INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 1

CANDIDATE NAME

CLASS | INDEX NUMBER

MATHEMATICS 8864/01

Paper 1

28 August 2017

3 hours

Additional Materials: Answer Paper
Cover Page
Graph Paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

Section A: Pure Mathematics [35 marks]

1 **(i)** Differentiate $\dfrac{3}{\sqrt{(2x-7)^3}}$. [2]

(ii) Find $\displaystyle\int \frac{1}{e^{4t-3}}+\frac{1}{2t-1}\,dt$. [3]

2 By considering $u=e^{2x}$ or otherwise, solve the equation

$$3-10e^{2x}-8e^{4x}=0,$$

leaving your answer in the form $a\ln b$, where a and b are integers to be determined.

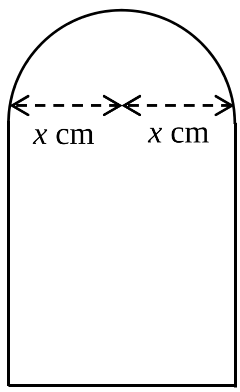
[4]

Hence solve the inequality

$$3-10e^{2x}-8e^{4x}\le 0,$$

leaving your answer in exact form. [2]

3 A window frame is in the shape of a rectangle joined to a semicircle of radius x cm.



(i) If the window frame is made using 300 cm of framework with negligible thickness, show that the total area of the window is given by $A=300x-2x^2-\frac{1}{2}\pi x^2$. [2]

(ii) Using differentiation, find the maximum area of the window. Leave your answer correct to the nearest integer. [4]

4 A curve C has equation $y = \ln(3 - x)$.

(i) Sketch C, indicating clearly the exact coordinates of any points of intersection with the axes and the equation of asymptote, if any. [2]

(ii) Find the equation of the tangent to C at the point P where $x = 1$, giving your answer in the form $y = mx + c$, where m and c are exact constants to be determined. [3]

(iii) R is the region bounded by the tangent at P, the curve C and the x-axis. By sketching the equation of the tangent at P on the diagram in part **(i)**, indicate the region R. Hence find the numerical value of the area of region R. [4]

5 The curve C has equation $y = x^4 - 4x^3 + \frac{9}{2}x^2 - 2x + 2$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence find the coordinates of the stationary points on the curve. [3]

(ii) Use a non-calculator method to determine the nature of each of the stationary points. [2]

(iii) Sketch the graph of C, stating the coordinates of the stationary points and any points where the curve crosses the axes. [2]

(iv) By adding a suitable graph in your diagram in part **(iii)**, solve the inequality $x^4 - 4x^3 + \frac{9}{2}x^2 + x - 9 < 0$, giving your answer correct to 4 decimal places. [2]

Section B: Statistics [60 marks]

6 The Physical Education (PE) Department of Sunflower Secondary School intends to carry out a survey to investigate the number of hours of exercise each student spends per week. The school has a total of 1200 students studying in four different levels. On one particular school day, a PE teacher selects a random sample of 80 students from those who enter the school via the main gate by

- choosing at random one of the first 15 students who enter the main gate,
- then choosing every 15^{th} student after the first student is chosen.

(i) State the sampling method used by the PE teacher. [1]

(ii) State one disadvantage of the sampling method used in this context. [1]

(iii) Describe briefly how, in this case, the PE teacher might choose a more appropriate random sample. [3]

7 At a lucky draw booth, each contestant will roll an unbiased die. If the die shows a "6", the contestant will pick a counter at random from Box *A*. Otherwise, he will pick a counter at random from Box *B*. Box *A* contains 3 red counters, 2 green counters and 3 yellow counters. Box *B* contains 5 red counters, 3 green counters and 2 yellow counters.

(a) A contestant will win a prize if a yellow counter is picked.

(i) Draw a tree diagram to represent this situation. [2]

(ii) Find the probability that a contestant wins a prize. [2]

(iii) Given that the contestant wins a prize, find the probability that it came from Box *A*. [2]

(b) The rule of winning a prize has now changed. Each contestant needs to pick two counters, without replacement, instead of one. A contestant will win a prize if both counters picked are yellow. Find the probability that a contestant wins a prize. [2]

8 In a neighbourhood, it is known that 9% of the residents use the bicycle-sharing platform, ShareBike. A sample of n residents is selected at random and the number of residents who use ShareBike in the sample is denoted by the random variable X.

(i) State, in context, an assumption needed for X to be well modelled by a binomial distribution. [1]

(ii) Explain why the assumption stated in part **(i)** may not hold in this context. [1]

Assume now the assumption stated in part **(i)** does in fact hold.

(iii) Find the greatest value of n such that the probability that there is at least 1 resident using ShareBike is less than 0.99. [3]

(iv) Given that $n = 40$, find the probability that more than 3 but at most 5 residents use ShareBike. [2]

(v) Given instead that $n = 70$, using a suitable approximation, find the probability that less than 7 residents use ShareBike. [4]

9 Two electrical components, Type A and Type B, have lifespans of A weeks and B weeks respectively. It is given that A and B are independent random variables with distributions $\mathrm{N}\left(43, 8^2\right)$ and $\mathrm{N}\left(40, 6^2\right)$ respectively.

