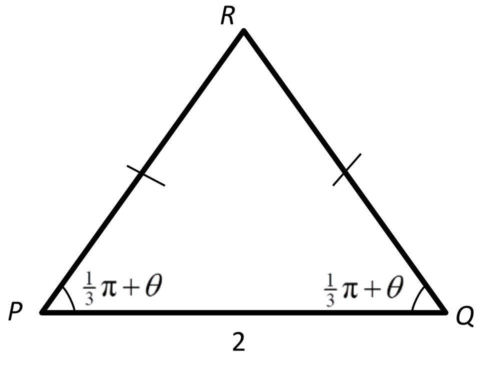
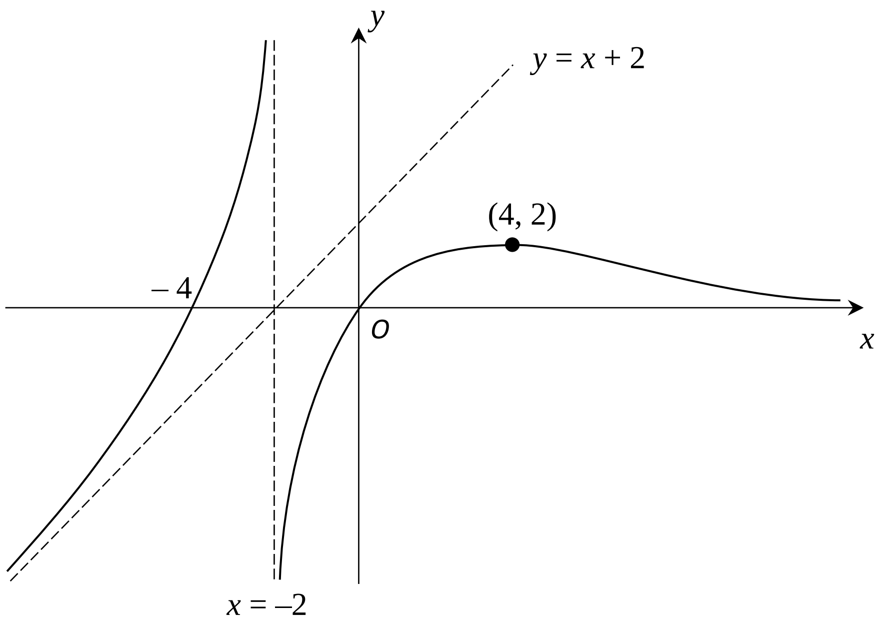
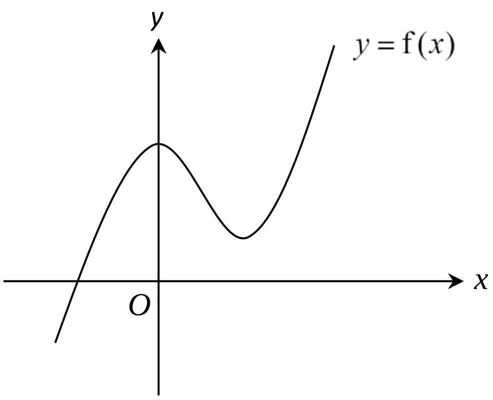
H2 Mathematics 2017 Prelim Exam Paper 1 Question
Answer all questions [100 marks].

1	Given that $\sum_{k=1}^{n} k!(k^2+1) = (n+1)!\,n$, find $\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)$. [3]
2	A geometric sequence $T_1, T_2, T_3, \ldots$ has a common ratio of e. Another sequence $U_1, U_2, U_3, \ldots$ is such that $U_1 = 1$ and $$U_r = \ln T_r - 3 \quad \text{for all } r \geq 1.$$ **(i)** Prove that the sequence $U_1, U_2, U_3, \ldots$ is arithmetic. [2] A third sequence $W_1, W_2, W_3, \ldots$ is such that $W_1 = \frac{1}{2}$ and $W_{r+1} = W_r + U_r$ for all $r \geq 1$. **(ii)** By considering $\sum_{r=1}^{n-1}(W_{r+1} - W_r)$, show that $W_n = \frac{1}{2}(n^2 - n + 1)$. [3]
3	Using an algebraic method, find the set of values of x that satisfies the inequality $$2 - x \leq \frac{x}{2-x}.$$ [3] Hence solve $2 - x^2 \leq \dfrac{x^2}{2-x^2}$. [2]
4	 In the isosceles triangle PQR, PQ = 2 and the angle QPR = angle $PQR = \left(\frac{1}{3}\pi + \theta\right)$ radians. The area of triangle PQR is denoted by A. Given that θ is a sufficiently small angle, show that $$A = \frac{\sqrt{3} + \tan\theta}{1 - \sqrt{3}\tan\theta} \approx a + b\theta + c\theta^2,$$ for constants a, b and c to be determined in exact form. [5]

5	**(a)** Given that $\operatorname{cosec} y = x$ for $0 < y < \frac{1}{2}\pi$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of y. Deduce that $\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{(x^2-1)}}$ for $x > 1$. [3] **(b)** The function f is such that $\mathrm{f}(x)$ and $\mathrm{f}'(x)$ exist for all real x. Sketch a possible graph of f which illustrates that the following statement is not necessarily true: "If the equation $\mathrm{f}'(x) = 0$ has exactly one root $x = 0$ and $\mathrm{f}''(0) > 0$, then $\mathrm{f}(x) \to \infty$ as $x \to \pm\infty$." [2]
6	**(a)** State a sequence of transformations that transform the graph of $x^2 + \frac{1}{3}(y-2)^2 = 1$ to the graph of $(x-2)^2 + y^2 = 1$. [3] **(b)** The diagram below shows the curve $y = \mathrm{f}(x)$. It has a maximum point at $(4, 2)$ and intersects the x-axis at $(-4, 0)$ and the origin. The curve has asymptotes $x = -2$, $y = 0$ and $y = x + 2$.  Sketch on separate diagrams, the graphs of **(i)** $y = \mathrm{f}'(x)$, [3] **(ii)** $y = \frac{1}{\mathrm{f}(x)}$, [3] including the coordinates of the points where the graphs cross the axes, the turning points and the equations of any asymptotes, where appropriate.
7	**(i)** Express $\sin x + \sqrt{3}\cos x$ as $R\sin(x+\alpha)$, where $R > 0$ and α is an acute angle. [1]

	The function f is defined by $$\mathrm{f} : x \mapsto \sin x + \sqrt{3}\cos x, \quad x \in \mathbb{R},\ -\tfrac{1}{3}\pi \le x \le \tfrac{1}{6}\pi.$$ **(ii)** Sketch the graph of $y = \mathrm{f}(x)$. [2] **(iii)** Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. On the same diagram as in part **(ii)**, sketch the graph of $y = \mathrm{f}^{-1}(x)$, indicating the equation of the line of symmetry. [4] **(iv)** Using integration, find the area of the region bounded by the graph of f^{-1} and the axes. [3] The function g is defined by $$\mathrm{g} : x \mapsto \lvert \ln(x+2) \rvert, \text{ for } x \in \mathbb{R},\ x > -2.$$ **(v)** Show that the composite function gf^{-1} exists, and find the range of gf^{-1}. [3]
8	**Do not use a graphic calculator in answering this question.** **(a)**  It is given that $\mathrm{f}(x)$ is a cubic polynomial with real coefficients. The diagram shows the curve with equation $y = \mathrm{f}(x)$. What can be said about all the roots of the equation $\mathrm{f}(x) = 0$? [2] **(b)** The equation $2z^2 - (7+6\mathrm{i})z + 11 + \mathrm{i}c = 0$, where c is a non-zero real number, has a root $z = 3 + 4\mathrm{i}$. Show that $c = -2$. Determine the other root of the equation in cartesian form. Hence find the roots of the equation $2w^2 + (-6+7\mathrm{i})w - 11 + 2\mathrm{i} = 0$. [6] **(c)** The complex number z is given by $z = 1 + \mathrm{e}^{\mathrm{i}\alpha}$. **(i)** Show that z can be expressed as $2\cos\left(\tfrac{1}{2}\alpha\right)\mathrm{e}^{\mathrm{i}\left(\frac{1}{2}\alpha\right)}$. [2]

(ii) Given $\alpha = \frac{1}{3}\pi$ and $w = -1 - \sqrt{3}\mathrm{i}$, find the exact modulus and argument of $\left(\dfrac{z}{w^3}\right)^*$. [5]

9 The line l_1 passes through the point A, whose position vector is $3\mathbf{i}+7\mathbf{j}+3\mathbf{k}$, and is parallel to the vector $3\mathbf{i}+4\mathbf{j}+\mathbf{k}$. The line l_2 is given by the cartesian equation $x-2=\dfrac{3-y}{2}=\dfrac{z-5}{2}$.

The plane p_1 contains l_1 and is parallel to l_2. Another plane p_2 also contains l_1 and is perpendicular to p_1.

(i) Find the cartesian equation of p_1. [3]

(ii) Find the distance of l_2 to p_1. [2]

(iii) Find the equation of p_2 in the scalar product form. [2]

A particle P moves along a straight line c which lies in the plane p_2 and c passes through a point $(5, \frac{1}{2}, -3)$. P hits the plane p_1 at A and rebounds to move along another straight line d in p_2. The angle between d and l_1 is the same as the angle between c and l_1.

(iv) Find the direction cosines of d. [6]

(v) Another particle, Q, is placed at the point $(\frac{25}{2}, \frac{21}{2}, -\frac{1}{2})$. Find the shortest distance PQ as P moves along d. [3]

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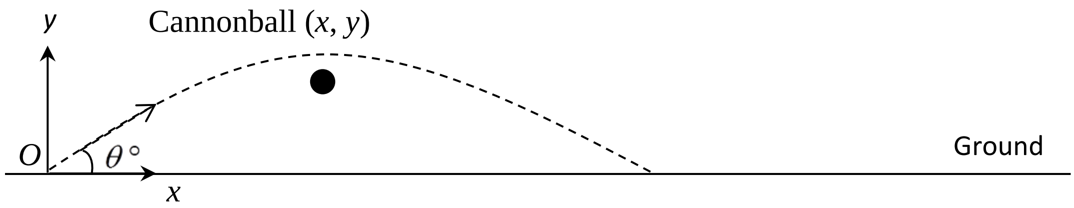

The diagram shows the trajectory of a cannonball fired off from an origin O with an initial speed of v ms^{-1} and at an angle of $\theta°$ above the ground. At time t seconds, the position of the cannonball can be modelled by the parametric equations

$$x = (v\cos\theta)t,\ \ y = (v\sin\theta)t - 5t^2,$$

where x m is the horizontal distance of the cannonball with respect to O and y m is the vertical distance of the cannonball with respect to ground level.

(i) Find the horizontal distance, d m, that a cannonball would have travelled by the time it hits the ground. Leave your answer in terms of v and θ. [4]

Use $v = 200$ to answer the remaining parts of the question.

An approaching target is travelling at a constant speed of 10 ms^{-1} along the ground. A cannonball is fired towards the target when it is 3000 m away. You may assume the height of the moving target is negligible.

	(ii) Show that in order to hit the target, the possible angles at which the cannonball should be fired are $22.7°$ and $69.5°$. [2] **(iii)** Explain at which angle the cannonball should be fired in order to hit the target earlier. [2] **(iv)** Given that $\theta = 22.7,$ find the angle that the tangent to the trajectory makes with the horizontal when $x = 370$. [4]
11	**For this question, you may leave your answers to the nearest dollar.**

(a) Mr Foo invested $25,000 in three different stocks *A*, *B* and *C*. After a year, the value of the stocks *A* and *B* grew by 2% and 6% respectively, while the value of stock *C* fell by 2%. Mr Foo did not gain or lose any money. Let *a*, *b* and *c* denote the amount of money he invested in stocks *A*, *B* and *C* respectively.

