_{1/2}^{1} \frac{2t}{t^2+1} - \frac{1}{t^2+1}\, dt$$

$$= \left[\ln\left(t^2+1\right) - \tan^{-1} t\right]_{1/2}^{1}$$

$$= \left[\left(\ln 2 - \frac{\pi}{4}\right) - \left(\ln\frac{5}{4} - \tan^{-1}\frac{1}{2}\right)\right]$$

$$= \ln\frac{8}{5} - \frac{\pi}{4} + \tan^{-1}\frac{1}{2}$$

4	(a)(i) Area of **unsown** ploughed land $=0.4\left[0.4(300)+100\right]$ $=88\text{ m}^2$

(a)(ii)

n	Beginning of week	End of week
1	300	$0.4(300)$
2	$0.4(300)+100$	$0.4\left[0.4(300)+100\right]$ $=0.4^2(300)+0.4(100)$
3	$0.4^2(300)+0.4(100)$ $+100$	$0.4\left[0.4^2(300)+0.4(100)+100\right]$ $=0.4^3(300)+0.4^2(100)+0.4(100)$
..	…	…
n	…	$0.4^n(300)+0.4^{n-1}(100)+...$ $+0.4^2(100)+0.4^1(100)$

Area of land **unsown** ploughed land at the end of nth week

$$=0.4^n(300)+100\left[\frac{0.4\left(1-0.4^{n-1}\right)}{1-0.4}\right]$$

$$=\left[0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)\right]\text{ m}^2$$

$\therefore$ the value of k is $\dfrac{200}{3}$.

(a)(iii)

<u>Method 1</u>

$$0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)<70$$

$$0.4^n(300)+\frac{200}{3}-\frac{200}{3}(0.4)^{-1}0.4^n<70$$

$$\frac{400}{3}\left(0.4^n\right)<\frac{10}{3}$$

$$0.4^n<\frac{1}{40}$$

$$n>\ln\left(\tfrac{1}{40}\right)\Big/\ln 0.4$$

$$n>4.02588$$

Hence the number of complete weeks required is 5.

Method 2

$0.4^n(300)+\frac{200}{3}\left(1-0.4^{n-1}\right)<70$

Using GC,
when n = 4, unsown ploughed land = 70.08 (> 70)
when n = 5, unsown ploughed land = 68.032 (< 70)
when n = 6, unsown ploughed land = 67.213 (< 70)

Hence the number of complete weeks required is 5.

(b)(i)

n	Beginning of week	End of week
1	300	$300-80$
2	$300+(100)-80$	$300+(100)-80-100$
3	$300+2(100)-80-100$	$300+2(100)-80-100-120$
..	$\cdots$	$\cdots$
n	$\cdots$	$300+(n-1)(100)-80-100$ $-\ \cdots\ -\left[80+20(n-1)\right]$

Area of **unsown** ploughed land at the end of nth week

$=300+100(n-1)-\frac{n}{2}\left[2(80)+20(n-1)\right]$

$=300+100n-100-\frac{n}{2}(140+20n)$

$=300+100n-100-70n-10n^2$

$=-10n^2+30n+200$

(b)(ii)
For the farmer to finish sowing all the ploughed farmland,
$-10n^2+30n+200\le 0$

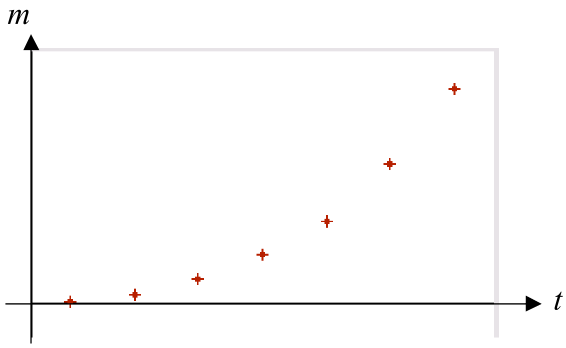
Method 1:
Solving the inequality,
$n\ge 6.21699$ or $n\le -3.21699$ (rejected)
Hence the number of complete weeks is 7.

Method 2:
Using GC to set up a table,
When $n=6$, area unsown $=20$
When $n=7$, area unsown $=-80$
When $n=8$, area unsown $=-200$
Hence the number of complete weeks is 7.