(i) Find the probability that the total lifespan of 3 randomly chosen Type A components is shorter than thrice the lifespan of a randomly chosen Type B component. [4]

(ii) Find the probability of the event that both the lifespan of a randomly chosen Type A component exceeds 38 weeks and the total lifespan of 2 randomly chosen Type B components exceeds 82 weeks. [4]

(iii) Find the probability that the total lifespan of a randomly chosen Type A component and 2 randomly chosen Type B components exceeds 120 weeks. [3]

(iv) Explain why the answer in part **(ii)** is smaller than the answer in part **(iii)**. [1]

10 Mr Lee recorded the length of time, t minutes, taken to travel to work when leaving home x minutes after 7 am on 10 mornings over two weeks. The results are as follows.

x	0	5	10	15	20	25	30	35	40	45
t	15	19	30	28	32	39	30	48	53	62

(i) Plot a scatter diagram on graph paper for this data, labelling the axes, using a scale of 2 cm to represent 10 minutes on the t-axis and an appropriate scale for the x-axis. [2]

(ii) Suggest a reason why one of the data points does not seem to follow the trend and indicate the corresponding point on your diagram by labelling it P. [2]

Omit the point P.

(iii) Calculate the product moment correlation coefficient and comment on this value. [2]

(iv) Find the equation of the least squares regression line of t on x, writing your answer in the form $t = ax + b$. [1]

(v) Sketch the regression line on your scatter diagram and interpret the meaning of the value of a in the context of the question. [2]

(vi) Mr Lee needs to arrive at work no later than 8.30 am. Estimate, to the nearest minute, the latest time that he has to leave home without arriving late at work. [3]

11 A large group of Health and Fitness Club members is known to have a mean mass of 85 kg. The trainer claims that the mean mass of his members has decreased under his strict routine. To investigate his claim, the mass, x kg, of 30 randomly chosen members are collated and the results are summarised below.

$$\sum x = 2526, \qquad \sum \left(x - \bar{x}\right)^2 = 544$$

(i) Find unbiased estimates of the population mean and variance. [3]

(ii) Test at the 10% level of significance whether the trainer's claim is valid. [5]

(iii) State the meaning of the p-value obtained in part **(ii)**. [1]

(iv) The trainer makes some adjustments to his training routine and the new population standard deviation is known to be 5 kg. A new sample of 30 members is randomly chosen and the mean mass of this sample is m kg. At the 10% level of significance, find the range of values of m for the trainer's claim to be valid, giving your answer correct to 2 decimal places. [3]

1

(i)

Let $y = \dfrac{3}{\sqrt{(2x-7)^3}} = 3(2x-7)^{-\frac{3}{2}}$

$$\frac{dy}{dx} = 3(2x-7)^{-\frac{5}{2}} \cdot \left(\frac{-3}{2}\right) \cdot (2)$$

$$= -9(2x-7)^{-\frac{5}{2}} \qquad \left(\text{or } -\frac{9}{(2x-7)^{\frac{5}{2}}}\right)$$

(ii)

$$\int \frac{1}{e^{4t-3}} + \frac{1}{2t-1}\, dt$$

$$= \int e^{-4t+3} + \frac{1}{2t-1}\, dt$$

$$= -\frac{1}{4}e^{-4t+3} + \frac{1}{2}\ln(2t-1) + c$$

2

(i)

$3 - 10e^{2x} - 8e^{4x} = 0$

Let $u = e^{2x}$. Then

$\Rightarrow \quad 3 - 10u - 8u^2 = 0$

$\Rightarrow \quad (2u+3)(1-4u) = 0$

$\Rightarrow \quad u = -\frac{3}{2} \quad$ or $\quad u = \frac{1}{4}$

$\Rightarrow \quad e^{2x} = -\frac{3}{2} \quad$ or $\quad e^{2x} = \frac{1}{4}$

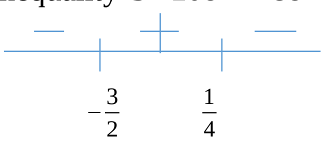
(rej $\because e^{2x} > 0$)

$\therefore \quad 2x = \ln\frac{1}{4} \Rightarrow x = \frac{1}{2}\ln\frac{1}{4} = -\ln 2$

$a = -1$ and $b = 2$

(ii)

For the inequality $3 - 10e^{2x} - 8e^{4x} \le 0$,

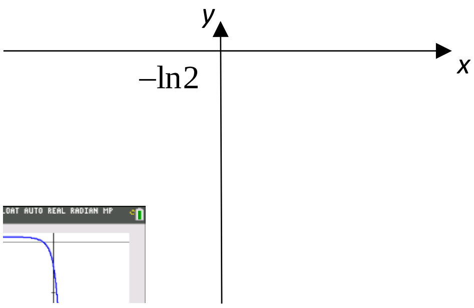


$e^{2x} \le -\frac{3}{2} \quad$ or $\quad e^{2x} \ge \frac{1}{4}$

(rej $\because e^{2x} > 0$)

$\therefore \quad x \geq -\ln 2$

Or use graphical method



Therefore,

$\therefore \quad x \geq -\ln 2$

3

(i)

Let A cm^2 be the area of the window and l cm be the length of the rectangle.