(i) Find expressions for *a* and *b*, in terms of *c*. [2]

(ii) Find the values between which *c* must lie. [2]

(b) Mr Lee is interested in growing his savings amount of $55,000 and is considering the Singapore Savings Bonds. He is able to enjoy a higher average return per year when he invests over a longer period of time as shown in the following table.

Number of years invested	1	2	3	4	5	6	7	8
Average return per year, %	1.04	1.21	1.35	1.48	1.60	1.71	1.82	1.92

For example, if Mr Lee invests for two years, he is able to enjoy compound interest at a rate of 1.21% per year.

(i) Calculate the compound interest earned by Mr Lee if he were to invest $55,000 in this bond for a period of five years. [2]

A bank offers a dual-savings account with the following scheme:

“ For every $1,000 deposited into the normal savings account, an individual can deposit $10,000 into the special savings account to enjoy a higher interest rate. The annual compound interest rates for the normal savings account and the special savings account are 0.19% and 1.8% respectively.”

Mr Lee is interested in setting up this dual-savings account and considers an *n*-year investment plan as such:

At the start of each year, he will place $1,000 in the normal savings account and $10,000 in the special savings account.

(ii) Find the respective amount of money in the normal savings account and special savings account at the end of *n* years. Leave your answers in terms of *n*. [4]

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	(iii) Find the least value of n such that the compound interest earned in dual-savings account is more than the compound interest earned in part **(i)**. [2]

End Of Paper −

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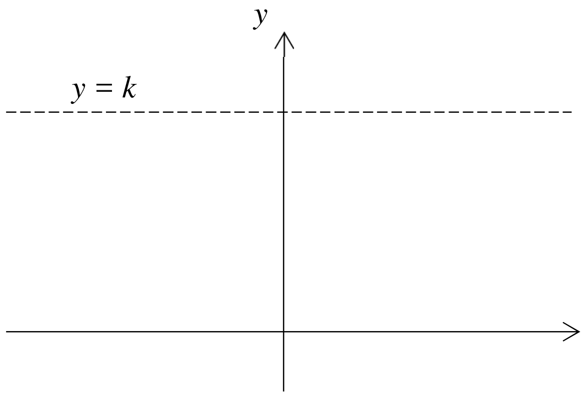
ANNEX B

DHS H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Sigma Notation and Method of Difference	$(n+1)!n-2$
2	Sigma Notation and Method of Difference	--
3	Equations and Inequalities	$\{1\le x<2 \text{ or } x\ge 4\}$; $x\le -2$ or $-\sqrt{2}<x\le -1$ or $1\le x<\sqrt{2}$ or $x\ge 2$
4	Maclaurin series	$\sqrt{3}+4\theta+\left(4\sqrt{3}\right)\theta^2$
5	Differentiation & Applications	(a) $\dfrac{dy}{dx}=-\dfrac{1}{\operatorname{cosec} y \cot y}$ (b) 
6	Graphs and Transformation	(a) 1. Translate 2 units in the positive x-direction 2. Translate 2 units in the negative y-direction 3. Scale by a factor of $\dfrac{1}{\sqrt{3}}$ parallel to the y-direction Alternative 2. Scale by a factor of $\dfrac{1}{\sqrt{3}}$ parallel to the y-direction 3. Translate $\dfrac{2}{\sqrt{3}}$ units in the negative y-direction

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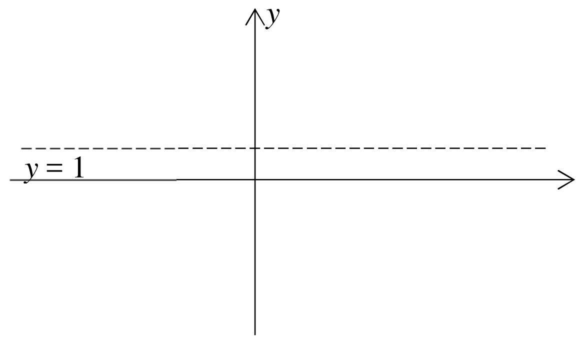
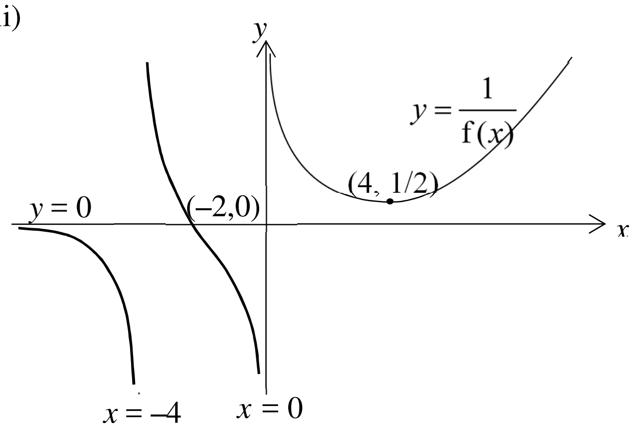
(b)(i)



(b) (ii)



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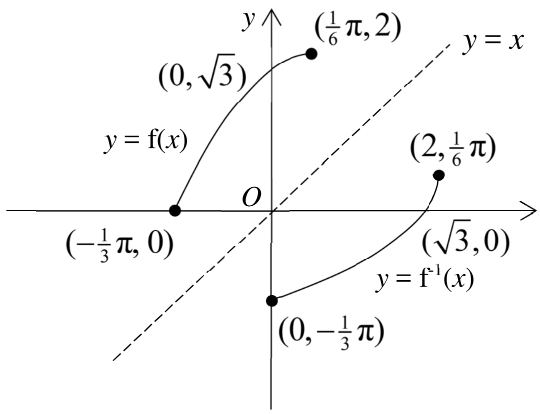
(i) $\sin x + \sqrt{3}\cos x = 2\sin(x + \tfrac{1}{3}\pi)$

(ii)



(iii) $f^{-1}(x) = -\tfrac{1}{3}\pi + \sin^{-1}(\tfrac{1}{2}x);\ D_{f^{-1}} = R_f = [0, 2]$

(iv) 1

(v) $R_{gf^{-1}} = [0,\ 0.926]$

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8	Complex numbers	(a) Since the curve shows only one x-intercept, it means that there is only one real root in the equation $\mathrm{f}(x)=0$. Since the equation has all real coefficients, then the two other roots must be non-real and they are a conjugate pair. (b) $\frac{1}{2}-\mathrm{i}$; $4-3\mathrm{i}$ and $-1-\frac{1}{2}\mathrm{i}$. (c) (ii) $\frac{\sqrt{3}}{8}$; $-\frac{\pi}{6}$
9	Vectors	(i) $2x-y-2z=-7$ (ii) $\frac{2}{3}$ (iii) $\mathbf{r}.\begin{pmatrix}-7\\8\\-11\end{pmatrix}=2$ (iv) $\frac{16}{\sqrt{329}},\frac{3}{\sqrt{329}}$ and $-\frac{8}{\sqrt{329}}$ or $-\frac{16}{\sqrt{329}},-\frac{3}{\sqrt{329}}$ and $\frac{8}{\sqrt{329}}$. (v) $\frac{1}{2\sqrt{329}}\left\|\begin{pmatrix}-35\\40\\-55\end{pmatrix}\right\|=2.11$ (3 s.f.)
10	Differentiation & Applications	(i) $\frac{v^2\sin\theta\cos\theta}{5}$ (ii) – (iii) 22.7° (iv) 17.2° (to 1dp)
11	AP and GP	(a)(i) $a=37500-2c,\ b=c-12500$ (ii) between 12500 and 18750 (b)(i) 4543 (to the nearest dollar) (ii) Normal savings account: $527315.79(1.0019^n-1)$ Special savings account: $565555.56\left(1.018^n-1\right)$ (iii) 7

H2 Mathematics 2017 Prelim Exam Paper 1 Solution

1	Method 1 Consider replace k by $(k-1)$: $\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)=\sum_{k-1=1}^{k-1=n-1}(k-1+1)!((k-1)^2+2(k-1)+2)$ $=\sum_{k=2}^{n}k!(k^2+1)$ $=\sum_{k=1}^{n}k!(k^2+1)-1!(1^2+1)$ $=(n+1)!n-2$ Method 2 $\sum_{k=1}^{n}k!\left(k^2+1\right)=\sum_{k=0}^{n-1}(k+1)!\left((k+1)^2+1\right)$ $=\sum_{k=0}^{n-1}(k+1)!\left(k^2+2k+2\right)$ $=(n+1)!n$ $\sum_{k=1}^{n-1}(k+1)!\left(k^2+2k+2\right)=\sum_{k=0}^{n-1}(k+1)!\left(k^2+2k+2\right)$ $+(0+1)!\left(0^2+2(0)+2\right)$ $=(n+1)!n+2$
2	(i) To prove AP, consider $U_{r+1}-U_r$ $=(\ln T_{r+1}-3)-(\ln T_r-3)$ $=\ln\left(\frac{T_{r+1}}{T_r}\right)$ $=\ln \mathrm{e}$ $=1$ Since difference is a **constant,** the sequence is arithmetic. (Proven) $\sum_{r=1}^{n-1}(W_{r+1}-W_r)=\sum_{r=1}^{n-1}U_r$

$$\text{LHS} = \sum_{r=1}^{n-1}(W_{r+1} - W_r)$$

$$= W_2 - W_1$$
$$+ \cancel{W_3} - W_2$$
$$+ \cancel{W_4} - \cancel{W_3}$$
$$\vdots$$
$$+ \cancel{W_n} - \cancel{W_{n-1}}$$
$$= W_n - W_1$$
$$= W_n - \tfrac{1}{2}$$

$$\text{RHS} = \sum_{r=1}^{n-1} U_r$$

$$= U_1 + U_2 + \ldots + U_{n-1}$$
$$= \frac{n-1}{2}(2(1) + (n-2)1)$$
$$= \frac{n(n-1)}{2}$$

Thus, $W_n - \dfrac{1}{2} = \dfrac{n(n-1)}{2}$

$\therefore W_n = \dfrac{1}{2}\left(n^2 - n + 1\right)$ (shown)