	In week 6, the area of **unsown** ploughed land $=-10(6)^2+30(6)+200=20\ \text{m}^2$ $\therefore$ area of ploughed land to be **sown** in week 7 (the final week) $=20+100=120\ \text{m}^2$
5	(i) Number of arrangements = $6!\times 2^6=46080$ (ii) Required probability $=\dfrac{{}^{6}C_5\times(5-1)!\times 2}{{}^{12}C_{10}\times(10-1)!}$ $=\dfrac{288}{23950080}$ $=0.0000120$ (3 sig fig)
6	(i) P(Clark wins in 3rd draw) $=\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{7}{9}\times\dfrac{2}{9}$ $=0.081322$ $=0.0813$ (ii) P(Kara wins) $=\dfrac{7}{9}\times\dfrac{2}{9}+\left(\dfrac{7}{9}\right)^3\times\dfrac{2}{9}+\left(\dfrac{7}{9}\right)^5\times\dfrac{2}{9}+....$ $=\dfrac{2}{9}\left[\dfrac{7}{9}+\left(\dfrac{7}{9}\right)^3+\left(\dfrac{7}{9}\right)^5+...\right]$ $=\dfrac{2}{9}\left(\dfrac{\frac{7}{9}}{1-\left(\frac{7}{9}\right)^2}\right)$ $=0.4375\ \ or\ \ \dfrac{7}{16}$
7	(i) Given that X is the number of points scored for one arrow shot. $\text{P}(X=50)=\dfrac{\pi(10)^2}{\pi(60)^2}=\dfrac{1}{36}$ $\text{P}(X=20)=\dfrac{\pi(20)^2-\pi(10)^2}{\pi(60)^2}=\dfrac{1}{12}$ $\text{P}(X=10)=\dfrac{\pi(40)^2-\pi(20)^2}{\pi(60)^2}=\dfrac{1}{3}$

	$E(X)=(10)\left(\frac{1}{3}\right)+(20)\left(\frac{1}{12}\right)+(50)\left(\frac{1}{36}\right)$ $=6.389$ (4 sig fig) (ii) If the archer is to shoot at the target board repeatedly, then in the long run his average score will be 6.389 points. (iii) $\text{Var}(X)=(10)^2\left(\frac{1}{3}\right)+(20)^2\left(\frac{1}{12}\right)+(50)^2\left(\frac{1}{36}\right)-(6.38888)^2$ $=95.2932$ Let $\overline{X}=\frac{X_1+X_2+...+X_{40}}{40}$. Since n = 40 is large, by Central Limit Theorem, $\overline{X}\sim N\left(6.38888,\frac{95.2932}{40}\right)$ approximately. Required probability $=P(10<\overline{X}<20)$ $=0.00965$ (3 sig fig)
8	(i) Whether a randomly chosen patient turns up for an appointment is independent of any other patient. (ii) Let X be the number of patients who turn up for their appointments, out of 20 appointments. $X\sim B(20,0.845)$ $P(X>15)$ $=1-P(X\le 15)$ $=0.812$ (3 sig fig) (iii) Required probability $=P(X\le 17 \mid X\ge 12)$ $=\frac{P(12\le X\le 17)}{P(X\ge 12)}$ $=\frac{P(X\le 17)-P(X\le 11)}{1-P(X\le 11)}$ $=0.618$ (3 sig fig)

	(iv) Let Y be the number of patients who turn up for their appointments, out of n appointments. $Y \sim \mathrm{B}(n, 0.845)$ $\mathrm{P}(Y \le 20) \ge 0.85$ --- (*) Using GC, When $n = 21$, $\mathrm{P}(Y \le 20) = 0.9709 \quad (> 0.85)$ When $n = 22$, $\mathrm{P}(Y \le 20) = 0.8762 \quad (> 0.85)$ When $n = 23$, $\mathrm{P}(Y \le 20) = 0.7146 \quad (< 0.85)$ $\therefore$ Largest n is 22.
9	(i)(a) Given: $L \sim \mathrm{N}(35.2, 5.2^2)$ $P \sim \mathrm{N}(24.6, 3.8^2)$ $C \sim \mathrm{N}(29.3, 4.3^2)$ Let $T = 3L + 2P$. $\mathrm{E}(T) = 3 \times 35.2 + 2 \times 24.6 = 154.8$ $\mathrm{Var}(T) = 3^2 \times 5.2^2 + 2^2 \times 3.8^2 = 301.12$ $\therefore T \sim \mathrm{N}(154.8, 301.12)$ Let a be the required score exceed by 1% of the candidates. $\mathrm{P}(T > a) = 0.01$ $\Rightarrow \mathrm{P}(T \le a) = 0.99$ Using GC, $a = 195.2$ (1 dec pl) (i)(b) Required probability $= [\mathrm{P}(T > 150)]^3 [\mathrm{P}(T < 140)]^2 \times \left(\frac{5!}{2!3!}\right)$ $= 0.0875$ (3 sig fig) (ii) Consider $A = 3L + 2P - 5C$ $\mathrm{E}(A) = 154.8 - 5(29.3) = 8.3$ $\mathrm{Var}(A) = 301.12 + 5^2(4.3^2) = 763.37$ $\therefore A \sim \mathrm{N}(8.3, 763.37)$ Required probability $= \mathrm{P}(\|A\| < 25)$ $= \mathrm{P}(-25 < A < 25)$ $= 0.613$ (3 sig fig) Required percentage = 61.3%