$\pi x + 2l + 2x = 300$

$\Rightarrow \; l = 150 - x - \dfrac{\pi x}{2}$

$$A = \frac{1}{2}\pi x^2 + 2xl$$

$$= \frac{1}{2}\pi x^2 + 2x\left(150 - x - \frac{\pi x}{2}\right)$$

$$= \frac{1}{2}\pi x^2 + 300x - 2x^2 - \pi x^2$$

$$A = 300x - 2x^2 - \frac{1}{2}\pi x^2$$

(ii)

For maximum area, $\dfrac{\mathrm{d}A}{\mathrm{d}x} = 0$

$300 - 4x - \pi x = 0$

$x = \dfrac{300}{4+\pi}$ or 42.007

Using 1st derivative test,

x	$\left(\dfrac{300}{4+\pi}\right)^-$	$\dfrac{300}{4+\pi}$	$\left(\dfrac{300}{4+\pi}\right)^+$
$\dfrac{dA}{dx}$	+ve	0	–ve

A is maximum when $x=\dfrac{300}{4+\pi}$ or 42.007.

Maximum A

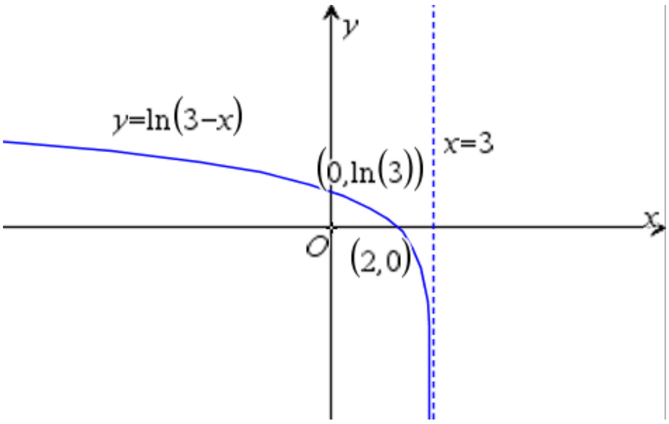
$$=300\left(\frac{300}{4+\pi}\right)-2\left(\frac{300}{4+\pi}\right)^2-\frac{1}{2}\pi\left(\frac{300}{4+\pi}\right)^2$$

$$=6301.115$$

$=6301\,\text{cm}^2$ (nearest whole number)

4

(i) $y=\ln(3-x)$



(ii)

$$\frac{dy}{dx}=-\frac{1}{3-x}$$

At point P, $x=1$; $\dfrac{dy}{dx}=-\dfrac{1}{2}$; $y=\ln(2)$

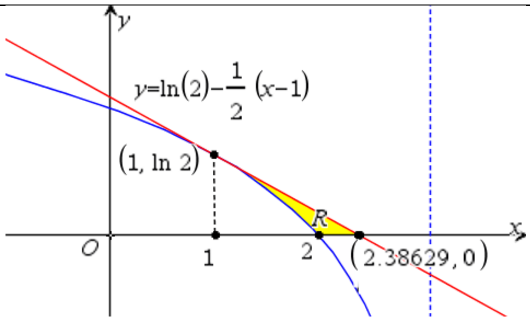
Eqn of tangent at P: $y-\ln(2)=-\dfrac{1}{2}(x-1)$

$$y=\ln(2)-\frac{1}{2}(x-1)$$

$$y=-\frac{1}{2}x+\ln 2+\frac{1}{2}$$

(iii)

$y=-\dfrac{1}{2}x+\ln 2+\dfrac{1}{2}$ $y=\ln 2-\dfrac{1}{2}(x-1)$



The area of *R*

= Area of triangle – area under the curve

$$=\frac{1}{2}(2.38629-1)(\ln 2)-\int_1^2 \ln(3-x)\,\mathrm{d}x$$

$$=0.0941586528$$

$$\approx 0.0942 \text{ units}^2$$

5

(i)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4x^3-12x^2+9x-2$$

Let $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$\Rightarrow 4x^3-12x^2+9x-2=0$$

Using GC, $x=2$ or $\frac{1}{2}$

When $x=2$, $y=(2)^4-4(2)^3+4.5(4)-2(2)+2$

$=0$

When $x=\frac{1}{2}$, $y=\left(\frac{1}{2}\right)^4-4\left(\frac{1}{2}\right)^3+4.5\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)+2$

$=\frac{27}{16}$ or 1.6875

The coordinates of the stationary points are $(2,0)$ and $\left(\frac{1}{2},\frac{27}{16}\right)$.