3

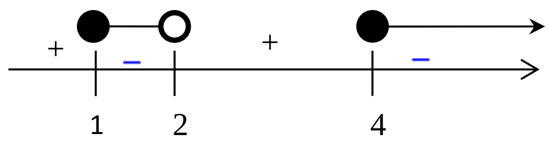
(i) $2 - x \le \dfrac{x}{2-x}$

$$2 - x - \frac{x}{2-x} \le 0$$

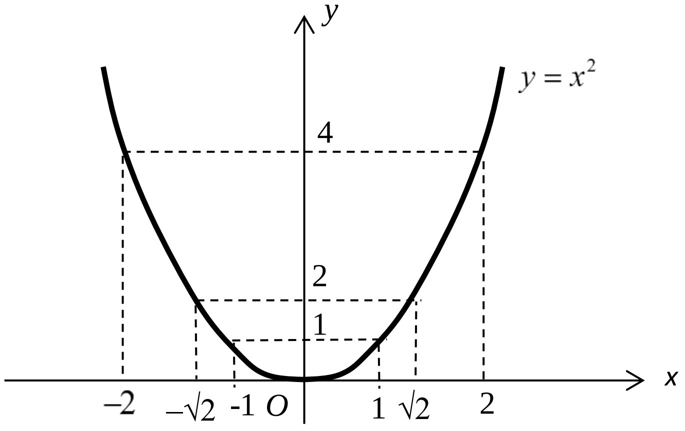
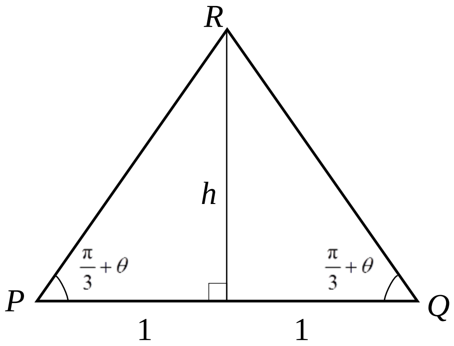
$$\frac{(2-x)^2 - x}{2-x} \le 0$$

$$\frac{x^2 - 5x + 4}{2-x} \le 0$$

$$\frac{(x-4)(x-1)}{2-x} \le 0$$



Set of values of x: $\{1 \le x < 2 \textbf{ or } x \ge 4\}$

	(ii) Let $y = x^2$. $2 - x^2 \le \dfrac{x^2}{2 - x^2} \Rightarrow 2 - y \le \dfrac{y}{2 - y}$ $1 \le y < 2$ or $y \ge 4$ Method 1: Using $y = x^2$ graph  The range of values of x is $x \le -2$ or $-\sqrt{2} < x \le -1$ or $1 \le x < \sqrt{2}$ or $x \ge 2$ Method 2: Using definition of $\|x\|$ Since $x^2 = \|x\|^2$ For $1 \le \|x\|^2 < 2$ $\Rightarrow 1 \le \|x\| < \sqrt{2} \Rightarrow -\sqrt{2} \le x < -1$ **or** $1 \le x < \sqrt{2}$ For $\|x\|^2 \ge 4$ $\|x\| \ge 2 \Rightarrow x \le -2$ **or** $x \ge 2$ Hence, the range of values of x is $x \le -2$ or $-\sqrt{2} < x \le -1$ or $1 \le x < \sqrt{2}$ or $x \ge 2$
4	

	$h=\tan\left(\frac{\pi}{3}+\theta\right)$ $A=\frac{1}{2}(2)\tan\left(\frac{\pi}{3}+\theta\right)=\tan\left(\frac{\pi}{3}+\theta\right)$ $=\frac{\tan\left(\frac{\pi}{3}\right)+\tan\theta}{1-\tan\left(\frac{\pi}{3}\right)\tan\theta}=\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}$ (shown) $\approx\frac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}$ $=\left(\sqrt{3}+\theta\right)\left(1-\theta\sqrt{3}\right)^{-1}$ $\approx\left(\sqrt{3}+\theta\right)\left(1+\theta\sqrt{3}+3\theta^2\right)$ $=\sqrt{3}+4\theta+\left(4\sqrt{3}\right)\theta^2$
5	$\operatorname{cosec} y=x$ Diff wrt x: $-\operatorname{cosec} y\ \cot y\frac{dy}{dx}=1$ $\therefore\frac{dy}{dx}=-\frac{1}{\operatorname{cosec} y\ \cot y}$ Using $\cot^2 y+1\equiv\operatorname{cosec}^2 y$, $\frac{dy}{dx}=-\frac{1}{\operatorname{cosec} y\sqrt{(\operatorname{cosec} y)^2-1}}$ [since $0<y<\frac{\pi}{2}\Rightarrow\tan y>0$ $\Rightarrow\cot y>0$ $\Rightarrow\cot y=\sqrt{(\operatorname{cosec} y)^2-1}$] $=-\frac{1}{x\sqrt{x^2-1}}$ (shown) Since $y=\operatorname{cosec}^{-1}x$, $\frac{dy}{dx}=\frac{d}{dx}(\operatorname{cosec}^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$

Alternative

$\operatorname{cosec} y = x$

Diff wrt x:

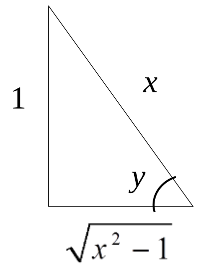
$-\operatorname{cosec} y \cot y \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$

$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{\operatorname{cosec} y \cot y}$

Since $\operatorname{cosec} y = x$,

$\therefore \dfrac{1}{\sin y} = x$

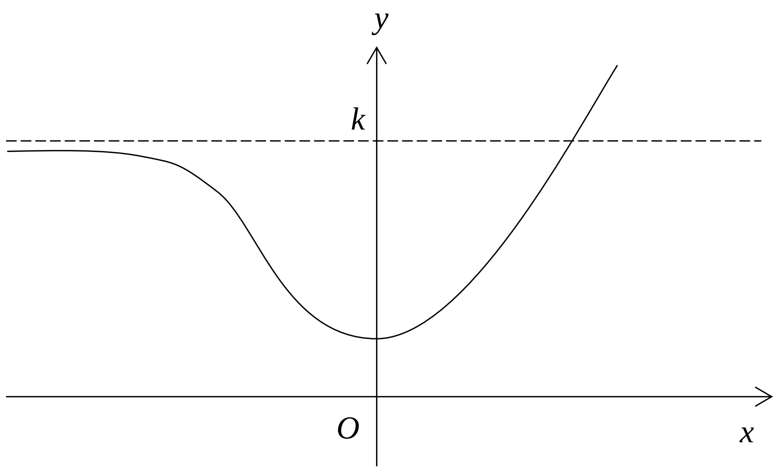
$\therefore \sin y = \dfrac{1}{x}$



By constructing the right angle triangle, $\tan y = \dfrac{1}{\sqrt{x^2 - 1}}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{\operatorname{cosec} y \cot y} = -\dfrac{\tan y}{\operatorname{cosec} y} = -\dfrac{1}{x\sqrt{x^2-1}}$ (shown)

(b)



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(a) $x^2 + \frac{1}{3}(y-2)^2 = 1$

$\downarrow$ Replace x by $x-2$

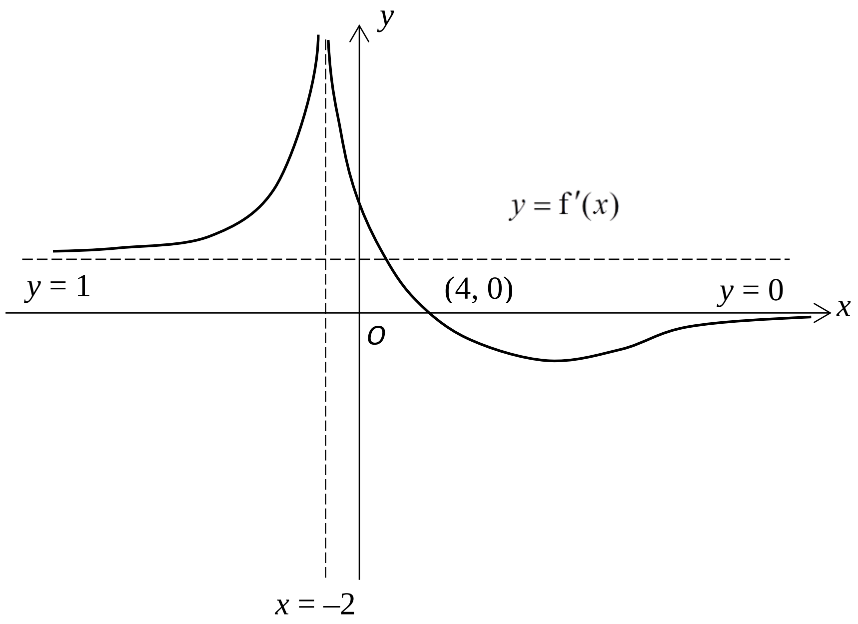
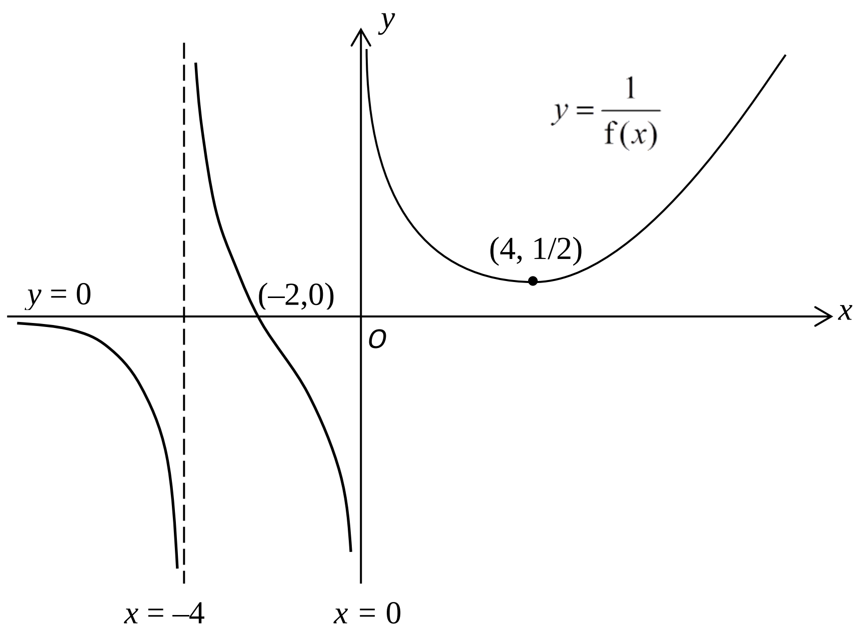
$(x-2)^2 + \frac{1}{3}(y-2)^2 = 1$

$\downarrow$ Replace y by $y+2$

$(x-2)^2 + \frac{1}{3}(y)^2 = 1$

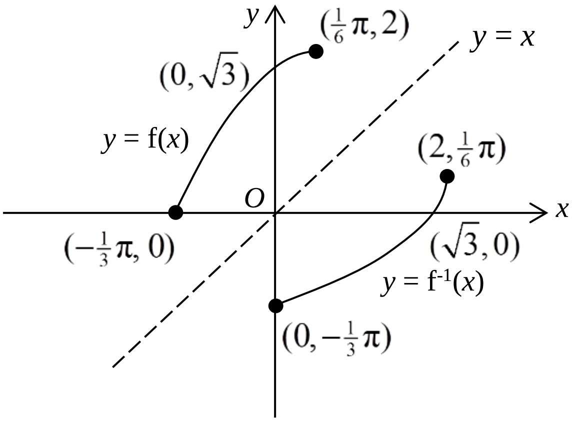
$\downarrow$ Replace y by $\sqrt{3}y$

$(x-2)^2 + y^2 = 1$

	1. Translate 2 units in the positive x-direction 2. Translate 2 units in the negative y-direction 3. Scale by a factor of $\frac{1}{\sqrt{3}}$ parallel to the y-direction <u>Alternative:</u> 2. Scale by a factor of $\frac{1}{\sqrt{3}}$ parallel to the y-direction 3. Translate $\frac{2}{\sqrt{3}}$ units in the negative y-direction (b) (i)  (b) (ii) 

7	(i) Using R formula, $\sin x+\sqrt{3}\cos x=2\sin(x+\frac{1}{3}\pi)$

(ii)



(iii) To find f^{-1}:

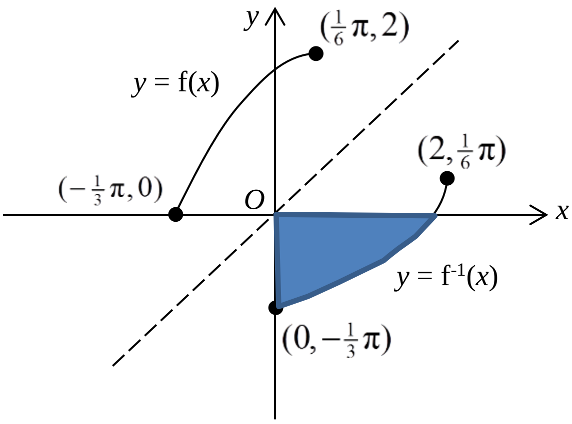
Let $y=2\sin(x+\frac{1}{3}\pi)$

$\therefore x=-\frac{1}{3}\pi+\sin^{-1}(\frac{1}{2}y)$

$f^{-1}(x)=-\frac{1}{3}\pi+\sin^{-1}(\frac{1}{2}x)$

$D_{f^{-1}}=R_f=[0,2]$

(iv) For the area bounded by the graph of f^{-1} and the axes:



By symmetry,

Area

$$=\int_{-\frac{\pi}{3}}^{0} f(x)\,dx=\int_{-\frac{\pi}{3}}^{0}(\sin x+\sqrt{3}\cos x)\,dx$$

$$=\left[-\cos x+\sqrt{3}\sin x\right]_{-\frac{\pi}{3}}^{0}=(-1+0)-\left(-\frac{1}{2}-\frac{3}{2}\right)=1$$

(v) gf^{-1} exists if $R_{f^{-1}} \subseteq D_g$.