10	(i)  (ii) The product moment correlation coefficient between t and m is $r = 0.94597$ (5 d.p.). A value of 0.94597 for r suggests that there is a strong positive linear correlation between t and m. However, the points on the scatter diagram **<u>show a curvilinear relationship</u>**. Therefore this value of r does not necessarily mean that the linear model is best model for the relationship between t and m. (iii) $m = at^b$ $\ln m = \ln\left(at^b\right)$ $\ln m = b\ln t + \ln a$ The product moment correlation coefficient between ln t and ln m is $r = 0.98967 = 0.990$ (3 sig fig) **Reason 1:** From the scatter diagram, as t increases, the **weight of the foetus increases at an increasing rate**. **Reason 2:** The value of r between ln t and ln m is 0.98967, which is closer to 1 as compared to that between t and m, hence indicating a **stronger positive linear correlation** between ln t and ln m. Hence $m = at^b$ is a better model. (iv) From GC, $\ln m = -8.3764 + 4.5938 \ \ln t$ (5 sig fig) $\ln a = -8.3764$ $a = 2.30\times10^{-4}$ and $b = 4.59$ (v) When $t = 26$, $\ln m = -8.3764 + 4.5938\ln 26$ $m = 728$ (nearest grams) Since the value of 26 is within the range of values of t and the value of r is close to 1, this estimate is reliable.

11	(i) Let X be the random variable denoting the mass of strawberry jam, in grams, in a randomly chosen jar. Unbiased estimate of population mean $\bar{x} = \frac{-66}{30} + 200 = 197.8$ Unbiased estimate of population variance $s^2 = \frac{1}{29}\left[958 - \frac{(-66)^2}{30}\right] = 28.02759$ $H_0 : \mu = 200$ $H_1 : \mu < 200$ Test at 2% significance level Assume H_0 is true. $\bar{X} \sim N\left(200, \frac{28.02759}{30}\right)$ Test statistic: $Z = \frac{\bar{X} - 200}{\sqrt{28.02759/30}} \sim N(0,1)$ Using GC, p-value = 0.011420121 < 0.02 Reject H_0 and conclude that there is sufficient evidence at 2% level of significance that the mean mass of strawberry jam in each jar is overstated. Therefore the retailer's suspicion is justifiable. (ii) At 2% significance level means that there is a probability of 0.02 that **<u>the test will indicate</u>** that the mean mass of the strawberry jam in the jar is less than 200 g when in fact it is 200 g. (iii) $H_0 : \mu = 200$ $H_1 : \mu \neq 200$ For a two tailed test, the p-value will be twice of 0.0114 which is 0.0228. This value is now more than the 0.02 where we do not reject H_0 at 2% significance level. As such this will result in a different conclusion. (iv) $H_0 : \mu = 200$ $H_1 : \mu \neq 200$ Test at 2% significance level Assume H_0 is true. $\bar{X} \sim N\left(200, \frac{3.5^2}{20}\right)$. Test statistic: $Z = \frac{\bar{X} - 200}{\sqrt{3.5^2/20}} \sim N(0,1)$

	For the retailer's suspicion that the mean mass differs from 200 g to be not justified, **do not reject H_0.** $\Rightarrow$ z-value falls outside the critical region $-2.32635 < z\text{-value} < 2.32635$ $-2.32635 < \dfrac{k-200}{3.5/\sqrt{20}} < 2.32635$ $-1.82066 < k-200 < 1.82066$ $198.17934 < k < 201.82066$ $\Rightarrow 198.2 < k < 201.8$ (to 1 d.p)