(ii)

When $x=2$,

x	$x=2^-$	$x=2$	$x=2^+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	$-$	0	+
slope	\	—	/

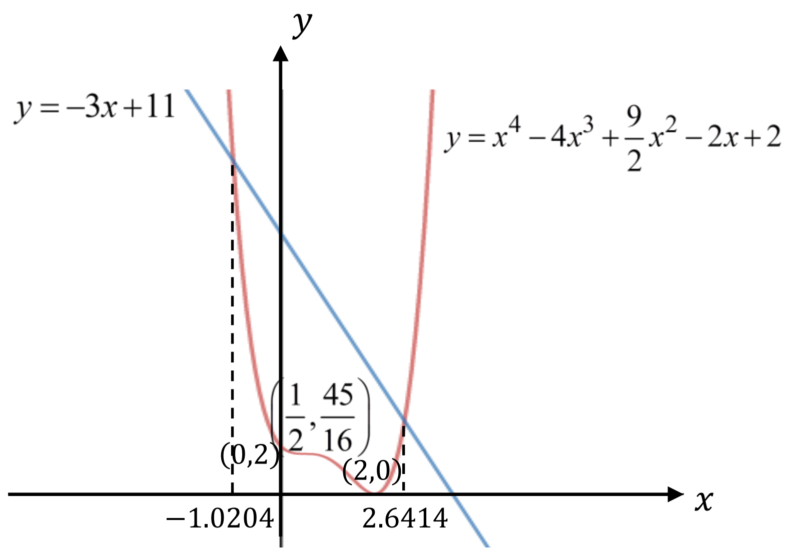
Thus $(2,0)$ is a minimum point.

When $x = \frac{1}{2}$,

x	$x = \frac{1}{2}^{-}$	$x = \frac{1}{2}$	$x = \frac{1}{2}^{+}$
$\frac{dy}{dx}$	$-$	0	$-$
slope	\	—	\

Thus $\left(\frac{1}{2}, \frac{27}{16}\right)$ is a stationary point of inflexion.

(iii)



(iv)

$$x^4 - 4x^3 + \frac{9}{2}x^2 + x - 9 < 0$$

$$x^4 - 4x^3 + \frac{9}{2}x^2 - 2x + 3x + 2 - 11 < 0$$

$$x^4 - 4x^3 + \frac{9}{2}x^2 - 2x + 2 < -3x + 11$$

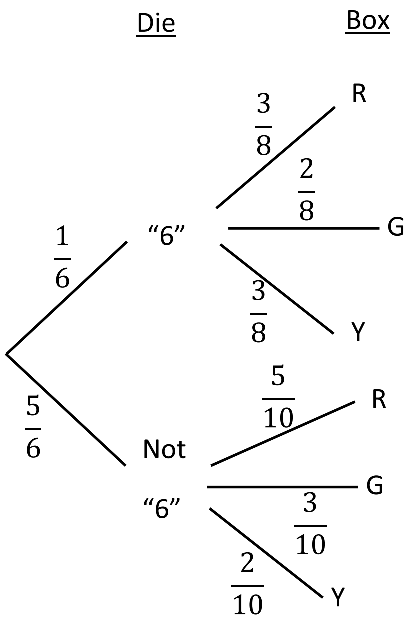
Using GC, the x-coordinates of intersection are $x = -1.0204$ or 2.6414

From the graph, $-1.0204 < x < 2.6414$.

<table>
<tr><td>6</td><td>(i) Systematic Sampling

(ii)
One disadvantage is that not all students get an equal chance of being selected as the students who are absent from school will not get a chance to be interviewed.

(iii)
Stratified Sampling

Split the school population into mutually exclusive subgroups/strata based on the four levels, Sec 1 to 4.
$\frac{80}{1200} \times 100\% \approx 6.67\%$
Choose random samples from each subgroup that is proportional to the size of the subgroup/stratum (6.67%) to make up the sample of 80 students.
or
Choose randomly 6.67% of students from each stratum to form a sample of 80 students.</td></tr>
<tr><td>7</td><td>

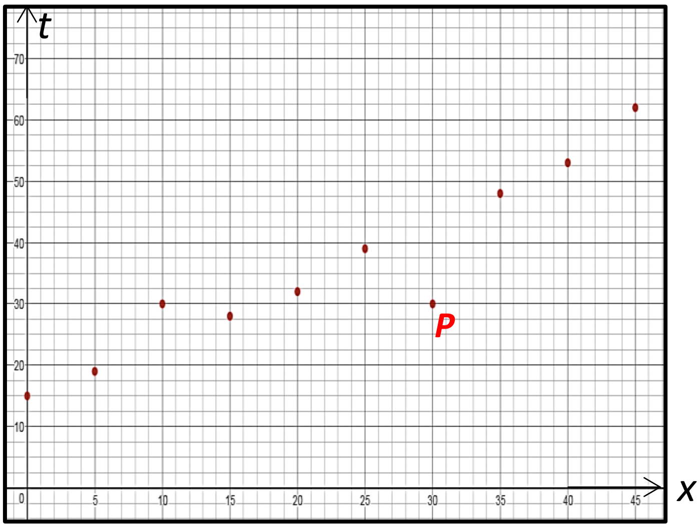
(a)(ii)
$\mathrm{P}(\text{wins a prize}) = \left(\frac{1}{6} \times \frac{3}{8}\right) + \left(\frac{5}{6} \times \frac{2}{10}\right)$
$= \frac{11}{48}$</td></tr>
</table>