Since

$R_{f^{-1}} = [-\frac{1}{3}\pi, \frac{1}{6}\pi]$

$D_g = (-2, \infty)$,

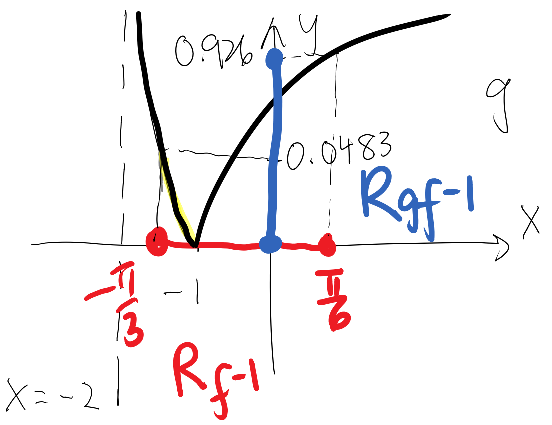
Ie. $R_{f^{-1}} \subseteq D_g \Rightarrow gf^{-1}$ exists

To find the range of gf^{-1}:

Method 1 (two stage mapping method)

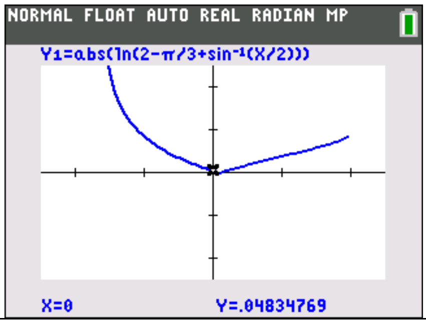
$$D_{f^{-1}} \xrightarrow{f^{-1}} R_{f^{-1}} \xrightarrow{g} R_{gf^{-1}}$$

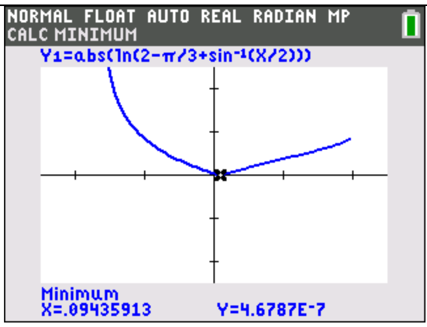
$[0, 2]$ $\quad$ $[-\frac{1}{3}\pi, \frac{1}{6}\pi]$ $\quad$?

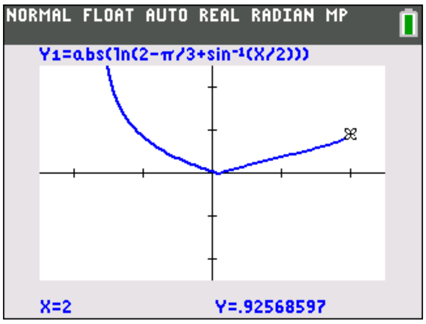


$R_{gf^{-1}} = [0,\ 0.926]$

Method 2 (find gf^{-1}) (need to use GC to see shape)







$\mathrm{gf}^{-1}(x) = \left|\ln(2 - \tfrac{1}{3}\pi + \sin^{-1}(\tfrac{1}{2}x))\right|$

$D_{\mathrm{gf}^{-1}} = D_{\mathrm{f}^{-1}} = [0, 2]$

$R_{\mathrm{gf}^{-1}} = [0, 0.926]$

8 (a) Since the curve shows only one x-intercept, it means that <u>there is only one real root</u> in the equation $\mathrm{f}(x) = 0$.

Since the equation has all real coefficients, then the two other roots must be <u>non-real and they are conjugate pair.</u>

(b) Since $z = 3 + 4\mathrm{i}$ is a root of $2z^2 - (7+6\mathrm{i})z + 11 + \mathrm{i}c = 0$,

$2(3+4\mathrm{i})^2 - (7+6\mathrm{i})(3+4\mathrm{i}) + 11 + \mathrm{i}c = 0$

$2(9 + 24\mathrm{i} - 16) - (21 + 28\mathrm{i} + 18\mathrm{i} - 24) + 11 + \mathrm{i}c = 0$

Comparing the Im - part,

$2 + c = 0$

$\therefore c = -2$ (shown)

Since $z = 3 + 4\mathrm{i}$ is a root of $2z^2 - (7+6\mathrm{i})z + 11 - 2\mathrm{i} = 0$,

$2z^2 - (7+6\mathrm{i})z + 11 - 2\mathrm{i} = \left[z - (3+4\mathrm{i})\right](2z - a)$, where $a \in \mathbb{C}$

Comparing the coefficient of constant term,

$11 - 2\mathrm{i} = a(3 + 4\mathrm{i})$

$$a = \frac{11-2\mathrm{i}}{3+4\mathrm{i}} = \frac{(11-2\mathrm{i})(3-4\mathrm{i})}{25} = \frac{25-50\mathrm{i}}{25} = 1 - 2\mathrm{i}$$

$$2z - (1-2\mathrm{i}) = 0 \Rightarrow z = \frac{1}{2} - \mathrm{i}$$

Therefore, the other root is $\dfrac{1}{2} - \mathrm{i}$.

Replace z by $\mathrm{i}w$

$$2(\mathrm{i}w)^2-(7+6\mathrm{i})(\mathrm{i}w)+11-2\mathrm{i}=0$$

$$-2w^2-(-6+7\mathrm{i})w+11-2\mathrm{i}=0$$

$$2w^2+(-6+7\mathrm{i})w-11+2\mathrm{i}=0$$

$\mathrm{i}w=3+4\mathrm{i}\ \Rightarrow w=4-3\mathrm{i}$ or $\mathrm{i}w=\frac{1}{2}-\mathrm{i}\ \Rightarrow w=-1-\frac{1}{2}\mathrm{i}$

$\therefore$ The roots of the equation are $4-3\mathrm{i}$ and $-1-\frac{1}{2}\mathrm{i}$.

Alternative Method:

$$2z^2-(7+6\mathrm{i})z+11-2\mathrm{i}=0$$

Let the other root be $a+b\mathrm{i}$.

Sum of the roots $=3+4\mathrm{i}+a+b\mathrm{i}=\frac{7+6\mathrm{i}}{2}=\frac{7}{2}+3\mathrm{i}$

Comparing real and imaginary parts:

$$a+3=\frac{7}{2}\Rightarrow a=\frac{1}{2}$$

$$4+b=3\Rightarrow b=-1$$

The other root is $\frac{1}{2}-\mathrm{i}$

(c) (i) $z=1+\mathrm{e}^{\mathrm{i}\alpha}$

$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}(\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}+\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}})$$

$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left[2\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\right)\right]$$

$$=2\cos\frac{\alpha}{2}\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\text{ (shown)}$$

Alternative Method:

$$z=1+\mathrm{e}^{\mathrm{i}\alpha}$$

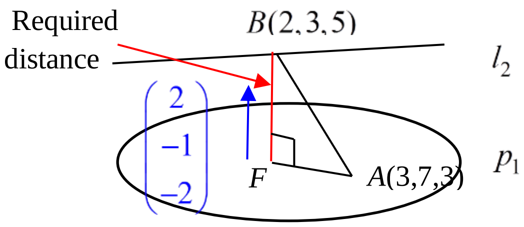
$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}(\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}+\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}})$$

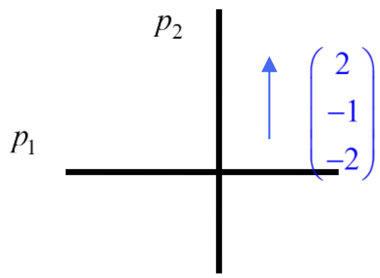
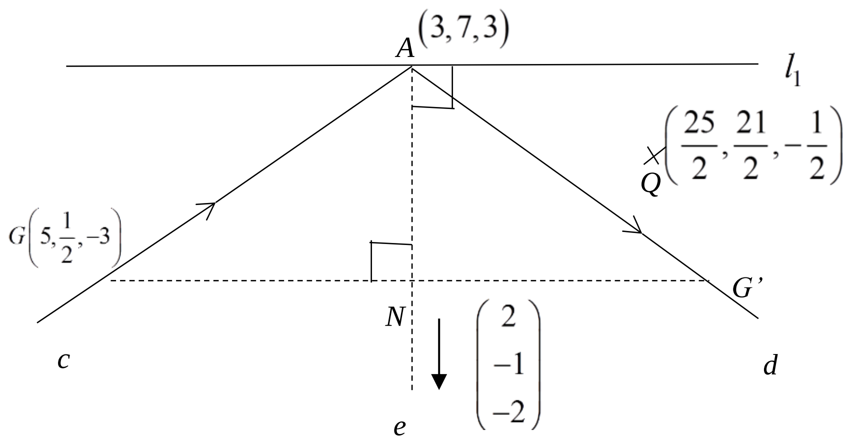
$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left(\cos\left(-\frac{\alpha}{2}\right)+\mathrm{i}\sin\left(-\frac{\alpha}{2}\right)+\cos\frac{\alpha}{2}+\mathrm{i}\sin\frac{\alpha}{2}\right)$$

$$=\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\left[\cos\frac{\alpha}{2}-\mathrm{i}\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}+\mathrm{i}\sin\frac{\alpha}{2}\right]$$

$$=2\cos\frac{\alpha}{2}\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}}\text{ (shown)}$$

	Alternative Method: $z = 1 + e^{i\alpha}$ $= 1 + \cos\alpha + i\sin\alpha$ $= 1 + 2\cos^2\frac{\alpha}{2} - 1 + i\left(2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)$ $= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$ $= 2\cos\frac{\alpha}{2}e^{i\frac{\alpha}{2}}$ (shown) (ii) $\left\lvert\left(\frac{z}{w^3}\right)^*\right\rvert = \left\lvert\left(\frac{z}{w^3}\right)\right\rvert = \frac{\lvert z\rvert}{\lvert w\rvert^3} = \frac{\left\lvert 2\cos\frac{\pi}{6}\right\rvert}{\left(\sqrt{1+3}\right)^3} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{(2)^3} = \frac{\sqrt{3}}{8}$ $\arg\left(\frac{z}{w^3}\right)^* = -\arg\left(\frac{z}{w^3}\right) = -\left[\arg(z) - 3\arg(w)\right]$ $= -\frac{\alpha}{2} + 3\left(-\frac{2\pi}{3}\right) = -\frac{\pi}{6} - 2\pi$ $\therefore \arg\left(\frac{z}{w^3}\right)^* = -\frac{\pi}{6}$
9	(i) A vector equation of l_1 is $\mathbf{r} = \begin{pmatrix}3\\7\\3\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\1\end{pmatrix}, \lambda \in \mathbb{R}$ Let $\mu = x - 2 = \frac{3-y}{2} = \frac{z-5}{2}$. $\therefore x = 2 + \mu, y = 3 - 2\mu, z = 5 + 2\mu$ Then a vector equation of l_2 is $\mathbf{r} = \begin{pmatrix}2\\3\\5\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\2\end{pmatrix}, \lambda \in \mathbb{R}$ A vector perpendicular to p_1 is $\begin{pmatrix}3\\4\\1\end{pmatrix} \times \begin{pmatrix}1\\-2\\2\end{pmatrix} = \begin{pmatrix}10\\-5\\-10\end{pmatrix} = 5\begin{pmatrix}2\\-1\\-2\end{pmatrix} // \begin{pmatrix}2\\-1\\-2\end{pmatrix}$

	Eqn of p_1: $\mathbf{r}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}3\\7\\3\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-7$ Cartesian eqn : $2x-y-2z=-7$ (ii) Distance of a line // to a plane is the distance between a point on this line to the plane  Distance of l_2 to $p_1 = BF = \dfrac{\left\lvert \overrightarrow{AB}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\rvert} = \dfrac{1}{3}\left\lvert\begin{pmatrix}-1\\-4\\2\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\rvert = \dfrac{2}{3}$ Alternative : Equation of line BF: $\mathbf{r}=\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ $\overrightarrow{OF}=\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ for some $\alpha\in\mathbb{R}$ As F lies in p_1, $\left[\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-7$ $-9+9\alpha=-7$ $\therefore \alpha=\dfrac{2}{9}$ Distance of l_2 to $p_1 = BF = \left\lvert\overrightarrow{OF}-\overrightarrow{OB}\right\rvert = \dfrac{2}{9}\left\lvert\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right\rvert = \dfrac{2}{9}\times 3 = \dfrac{2}{3}$

	(iii)  A vector perpendicular to p_2 $=\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}-7\\8\\-11\end{pmatrix}$ Equation of p_2 $\mathbf{r}.\begin{pmatrix}-7\\8\\-11\end{pmatrix}=\begin{pmatrix}3\\7\\3\end{pmatrix}.\begin{pmatrix}-7\\8\\-11\end{pmatrix}=2$ (iv)  line d is a reflection of line c in the line e which passes through A, is perpendicular to l_1 and p_1 and lying in p_2. Eqn of line e : $\mathbf{r}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ Foot of perpendicular, N, from $\left(5,\frac{1}{2},-3\right)$ to line e

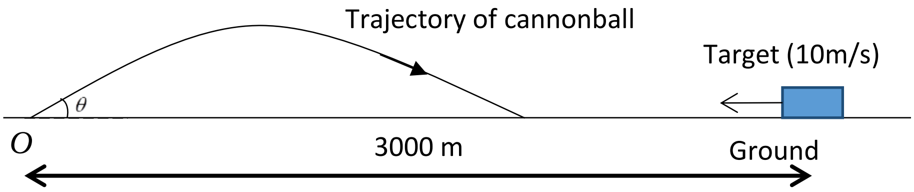
	$\left[\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}-\begin{pmatrix}5\\0.5\\-3\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$ $\left[\begin{pmatrix}-2\\6.5\\6\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$ $-4-6.5-12+\mu(4+1+4)=0$ $\mu=\frac{45}{18}=\frac{5}{2}$ $\overrightarrow{ON}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}8\\4.5\\-2\end{pmatrix}$ $\overrightarrow{ON}=\frac{1}{2}\left(\overrightarrow{OG}+\overrightarrow{OG'}\right)$ $\overrightarrow{OG'}=2\begin{pmatrix}8\\4.5\\-2\end{pmatrix}-\begin{pmatrix}5\\0.5\\-3\end{pmatrix}=\begin{pmatrix}11\\8.5\\-1\end{pmatrix}$ $\overrightarrow{AG'}=\begin{pmatrix}11\\8.5\\-1\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}=\begin{pmatrix}8\\1.5\\-4\end{pmatrix}//\begin{pmatrix}16\\3\\-8\end{pmatrix}$ Direction cosines of line d are $\frac{16}{\sqrt{329}},\frac{3}{\sqrt{329}}$ and $-\frac{8}{\sqrt{329}}$ or $-\frac{16}{\sqrt{329}},-\frac{3}{\sqrt{329}}$ and $\frac{8}{\sqrt{329}}$. **[Alternative** to find $\overrightarrow{ON}$ - intersection of 2 lines] Eqn of line e : $\mathbf{r}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ Eqn of line GN: $\mathbf{r}=\begin{pmatrix}5\\0.5\\-3\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$ At N,

	$\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}5\\0.5\\-3\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$ $2\mu-3\alpha=2$ $\mu+4\alpha=6.5$ $2\mu+\alpha=6$ Solving, $\mu=2.5,\alpha=1$ $\overrightarrow{ON}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}8\\4.5\\-2\end{pmatrix}$ (v) Shortest distance from Q to line d $=\left\|\frac{\overrightarrow{AQ}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}}{\sqrt{329}}\right\|$ $=\frac{1}{\sqrt{329}}\left\|\left[\begin{pmatrix}\frac{25}{2}\\\frac{21}{2}\\-\frac{1}{2}\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}\right]\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\|$ $=\frac{1}{\sqrt{329}}\left\|\begin{pmatrix}\frac{19}{2}\\\frac{7}{2}\\-\frac{7}{2}\end{pmatrix}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\|=\frac{1}{2\sqrt{329}}\left\|\begin{pmatrix}19\\7\\-7\end{pmatrix}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right\|$ $=\frac{1}{2\sqrt{329}}\left\|\begin{pmatrix}-35\\40\\-55\end{pmatrix}\right\|=2.11$ (3 s.f.)
10	(i) To determine range of cannonball, we consider y = 0: $0=(v\sin\theta)t-5t^2$ $0=t\left[v\sin\theta-5t\right]$ $\therefore t=0$ (rejected) or $v\sin\theta-5t=0$ $\therefore t=\frac{v\sin\theta}{5}$

When $t = \dfrac{v\sin\theta}{5}$,

$$x = (v\cos\theta)t$$
$$= (v\cos\theta)\frac{v\sin\theta}{5}$$
$$= \frac{v^2\sin\theta\cos\theta}{5} \qquad \therefore d = \frac{v^2\sin\theta\cos\theta}{5}$$

(ii)



Time taken for cannonball to hit the ground = time taken for the target to reach the point of impact of the cannonball.

$$\frac{v\sin\theta}{5} = \frac{3000-d}{10}$$
$$2v\sin\theta = 3000 - \frac{v^2\sin\theta\cos\theta}{5}$$
$$\frac{(200)^2\sin\theta\cos\theta}{5} + 400\sin\theta = 3000$$

Possible angles are $22.7°$ (to 1 dp) or $69.5°$ (to 1 dp). (shown)

(iii) Since $t = \dfrac{v\sin\theta}{5}$ when cannon hits target and $\dfrac{v\sin 22.7°}{5} < \dfrac{v\sin 69.5°}{5}$

Therefore to hit target earlier, cannonball should be fired at $22.7°$.

(iv) $x = (200\cos 22.7°)t$ $\qquad$ $y = (200\sin 22.7°)t - 5t^2$

$$\frac{dx}{dt} = 184.51 \qquad \frac{dy}{dt} = 77.181 - 10t$$
$$\therefore \frac{dy}{dx} = \frac{77.181-10t}{184.51}$$

When $x = 370$, $184.51t = 370 \Rightarrow t = 2.0053$

$$\therefore \frac{dy}{dx} = \frac{77.181-10(2.0053)}{184.51} = 0.30962$$

Let the required angle be α.

$$\tan\alpha = 0.30962 \Rightarrow \alpha = 17.2° \text{ (to 1dp)}$$

11	(a)(i) $a+b+c=25000$ ----------(1) $0.02a+0.06b-0.02c=0$ ----------(2) [or $1.02a+1.06b+0.98c=25000$] Solving SLE, $a=37500-2c$ $b=c-12500$ (ii) Since a and b must both be positive, it implies that c must lie between 12500 and 18750. (b)(i) Since Mr Lee invested in a period of five years, the average return per year will be 1.6%. Total amount of interest earned $=(1.016)^5(55000)-55000$ $=4543$ (to the nearest dollar) (ii) Amount of money in the normal savings account at the end of n years $=1000(1.0019+1.0019^2+1.0019^3+...+1.0019^n)$ $=1000(1.0019)\left(\dfrac{1.0019^n-1}{1.0019-1}\right)$ $=527315.79(1.0019^n-1)$ Amount of money in the special savings account at the end of n years $=10000(1.018)\left(\dfrac{1.018^n-1}{1.018-1}\right)$ $=565555.56\left(1.018^n-1\right)$ (iii) Total interest earned from dual-savings account $=527315.79(1.0019^n-1)+565555.56\left(1.018^n-1\right)-11000n$ $527315.79(1.0019^n-1)+565555.56\left(1.018^n-1\right)-11000n>4543$ From GC, $n\geq 7$ Least value of n is 7.

1	**(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^2 x$. Hence evaluate $\displaystyle\int_0^{\frac{1}{4}\pi} \sec^2 x \tan x \, \mathrm{e}^{\tan^2 x} \, \mathrm{d}x$, leaving your answer in exact form. [3] **(ii)** By expressing $1+72x-32x^3$ as $1+mx(9-4x^2)$ where m is a constant, find $\displaystyle\int \frac{1+72x-32x^3}{\sqrt{(9-4x^2)}} \, \mathrm{d}x$. [2]
2	The curve C with equation $y=\dfrac{x^2+(a-1)x-a-1}{x-1}$, where a is a constant, has the oblique asymptote $y=x+1$. **(i)** Show that $a = 1$. Hence sketch C, giving the equations of any asymptotes and the exact coordinates of any points of intersection with the axes. [3] **(ii)** The region bounded by C for $x>1$ and the lines $y=x+1$, $y=2$ and $y=4$ is rotated through 2π radians about the line $x=1$. By considering a translation of C, or otherwise, find the volume of revolution formed. [5]
3	The variables y and x satisfy the differential equation $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-\ln x}{x\ln x+2x^2}.$$ **(i)** Show that the substitution $u=\dfrac{\ln x}{x}$ reduces the differential equation to $\dfrac{\mathrm{d}u}{\mathrm{d}y}=u+2$. Given that $y=0$ when $x=1$, show that $y=\ln\left(\dfrac{\ln x}{2x}+1\right)$. [6] The curve C has equation $y=\ln\left(\dfrac{\ln x}{2x}+1\right)$. It is given that C has a maximum point and two asymptotes $y=a$ and $x=b$. **(ii)** Find the exact coordinates of the maximum point. [2] **(iii)** Explain why $a=0$. [You may assume that as $x\to\infty$, $\dfrac{\ln x}{x}\to 0$.] [1] **(iv)** Determine the value of b, giving your answer correct to 4 decimal places. [2] **(v)** Sketch C. [2]

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4		Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel. The point C lies on OB produced such that $3OC = 5OB$. It is given that $\lvert\mathbf{a}\rvert = 2\lvert\mathbf{b}\rvert$ and $\cos\angle AOB = -\frac{1}{4}$. **(a)** **(i)** Show that a vector equation of the line AC is $\mathbf{r} = \mathbf{a} + \lambda(3\mathbf{a} - 5\mathbf{b})$, where λ is a real parameter. [2] The line l lies in the plane containing O, A and B. **(ii)** Explain why the direction vector of l can be expressed as $s\mathbf{a} + t\mathbf{b}$, where s and t are real numbers. [1] Given that l is perpendicular to AB, show that $t = 3s$. [4] Given further that l passes through B, write down a vector equation of l, in a similar form as part **(i)**. [1] **(iii)** Find the position vector of the point of intersection of line AC and l, in terms of $\mathbf{a}$ and $\mathbf{b}$. [2] **(b)** Explain why, for any constant k, $\lvert(\mathbf{a} + k\,\mathbf{b}) \times \mathbf{b}\rvert$ gives the area of the parallelogram with sides OA and OB. Find the area of the parallelogram, leaving your answer in terms of $\lvert\mathbf{a}\rvert$. [4]
5		A new game has been designed for a particular casino using two fair die. In each round of the game, a player places a bet of \$2 before proceeding to roll the two die. The player's score is the sum of the results from both die. For the scores in the following table, the player keeps his bet and receives a payout as indicated. <table><tr><th>Score</th><th>Payout</th></tr><tr><td>9 or 10</td><td>\$1</td></tr><tr><td>2 or 4</td><td>\$5</td></tr><tr><td>11</td><td>\$8</td></tr></table> For any other scores, the player loses his bet. Let X be the random variable denoting the winnings of the casino from each round of the game. **(i)** Show that $\mathrm{E}(X) = \dfrac{1}{12}$ and find $\mathrm{Var}(X)$. [4] **(ii)** $\bar{X}$ is the mean winnings of the casino from n rounds of this game. Find $\mathrm{P}(\bar{X} > 0)$ when $n = 30$ and $n = 50\,000$. Make a comparison of these probabilities and comment in context of the question. [3]

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6 The students in a college are separated into two groups of comparable sizes, Group X and Group Y. The marks for their Mathematics examination are normally distributed with means and variances as shown in the following table.