	(a)(iii) $\text{P}\left(\text{from box A} \mid \text{wins the prize}\right) = \dfrac{\frac{1}{6}\times\frac{3}{8}}{\frac{11}{48}}$ $= \dfrac{3}{11}$ (b) $\text{P}\left(\text{wins the grand prize}\right) = \left(\frac{1}{6}\times\frac{3}{8}\times\frac{2}{7}\right) + \left(\frac{5}{6}\times\frac{2}{10}\times\frac{1}{9}\right)$ $= \dfrac{55}{1512}$
8	(i) The assumption is that the event of a resident using ShareBike or not is independent of any other residents in the neighbourhood. (ii) The assumption may not hold as usually families may use ShareBike together as they are going for the activity together. (iii) $X \sim \text{B}(n, 0.09)$ $\text{P}(X \geq 1) < 0.99$ $\text{P}(X = 0) > 0.01$ $\binom{n}{0}(0.09)^0(1-0.09)^n > 0.01$ $(0.91)^n > 0.01$ $n < \dfrac{\ln 0.01}{\ln 0.91}$ $n < 48.830$ $\therefore$ Greatest value of n is 48. **<u>Alternative method</u>** Using GC, When $n = 47$, $\text{P}(X = 0) = 0.0119$ (> 0.01) When $n = 48$, $\text{P}(X = 0) = 0.0108$ (> 0.01) When $n = 49$, $\text{P}(X = 0) = 0.0098$ (<0.01) $\therefore$ Greatest value of n is 48. (iv) $X \sim \text{B}(40, 0.09)$ $\text{P}(3 < X \leq 5) = \text{P}(X \leq 5) - \text{P}(X \leq 3)$ $= 0.344$ (3 s.f.)

	(v) $X \sim \mathrm{B}(70, 0.09)$ Since $n = 70$ is large, $np = 70 \times 0.09 = 6.3 \ \ (> 5)$ $nq = 70 \times (1 - 0.09) = 63.7 \ \ (> 5)$ $X \sim \mathrm{N}(6.3, 5.733)$ approximately $\mathrm{P}(X < 7) = \mathrm{P}(X < 6.5)$ (continuity correction) $= 0.533$ (3 s.f.)
9	(i) $A \sim \mathrm{N}\left(43, 8^2\right) \quad B \sim \mathrm{N}\left(40, 6^2\right)$ $\mathrm{P}\left(A_1 + A_2 + A_3 < 3B\right) = \mathrm{P}\left(A_1 + A_2 + A_3 - 3B < 0\right)$ Let $S = A_1 + A_2 + A_5 - 3B$ $\mathrm{E}(S) = 3(43) - 3(40) = 9$ $\mathrm{Var}(S) = 3\left(8^2\right) + 3^2\left(6^2\right) = 516$ $\mathrm{P}\left(S < 0\right) = 0.346$ (3 s.f.) (ii) $B \sim \mathrm{N}\left(40, 6^2\right)$ $\mathrm{E}\left(B_1 + B_2\right) = 2(40) = 80$ $\mathrm{Var}\left(B_1 + B_2\right) = 2\left(6^2\right) = 72$ $B_1 + B_2 \sim \mathrm{N}(80, 72)$ Required probability $= \mathrm{P}\left(A > 38\right) \times \mathrm{P}\left(B_1 + B_2 > 82\right)$ $= 0.299$ (3 s.f.) (iii) $\mathrm{E}\left(A + B_1 + B_2\right) = 43 + 2(40) = 123$ $\mathrm{Var}\left(A + B_1 + B_2\right) = 8^2 + 2\left(6^2\right) = 136$ $A + B_1 + B_2 \sim \mathrm{N}(123, 136)$ Required probability $= \mathrm{P}\left(A + B_1 + B_2 > 120\right)$ $= 0.602$ (3 s.f.) (iv) Because the case in (ii) is a proper subset of the case in (iii). For eg, Part iii contains cases whereby the lifespan of component A may not exceed 38 weeks (eg. 36 weeks) but total lifespan of 2 components of B exceeds 82 weeks (eg. 84 weeks), and yet the total lifespan is more than 120 weeks.