	Mean	Variance
Group X	55	20
Group Y	34	25

(i) Explain why it may not be appropriate for the mark of a randomly chosen student from the college population to be modelled by a normal distribution. [1]

(ii) In order to pass the examination, students from Group Y must obtain at least d marks. Find, correct to 1 decimal place, the maximum value of d if at least 60% of them pass. [3]

(iii) Find the probability that the total marks of 4 students from Group Y is less than three times the mark of a student from Group X. State clearly the mean and variance of the distribution you use in your calculation. [3]

(iv) The marks of 40 students, with 20 each randomly selected from Group X and Group Y, are used to compute a new mean mark , $\bar{M}$. Given that $\mathrm{P}(|\bar{M}-44.5|<k)=0.9545$, find the value of k. [4]

State a necessary assumption for your calculations to hold in parts **(iii)** and **(iv)**. [1]

7 The company Snatch provides a ride-hailing service comprising taxis and private cars in Singapore. Snatch claims that the mean waiting time for a passenger from the booking time to the time of the vehicle's arrival is 7 minutes.

To test whether the claim is true, a random sample of 30 passengers' waiting times is obtained. The standard deviation of the sample is 2 minutes. A hypothesis test conducted concludes that there is sufficient evidence at the 1% significance level to reject the claim.

(i) State appropriate hypotheses and the distribution of the test statistic used. [3]

(ii) Find the range of values of the sample mean waiting time, $\bar{t}$. [3]

(iii) A hypothesis test is conducted at the 1% significance level whether the mean waiting time of passengers is more than 7 minutes. Using the existing sample, deduce the conclusion of this test if the sample mean waiting time is more than 7 minutes. [2]

8	A retail manager of a large electrical appliances store wants to investigate the relationship between the monthly advertising expenditure, x hundred dollars, and the monthly sales of their refrigerators, y thousand dollars. The table below shows the results of the investigation.

x	5	8	12	16	18	20	23
y	12.5	12.9	13.6	14.8	17.0	19.3	25.1

(i) The manager concludes that an increase in monthly advertising expenditure will result in an increase in the monthly sales of refrigerators. State, with a reason, whether you agree with his conclusion. [1]

(ii) Draw a scatter diagram to illustrate the above data. Explain why a linear model is not likely to be appropriate. [2]

It is thought that the monthly sales y thousand dollars can be modelled by one of the formulae

$$y = a + b\mathrm{e}^{\sqrt{x}} \quad \text{or} \quad y = a + bx^2$$

where a and b are constants.

(iii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(A) $\mathrm{e}^{\sqrt{x}}$ and y,

(B) x^2 and y.

Explain which of $y = a + b\mathrm{e}^{\sqrt{x}}$ or $y = a + bx^2$ is the better model. [2]

Assume that the better model in part **(iii)** holds for part **(iv)**.

(iv) The manager forgot to record the monthly advertising expenditure when the monthly sales of refrigerators was \$11300. Combining this with the above data set, it is found that a = 10.876 and b = 0.09906 for the model. Find the monthly advertising expenditure that the manager forgot to record, leaving your answer to the nearest hundred. [3]

9	A sample of 5 people is chosen from a village of large population.

(i) The number of people in the sample who are underweight is denoted by X. State, in context, the assumption required for X to be well modelled by a binomial distribution. [1]

(ii) On average, the proportion of people in the village who are underweight is p. It is known that the mode of X is 2. Use this information to show that $\frac{1}{3} < p < \frac{1}{2}$. [3]

1000 samples of 5 people are chosen at random from the village and the results are shown in the table below.

x	0	1	2	3	4	5
Number of groups	93	252	349	220	75	11

(iii) Using the above results, find $\bar{x}$. Hence estimate the value of p. [2]

You may now use your estimate in part **(iii)** as the value of p.

(iv) Two random samples of 5 people are chosen. Find the probability that the first sample has at least 4 people who are underweight and has more people who are underweight than the second sample. [3]

10 **(a)** The word DISTRIBUTION has 12 letters.

(i) Find the number of different arrangements of the 12 letters that can be made. [1]

(ii) Find the number of different arrangements which can be made if there are exactly 8 letters between the two Ts. [3]

One of the Is is removed from the word and the remaining letters are arranged randomly.

(iii) Find the probability that no adjacent letters are the same. [4]

(b) The insurance company Adiva classifies 10% of their car policy holders as ‘low risk’, 60% as ‘average risk’ and 30% as ‘high risk’. Its statistical database has shown that of those classified as ‘low risk’, ‘average risk’ and ‘high risk’, 1%, 15% and 25% are involved in at least one accident respectively.

Find the probability that

(i) a randomly chosen policy holder is not involved in any accident if the holder is classified as ‘average risk’, [1]

(ii) a randomly chosen policy holder is not involved in any accident, [2]

(iii) a randomly chosen policy holder is classified as ‘low risk’ if the holder is involved in at least one accident. [2]

It is known that the cost of repairing a car when it meets with an accident has the following probability distribution.

Cost incurred (in thousand dollars)	5	10	50	100
Probability	0.75	0.15	0.08	0.02

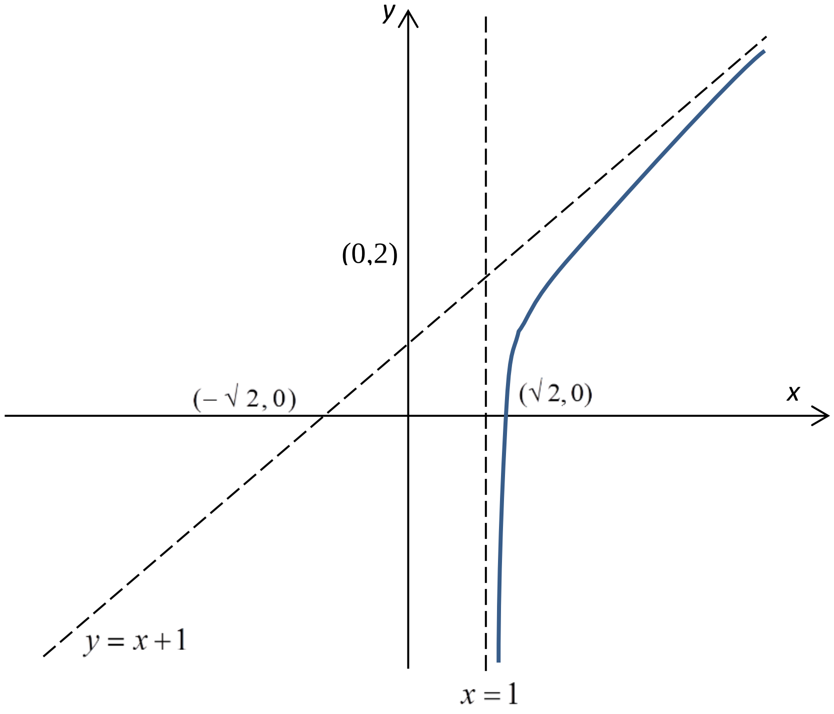
It is known that a 'low risk' policy holder will not be involved in more than one accident in a year. You may assume that there will be no cost incurred by the company in insuring a holder whose car is not involved in any accident.

(iv) Construct the probability distribution table of the cost incurred by Adiva in insuring a 'low risk' policy holder assuming that the cost of repairing a car is independent of a 'low risk' policy holder meeting an accident. [1]

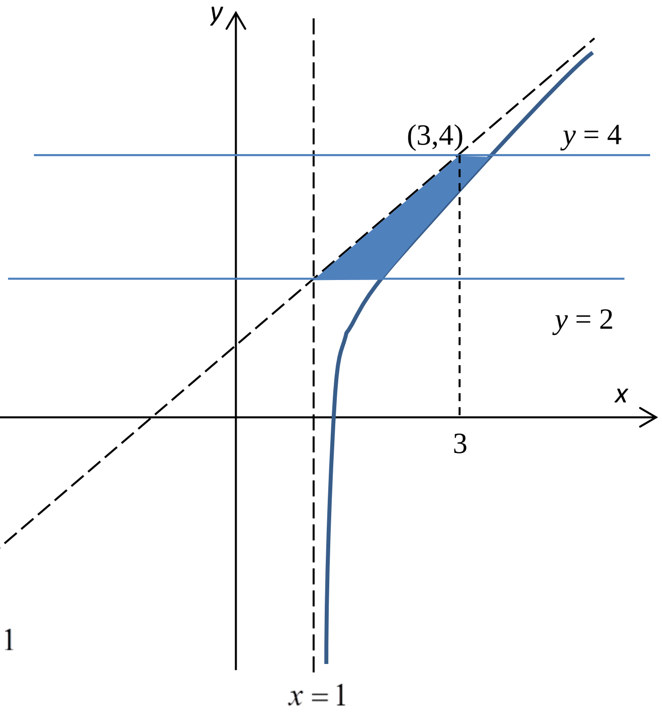
(v) In order to have an expected profit of $200 from each policy holder, find the amount that Adiva should charge a 'low risk' policy holder when he renews his annual policy. [2]

H2 Mathematics 2017 Preliminary Exam Paper 2 Solution

1	(i) $\int_0^{\frac{\pi}{4}} \sec^2 x \tan x\, \mathrm{e}^{\tan^2 x}\, \mathrm{d}x = \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sec^2 x \tan x\, \mathrm{e}^{\tan^2 x}\, \mathrm{d}x$ $= \frac{1}{2}\left[\mathrm{e}^{\tan^2 x}\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}\left(\mathrm{e}^{\tan^2 \frac{\pi}{4}} - \mathrm{e}^{\tan^2 0}\right)$ $= \frac{1}{2}(\mathrm{e}-1)$ (ii) $\int \frac{1+72x-32x^3}{\sqrt{9-4x^2}}\,\mathrm{d}x = \int \frac{1+8x\left(9-4x^2\right)}{\sqrt{9-4x^2}}\,\mathrm{d}x$ $= \int \frac{1}{\sqrt{9-4x^2}} + 8x\left(9-4x^2\right)^{\frac{1}{2}}\,\mathrm{d}x$ $= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) - \frac{2}{3}\left(9-4x^2\right)^{\frac{3}{2}} + C$
2	(i) $y = \frac{x^2+(a-1)x-a-1}{x-1}$ $= \frac{(x+a)(x-1)-1}{x-1}$ $= (x+a) - \frac{1}{x-1}$ Given that oblique asymptote is $y = x+1$, $\therefore a = 1$ (shown) <u>Alternative</u> Let $\frac{x^2+(a-1)x-a-1}{x-1} = (x+1) + \frac{b}{x-1}$ $\Rightarrow x^2+(a-1)x-a-1 = x^2-1+b$ Comparing coeff of x: $a-1=0$ $\therefore a=1$ (shown) and $b=-1$ $\therefore y = (x+1) - \frac{1}{x-1} = \frac{x^2-2}{x-1}$ HA: $x=1$ OA: $y = x+1$(given)



(ii)



$$y=\frac{x^2-2}{x-1}\xrightarrow{\textbf{replace } x \textbf{ with } (x+1)} y=\frac{(x+1)^2-2}{x}$$

	$y=\dfrac{(x+1)^2-2}{x}$ $xy=x^2+2x-1$ $x^2+(2-y)x-1=0$ $x=\dfrac{-(2-y)\pm\sqrt{(2-y)^2+4(1)(1)}}{2}$ $\therefore\ x=\dfrac{(y-2)+\sqrt{y^2-4y+8}}{2}$ **(reject -ve root)** Volume = $\pi\int_2^4\left(\dfrac{(y-2)+\sqrt{y^2-4y+8}}{2}\right)^2\ \mathrm{d}y-\dfrac{1}{3}\pi\left(2\right)^2(2)$ $= 9.75$ units^3 (3 s.f)
3	(i) $u=\dfrac{\ln x}{x}\Rightarrow\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{1-\ln x}{x^2}$ $\dfrac{\mathrm{d}u}{\mathrm{d}y}=\dfrac{\mathrm{d}u}{\mathrm{d}x}\times\dfrac{\mathrm{d}x}{\mathrm{d}y}=\dfrac{1-\ln x}{x^2}\times\dfrac{x\ln x+2x^2}{1-\ln x}=\dfrac{\ln x+2x}{x}$ $\dfrac{\mathrm{d}u}{\mathrm{d}y}=u+2$ (shown) $\dfrac{1}{u+2}\dfrac{\mathrm{d}u}{\mathrm{d}y}=1\Rightarrow\ln\lvert u+2\rvert=y+c$, c is an arbitrary constant $\lvert u+2\rvert=\mathrm{e}^{y+c}=\mathrm{e}^c\mathrm{e}^y$ $u+2=A\mathrm{e}^y$, A is an arbitrary constant $\dfrac{\ln x}{x}+2=A\mathrm{e}^y$ $y=0, x=1:\quad A=2$ $\dfrac{\ln x}{2x}+1=\mathrm{e}^y$ $y=\ln\left(\dfrac{\ln x}{2x}+1\right)$ (shown) <u>Alternative</u> $\dfrac{1}{u+2}\dfrac{\mathrm{d}u}{\mathrm{d}y}=1\Rightarrow\ln\lvert u+2\rvert=y+c$, c is an arbitrary constant With the boundary condition $u=0, y=0,$ we see that $u+2>0$ Thus $\ln\lvert u+2\rvert=y+c$ and $c=\ln 2$ (ii) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1-\ln x}{x\ln x+2x^2}$

When $\frac{\mathrm{d}y}{\mathrm{d}x}=0,\quad 1-\ln x=0\quad \Rightarrow x=\mathrm{e},\ y=\ln\left(\frac{1}{2\mathrm{e}}+1\right)$

Therefore the maximum point is $\left(\mathrm{e},\ln\left(\frac{1}{2\mathrm{e}}+1\right)\right)$.

(iii) $y=\ln\left(\frac{\ln x}{2x}+1\right)$

When $x\to\infty, \frac{\ln x}{2x}\to 0.$

$y=\ln\left(\frac{\ln x}{2x}+1\right)\to \ln 1=0.$

Thus $a=0$ (shown)

(iv) For $y\to-\infty$, $\frac{\ln x}{2x}+1\to 0$

$\ln x+2x\to 0$

$x\to 0.4263$

$\therefore b=0.4263$

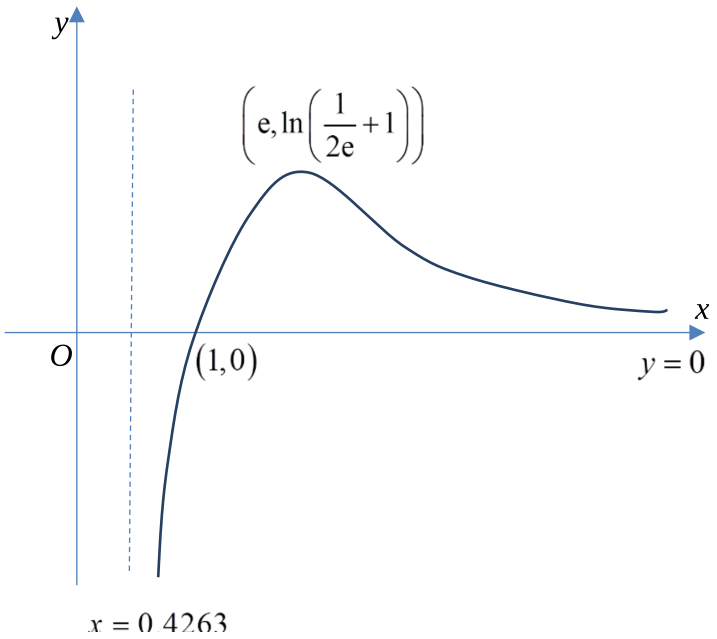
<u>Alternative</u>

When y is undefined, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-\ln x}{x\ln x+2x^2}$ is undefined.

Thus $x\ln x+2x^2=0$.

Since $x>0$ for $\ln x$ to be defined, $\ln x+2x=0$.

(v)



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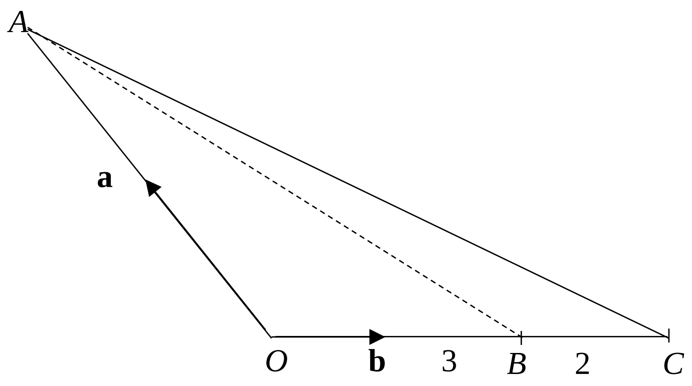
(a)



(i) $\overrightarrow{OC}=\dfrac{5}{3}\mathbf{b}\quad\therefore\overrightarrow{AC}=\dfrac{5}{3}\mathbf{b}-\mathbf{a}=\dfrac{1}{3}(5\mathbf{b}-3\mathbf{a})//5\mathbf{b}-3\mathbf{a}$

Equation of line AC: $\mathbf{r}=\mathbf{a}+\lambda(3\mathbf{a}-5\mathbf{b}),\quad \lambda\in\mathbb{R}$

(ii) Since l lies on the plane containing O, A and B, its direction vector is coplanar with a and b, thus it will be a linear combination of a and b, i.e. $s\mathbf{a}+t\mathbf{b}$ is a direction vector for l.

(iii) Let intersection point be D.

At D,

$\mathbf{a}+\lambda(3\mathbf{a}-5\mathbf{b})=\mathbf{b}+\mu(\mathbf{a}+3\mathbf{b})$

Since a and b are non-zero, non-parallel vectors,

$1+3\lambda=\mu-----(1)$

$-5\lambda=1+3\mu----(2)$

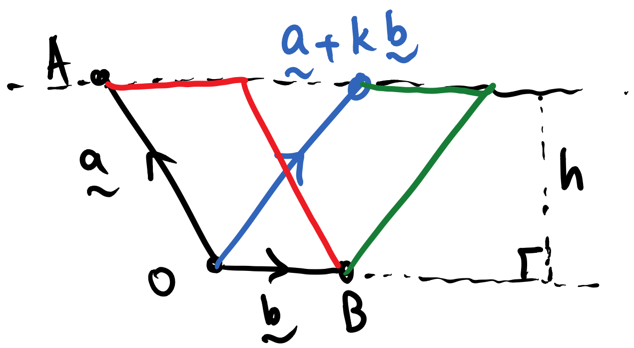
Solving,

$-5\lambda=1+3(1+3\lambda)$

$14\lambda=-4$

$\therefore\lambda=-\dfrac{2}{7},\mu=\dfrac{1}{7}\qquad\qquad \therefore\overrightarrow{OD}=\dfrac{1}{7}\mathbf{a}+\dfrac{10}{7}\mathbf{b}$

(b)



	<u>Method 1</u> Since the base length (*OB*) and perpendicular height remain the same, the area of parallelograms formed by different *k* remains the same as the area of the parallelogram with sides *OA* and *OB*. <u>Method 2</u> $\lvert(\mathbf{a}+k\mathbf{b})\times\mathbf{b}\rvert=\lvert\mathbf{a}\times\mathbf{b}+k\mathbf{b}\times\mathbf{b}\rvert=\lvert\mathbf{a}\times\mathbf{b}+\mathbf{0}\rvert=\lvert\mathbf{a}\times\mathbf{b}\rvert$ Area of parallelogram $=\lvert\mathbf{a}\times\mathbf{b}\rvert=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ $=\lvert\mathbf{a}\rvert\left(\frac{1}{2}\lvert\mathbf{a}\rvert\right)\sqrt{1-\left(-\frac{1}{4}\right)^2}$ $=\frac{\sqrt{15}}{8}\lvert\mathbf{a}\rvert^2$
5	(i) <table><tr><td>*x*</td><td>- 8</td><td>- 5</td><td>- 1</td><td>2</td></tr><tr><td>P(*X* = *x*)</td><td>$\frac{2}{36}=\frac{1}{18}$</td><td>$\frac{4}{36}=\frac{2}{18}$</td><td>$\frac{7}{36}$</td><td>$\frac{23}{36}$</td></tr></table> $\mathrm{E}(X)=\frac{46}{36}-\frac{7}{36}-\frac{20}{36}-\frac{16}{36}=\frac{1}{12}$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2=\frac{92}{36}+\frac{7}{36}+\frac{50}{18}+\frac{64}{18}-\frac{1}{12^2}$ $=\frac{1307}{144}$ or 9.08 (to 3sf) (ii) Since *n* is large, $\overline{X}\sim \mathrm{N}\left(\frac{1}{12},\frac{1307}{144n}\right)$ approximately by Central Limit Theorem. For *n* = 30, $\mathrm{P}\left(\overline{X}>0\right)=0.560$ (to 3sf) For *n* = 50000, $\mathrm{P}\left(\overline{X}>0\right)=1.00$ (to 3sf) The more rounds this game is played, the higher the chance of casino receiving a positive average winnings. In other words, it is almost certain that casino will win in the long run.
6	(i) The distribution may become <u>bimodal</u> when the data for both groups are combined (ii) Let *Y* be the score of a random student from Group Y. $Y\sim \mathrm{N}(34,\ 25)$ $\mathrm{P}(Y\ge d)\ge 0.6$ $\mathrm{P}(Y\le d)\le 0.4$

When $\mathrm{P}(Y \le d_c) = 0.4$, $d_c = 32.733$.
Thus $d < 32.733$. The maximum mark is 32.7

(iii) $\mathrm{E}\left(\sum_1^4 Y_i - 3X\right) = 4\mathrm{E}(Y) - 3\mathrm{E}(X) = -29$

$\mathrm{Var}\left(\sum_1^4 Y_i - 3X\right) = 4\mathrm{Var}(Y) + 9\mathrm{Var}(X) = 280$

$\therefore \sum_1^4 Y_i - 3X \sim \mathrm{N}(-29, 280)$

$\mathrm{P}(\sum_1^4 Y_i < 3X) = \mathrm{P}(\sum_1^4 Y_i - 3X < 0) = 0.958$ (to 3sf)

(iv) $\overline{M} = \dfrac{\sum_{i=1}^{20} X_i + \sum_{i=1}^{20} Y_i}{40}$

$\mathrm{E}\left(\overline{M}\right) = \dfrac{20\mathrm{E}(X) + 20\mathrm{E}(Y)}{40} = \dfrac{1}{2}\left(\mathrm{E}(X) + \mathrm{E}(Y)\right) = 44.5$

Let $\sigma^2 = \mathrm{Var}\left(\overline{M}\right)$

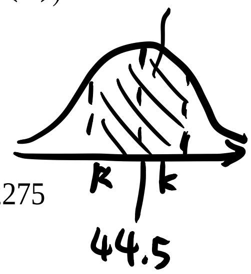
$= \dfrac{1}{1600}\left(20\mathrm{Var}(X) + 20\mathrm{Var}(Y)\right)$

$= \dfrac{1}{80}\left(\mathrm{Var}(X) + \mathrm{Var}(Y)\right) = 0.5625$

$\overline{M} \sim N(44.5, 0.5625)$

Since $\mathrm{P}(\left|\overline{M} - 44.5\right| < k) = 0.9545$

$\Rightarrow \mathrm{P}(\overline{M} < 44.5 - k) = \dfrac{1 - 0.9545}{2} = 0.02275$



$\therefore 44.5 - k = 43.000$

$\Rightarrow k = 1.50$ (3 s.f)

<u>Alternative</u>

$\overline{M} \sim N(44.5, \sigma^2)$

Since $\mathrm{P}(\left|\overline{M} - 44.5\right| < 2\sigma) = 0.9545$

$\therefore k = 2\sigma = 2\sqrt{0.5625} = 1.50$ (3sf)

<u>*Marks of students*</u> are independent of one another.