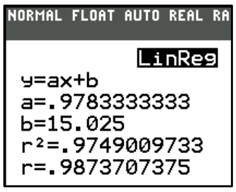
10

(i)



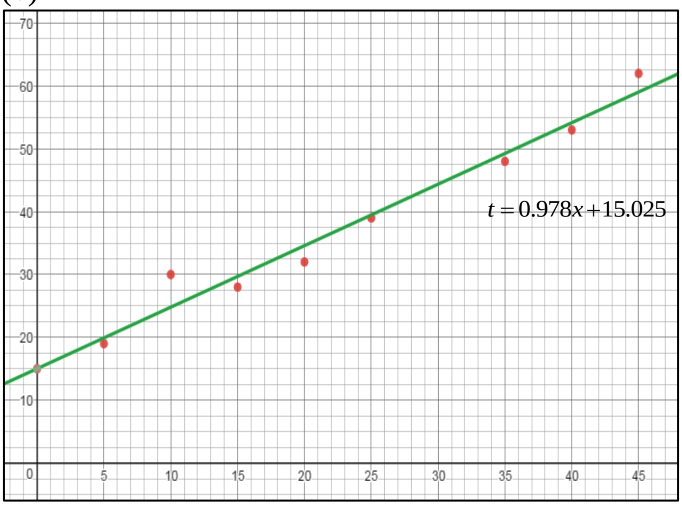
(ii)
Acceptable reasons:
The traffic condition on the road was good (Lesser cars on the road, no traffic jam) and thus he required much shorter travelling time though he left home only at 7.30am.
It was a public holiday/school holiday/Sunday and yet Mr Lee has to work.

(iii)
$r \approx 0.987$
The pmcc is close to 1, indicating a strong positive linear correlation between x and t. I.e. the later Mr Lee leaves home after 7 am, the longer the travelling time would take.



(iv) $t = 0.978x + 15.025$

(v)



$a = 0.978$ means that for every additional minute that Mr Lee delays in leaving home after 7am, his travelling time will increase by 0.978 minutes.

(vi)

Method 1:

There are 90 minutes from 7 am to 8.30 am.

$$x+t\le 90$$

$$x+(0.97833x+15.025)\le 90$$

$$1.97833x \le 74.975$$

$$x\le 37.898$$

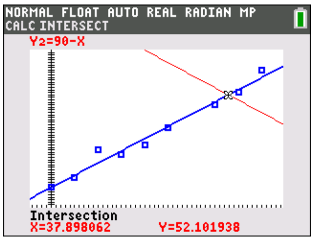
The largest possible value of x is 37 (correct to the nearest minute)

The latest time Mr Lee could leave home without being late for work is 7.37 am.

Method 2:

Sketch the line $x+t=90$ and find x-coordinate of the point of intersection with the regression line.

If x = 38, Mr Lee will arrive late for work. Thus the latest time he needs to leave home is 7.37am.



Method 3:

By Trial & Error, using GC

From part (vi),

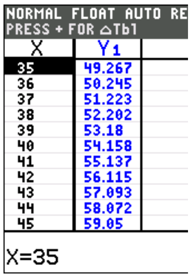
if $x=40, t=54.158, x+t>90$

$x=39, t=53.18,\quad x+t>90$

$x=38, t=52.202,\ x+t>90$

$x=37, t=51.223, x+t<90$

Thus the latest time he needs to leave home is 7.37am.



11

(i) $\sum x=2526,\qquad \sum(x-\bar{x})^2=544$

Unbiased estimates of the population mean μ is

$$\bar{x}=\frac{2526}{30}=84.2$$

Unbiased estimates of the population variance σ^2 is

$$s^2=\frac{30}{29}\left[\frac{544}{30}\right]$$

$$=18.75862$$

$$=18.8\ (3\text{ s.f})$$

(ii)

$H_0:\ \mu=85$

$H_1:\ \mu<85$

Test at 10% significance level

	Assuming that H_0 is true, Since $n = 30$ is sufficiently large, by the Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(85, \frac{s^2}{30}\right)$ approximately. Test statistic: $Z = \dfrac{\bar{X} - 85}{\sqrt{\dfrac{18.75862}{30}}} \sim \mathrm{N}(0,1)$ approximately. Using GC, *p*-value = 0.15584 = 0.156 (3 s.f) $\left(\text{or } z = \dfrac{84.2 - 85}{\sqrt{\dfrac{18.75862}{30}}} = -1.0117 \right)$ Since *p*-value = 0.15584 > 0.1 (or $z = -1.0117 > -1.28155$), we do not reject H_0 and conclude that there is insufficient evidence at 10% level, that the mean mass of the Health and Fitness Club members has decreased. (or that the trainer's claim is invalid.) (iii) There is 0.15584 probability of drawing **a random sample of 30** Health and Fitness Club members with **sample mean less than 84.2 kg**, assuming that the population mean weight is 85 kg. (iv) $H_0: \mu = 85$ $H_1: \mu < 85$ Test at 10% level significance level. Assuming that H_0 is true, Since *n* is large, by the Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(85, \frac{5^2}{30}\right)$ approximately. Test statistic: $Z = \dfrac{\bar{X} - 85}{5/\sqrt{30}} \sim \mathrm{N}(0,1)$ approximately. Since the null hypothesis is rejected, $\Rightarrow$ *z*-value falls inside critical region $\Rightarrow$ $z\text{-value} < -1.28155$