7 (i) Let μ be the mean of X.

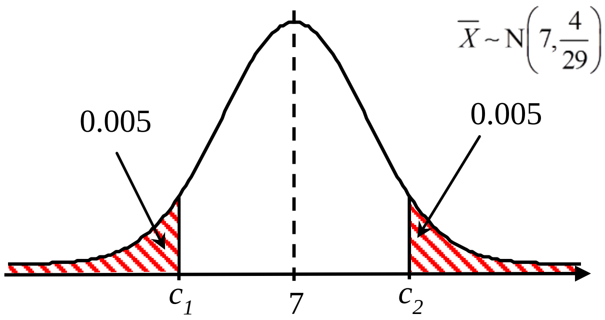
$\mathrm{H}_0: \ \mu = 7$

$\mathrm{H}_1: \ \mu \ne 7$

$s^2 = \frac{30}{29}(\text{sample variance}) = \frac{30}{29}(4) = \frac{120}{29}$

Under H_0, since the sample size is large, the test statistic is
$\overline{T} \sim \mathrm{N}\left(7, \frac{4}{29}\right)$ approximately by Central Limit Theorem.

(ii) Since the claim is rejected i.e. to reject H_0 at 1% significance level.



From GC, $c_1 = 6.04$ and $c_2 = 7.96$.
$\bar{t} \le 6.04$ or $\bar{t} \ge 7.96$

(iii) From the two tail test, we know that $\text{p-value (two tail)} \le 0.01$. For a one-tail test, $\text{p-value(one tail)} = \frac{\text{p-value (two tail)}}{2} \le 0.005 < 0.01$,
therefore we reject H_0 and conclude that there is sufficient evidence at 1% significance level to say that mean waiting time is more than 7 minutes.

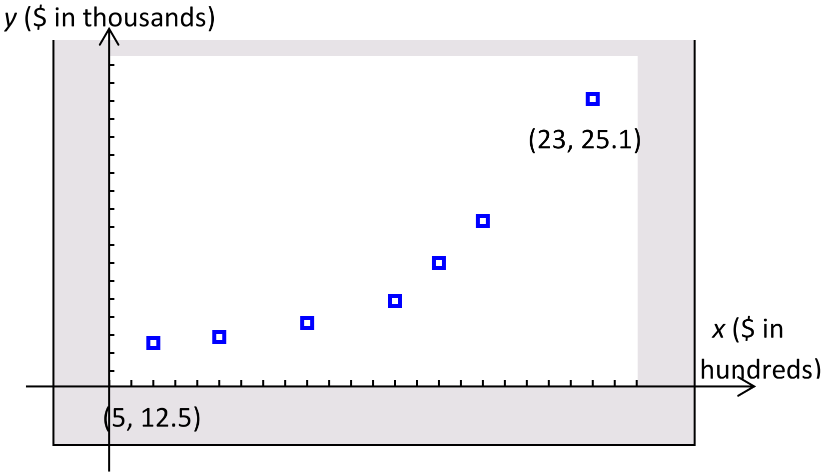
Alternatively,
From the two tail test and $\bar{t} > 7$, $\mathrm{P}(\overline{T} > \bar{t}) < 0.005$.

Thus, $\mathrm{P}(\overline{T} > \bar{t}) < 0.005 < 0.01$.

p-value for one-tail test = $\mathrm{P}(\overline{T} > \bar{t}) < 0.01$. Therefore we reject H_0 and conclude that there is sufficient evidence at 1% significance level to say that mean waiting time is more than 7 minutes.

8 **(i)** No, because correlation does not imply causation /
The increase in the monthly sales of refrigerators could be due to other factors such as a rise in the income level.

(ii)



There appears to be a curvilinear/non-linear relationship between x and y, thus a linear model is not likely to be appropriate.

(iii)

(A) $r = 0.9684$ (to 4dp)

(B) $r = 0.9495$ (to 4dp)

Since the r value between $e^{\sqrt{x}}$ and y has an absolute value closer to 1, $y = a + be^{\sqrt{x}}$ is the better model.

(iv) New regression line (8 data points) for y on $e^{(\sqrt{x})}$ is $y = 10.876 + 0.09906e^{(\sqrt{x})}$

$$\bar{y} = \frac{11.3+12.5+12.9+13.6+14.8+17+19.3+25.1}{8} = 15.813$$

Since $\overline{e^{(\sqrt{x})}}$ and $\bar{y}$ lie on the new regression line y on $e^{(\sqrt{x})}$,

and letting $x = m$ when $y = 11.3$,

$$15.813 = 10.876 + 0.09906\left(\frac{\sum_{i=1}^{7} e^{(\sqrt{x_i})} + e^{(\sqrt{m})}}{8}\right)$$

Using GC (1-var stats), $\sum_{i=1}^{7} e^{(\sqrt{x_i})} = 390.96$

$$\therefore 15.813 = 10.876 + 0.09906\left(\frac{390.96 + e^{(\sqrt{m})}}{8}\right)$$

$\therefore e^{(\sqrt{m})} = 7.7479 \quad \Rightarrow \quad m = 4.19 \approx 4$

Monthly advertising expenditure = $400 (nearest hundred)

9	(i) Assume that the: • weights of the 5 people chosen are independent of each other • sample is chosen randomly. (ii) $\mathrm{P}(X=1)<\mathrm{P}(X=2)$ and $\mathrm{P}(X=2)>\mathrm{P}(X=3)$ ${}^5C_1 p(1-p)^4 < {}^5C_2 p^2(1-p)^3$ and ${}^5C_2 p^2(1-p)^3 > {}^5C_3 p^3(1-p)^2$ Since $(1-p)>0$ and $p>0$, $1-p<2p$ and $1-p>p$ $p>\frac{1}{3}$ and $p<\frac{1}{2}$ $\therefore \frac{1}{3}<p<\frac{1}{2}$ (shown) (iii) $\bar{x}=1.965$ (from GC) Since $n=5$, $np\approx 1.965 \Rightarrow p\approx 0.393$ (iv) $X\sim \mathrm{B}(5,\ 0.393)$ $\mathrm{P}((X_1\geq 4)\cap(X_1>X_2))$ $=\mathrm{P}(X_1=4)\mathrm{P}(X_2\leq 3)+\mathrm{P}(X_1=5)\mathrm{P}(X_2\leq 4)$ $=0.0724(0.91823)+(0.00937)(0.99063)$ $=0.0758$ (3 sf)
10	(i) Number of ways = $\dfrac{12!}{3!2!}=39916800$ (ii) Method 1 I S D R I B U I O N I S D R I B U I O N I S D R I B U I O N There are 3 ways to slot in the 2 T's Total number of ways = total number of ways to arrange the remaining ten letters $\times 3=\dfrac{10!}{3!}\times 3=1814400$

Method 2

T_ _ _ _ _ _ _ _ _ T | _ | _

Case 1: One I included between the two T's

Number of ways = ${}^{7}C_{6}\times 8!\times\frac{3!}{2!}=120960$

Case 2: Two I's included between the two T's

Number of ways = ${}^{7}C_{6}\times\frac{8!}{2!}\times 3!=846720$

Case 3: Three I's included between the two T's

Number of ways = ${}^{7}C_{5}\times\frac{8!}{3!}\times 3!=846720$

Total number of ways $=120960+2\left(846720\right)=1814400$

(iii) Method 1

Case 1: Both I together but both T separated

II D S R B U O N

7 single letters (excluding Is and Ts) and 1 block of 2Is.

Number of ways = $8!\times{}^{9}C_{2}=1451520$

Case 2: Both T together but I separated

Number of ways = $8!\times{}^{9}C_{2}=1451520$ (same approach as case 1)

Case 3: Both I together and both T together

II TT D S R B U O N

7 single letters (excluding Is and Ts),1 block of Is and 1 block of Ts.

Number of ways = $9!=362880$

Total number of ways in complement

= $(1451520\times 2)+362880$ = 3265920

Method 2

Number of ways in which both T are together = $\frac{10!}{2!}$

Number of ways in which both I are together = $\frac{10!}{2!}$

Number of ways in which both pairs of identical letters are together = 9!

Total number of ways in complement $=2\times\frac{10!}{2!}-9!=3265920$

	Required probability = $1-\frac{3265920}{\frac{11!}{2!2!}}=0.673$ (b)(i) P(holder is not involved in any accident | the holder is classified as 'average risk') $=100\%-15\%=85\%=0.85$ (ii) Probability of a randomly chosen policy holder not involved in any car accident = (0.1)(0.99) + (0.6)(0.85) + (0.3)(0.75) = 0.834 or $\frac{417}{500}$ (iii) P(policy holder is 'low risk' | has met at least one car accident) $=\frac{\text{P(holder is classified as 'low risk' and met with at least 1 accident)}}{\text{P(holder meets with at least 1 accident)}}$ $=\frac{0.1(0.01)}{1-0.834}$ $=0.00602$ (to 3sf) or $\frac{1}{166}$ (iv) Let C be the cost of insuring a randomly chosen 'low risk' policy holder (in thousands). <table><tr><td>c</td><td>0</td><td>5</td><td>10</td><td>50</td><td>100</td></tr><tr><td>P(C = c)</td><td>0.99</td><td>(0.01)(0.75) = 0.0075</td><td>(0.01)(0.15) = 0.0015</td><td>(0.01)(0.08) = 0.0008</td><td>(0.01)(0.02) = 0.0002</td></tr></table> (v) E(C) = 100(0.0002)+50(0.0008)+10(0.0015)+5(0.0075)= 0.1125 Note: "Profit = Premium Charged – Cost Incurred for Repair" 1000(0.1125) + 200 = 312.5 The company should charge $312.50 for a car insurance plan for 'low risk'. <u>Alternative (Using "Profit = Premium – Cost Incurred")</u> Let P be the premium charged by Adiva for a 'low risk' holder. $P(0.99)+(P-5000)(0.0075)+(P-10000)(0.0015)$ $+(P-50000)(0.0008)+(P-100000)(0.0002)=200$ Solving, $P=312.5$ The company should charge $312.50 for a car insurance plan for 'low risk'.