	$\Rightarrow \quad \dfrac{m-85}{5/\sqrt{30}} < -1.28155$ $m-85 < -1.1699$ $m < 83.83$ $\therefore \quad 0 < m < 83.83$ **Alternate method** Using $\bar{X} \sim \mathrm{N}\left(85, \dfrac{5^2}{30}\right)$ $\mathrm{P}\left(\bar{X} \le m\right) < 0.1$ $\Rightarrow \; m < 83.83$ $\therefore \quad 0 < m < 83.83$

INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
in preparation for General Certificate of Education Advanced Level
Higher 1

CANDIDATE NAME			
CLASS		INDEX NUMBER	

MATHEMATICS 8865/01

Paper 1

28 August 2017

3 hours

Additional Materials: Answer Paper
Cover Page
Graph Paper
List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.

Write your name, class and index number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper. You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

Section A: Pure Mathematics [40 marks]

1 At the grand opening of a new indoor playground in town, visitors are given discount on entrance fees based on different promotions. The entrance fees to the playground are divided into three categories, namely Toddler (below 3 year old), Child (3 to 12 years old) and Adult (13 years old and above). Three families visit the playground and the number of family members are shown in the table below.

Members	Toddler	Child	Adult
Tan Family	0	2	2
Ng Family	1	1	2
Lim Family	1	3	3

The Tan family paid \$22.50 for their entrance fees after a 25% discount. The Ng family paid \$20 for their entrance fees after a 20% discount. The Lim family bought the membership priced at \$15 nett to enjoy a 50% discount on entrance fees and paid \$41.25 in total. By forming a system of linear equations, find the original entrance fee for each of the categories. [4]

2 **(a)** Differentiate $\dfrac{3}{\sqrt{(2x-7)^3}}$. [2]

(b) Use a non-calculator method to find the exact value of $\displaystyle\int_1^3 \frac{1}{e^{4t-3}} + \frac{1}{2t-1}\,dt$. [4]

3 By considering $u = e^{2x}$ or otherwise, solve the equation

$$3 - 10e^{2x} - 8e^{4x} = 0,$$

leaving your answer in the form $a\ln b$, where a and b are integers to be determined. [4]

Hence solve the inequality

$$3 - 10e^{2x} - 8e^{4x} \le 0,$$

leaving your answer in exact form. [2]

4 A curve C has equation $y = \ln(3-x)$.

(i) Sketch C, indicating clearly the exact coordinates of any points of intersection with the axes and the equation of the asymptote. [2]

(ii) Find the equation of the tangent to C at the point P where $x = 1$, giving your answer in the form $y = mx + c$, where m and c are exact constants to be determined. [3]

(iii) R is the region bounded by the tangent at P, the curve C and the x-axis. By sketching the equation of the tangent at P on the diagram in part **(i)**, indicate the region R. Hence find the numerical value of the area of region R. [4]

5 An electronic company manufactures smart phones and the manager of this company monitors how the rate of the total manufacturing costs, x million dollars per month, of their new smart phone model changes over a period of t months. The company's financial analyst believes that the relationship between x and t can be modelled by the equation

$$x = t^3 - 13t^2 + 40t + 35, \text{ for } 0 \le t \le 12.$$

(i) Using differentiation, find the minimum value of x, justifying that this value is a minimum. [6]

(ii) Sketch the graph of x against t, giving the coordinates of any intersections with the axes. [2]

(iii) Find the area of the region bounded by the curve, the line $t = 12$ and both t- and x-axes. Give an interpretation of the area you have found, in the context of the question. [3]

The company's accountant believes that the connection between the profit per month, $$P$ million, is related to x, by the equation

$$P = 45 + 20\ln(3x+4)$$

(iv) Find the exact value of $\frac{\mathrm{d}P}{\mathrm{d}x}$ for which $t = 8$. [2]

(v) Hence find the rate of increase in profit per month when $t = 8$. [2]

Section B: Statistics [60 marks]

6 A 4-digit number is chosen at random from the digits $\{1,2,3,4,5\}$ where repetition of digits is not allowed.

Find the probability that the 4-digit number chosen

(i) is an even number, [3]

(ii) is greater than 3000 given that the number is an even number. [3]

7 At a lucky draw booth, each contestant will roll an unbiased die. If the die shows a "6", the contestant will pick a counter at random from Box *A*. Otherwise, he will pick a counter at random from Box *B*. Box *A* contains 3 red counters, 2 green counters and 3 yellow counters. Box *B* contains 5 red counters, 3 green counters and 2 yellow counters.

(a) A contestant will win a prize if a yellow counter is picked.

(i) Draw a tree diagram to represent this situation. [2]

(ii) Find the probability that a contestant wins a prize. [2]

(iii) Given that the contestant wins a prize, find the probability that it came from Box *A*. [2]

(b) The rule of winning a prize has now changed. Each contestant needs to pick two counters, without replacement, instead of one. A contestant will win a prize if both counters picked are yellow. Find the probability that a contestant wins a prize. [2]

8 In a neighbourhood, it is known that 9% of the residents use the bicycle-sharing platform, ShareBike. A sample of n residents is selected at random and the number of residents who use ShareBike in the sample is denoted by the random variable X.

(i) State, in context, an assumption needed for X to be well modelled by a binomial distribution. [1]

(ii) Explain why the assumption stated in part **(i)** may not hold in this context. [1]

Assume now the assumption stated in part **(i)** does in fact hold.

(iii) Find the greatest value of n such that the probability that there is at least 1 resident using ShareBike is less than 0.99. [3]

It is now given that $n = 20$.

(iv) Find the probability that more than 2 but at most 5 residents use ShareBike. [2]

(v) 40 such random samples are taken and the number of residents using ShareBike is being observed in each sample. Find the probability that the mean number of residents using ShareBike of these observations exceeds 2. [3]

9 Two electrical components, Type A and Type B, have lifespans of A weeks and B weeks respectively. It is given that A and B are independent random variables with distributions $\mathrm{N}\left(43, 8^2\right)$ and $\mathrm{N}\left(40, 6^2\right)$ respectively.

(i) Find the probability that the total lifespan of 3 randomly chosen Type A components is shorter than thrice the lifespan of a randomly chosen Type B component. [4]

(ii) Find the probability of the event that both the lifespan of a randomly chosen Type A component exceeds 38 weeks and the total lifespan of 2 randomly chosen Type B components exceeds 82 weeks. [4]

(iii) Find the probability that the total lifespan of a randomly chosen Type A component and 2 randomly chosen Type B components exceeds 120 weeks. [3]

(iv) Explain why the answer in part **(ii)** is smaller than the answer in part **(iii)**. [1]

[Turn over

10 Mr Lee recorded the length of time, t minutes, taken to travel to work when leaving home x minutes after 7 am on 10 mornings over two weeks. The results are as follows.

x	0	5	10	15	20	25	30	35	40	45
t	15	19	30	28	32	39	30	48	53	62

(i) Plot a scatter diagram on graph paper for this data, labelling the axes, using a scale of 2 cm to represent 10 minutes on the t-axis and an appropriate scale for the x-axis. [2]

(ii) Suggest a reason why one of the data points does not seem to follow the trend and indicate the corresponding point on your diagram by labelling it P. [2]

Omit the point P.

(iii) Calculate the product moment correlation coefficient and comment on this value. [2]

(iv) Find the equation of the least squares regression line of t on x, writing your answer in the form $t = ax + b$. [1]

(v) Sketch the regression line on your scatter diagram and interpret the meaning of the value of a in the context of the question. [2]

(vi) Mr Lee needs to arrive at work no later than 8.30 am. Estimate, to the nearest minute, the latest time that he has to leave home without arriving late at work. [3]

11 A large group of Health and Fitness Club members is known to have a mean mass of 85 kg. The trainer claims that the mean mass of his members has decreased under his strict routine. To investigate his claim, the mass, x kg, of 30 randomly chosen members are collated and the results are summarised below.

$$\sum x = 2526, \qquad \sum \left(x - \overline{x}\right)^2 = 544$$

(i) Find unbiased estimates of the population mean and variance. [3]

(ii) Test at the 10% level of significance whether the trainer's claim is valid. [5]

(iii) State the meaning of the p-value obtained in part **(ii)**. [1]

(iv) The trainer makes some adjustments to his training routine and the new population standard deviation is known to be 5 kg. A new sample of 30 members is randomly chosen and the mean mass of this sample is m kg. At the 10% level of significance, find the range of values of m for the trainer's claim to be valid, giving your answer correct to 2 decimal places. [3]

1	Let \$$x$ be the entry rates for toddlers. Let \$$y$ be the entry rates for children. Let \$$z$ be the entry rates for adults. $0.75(2y+2z)=22.50$ $2y+2z=30 \quad ----(1)$ $0.8(x+y+2z)=20$ $x+y+2z=25 \quad ----(2)$ $0.5(x+3y+3z)+15=41.25$ $x+3y+3z=52.50 \quad ----(3)$ Using GC to solve eq (1), (2) and (3) $x=7.5$ $y=12.5$ $z=2.5$ $\therefore$ the entry rates are \$7.50 (Toddler), \$12.50 (children) and \$2.50 (adult) respectively.
2(i)	(a) Let $y=\dfrac{3}{\sqrt{(2x-7)^3}}=3(2x-7)^{-\frac{3}{2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=3(2x-7)^{-\frac{5}{2}}\cdot\left(\dfrac{-3}{2}\right)\cdot(2)$ $=-9(2x-7)^{-\frac{5}{2}} \qquad \left(\text{or } -\dfrac{9}{(2x-7)^{\frac{5}{2}}}\right)$ (b) $\displaystyle\int_1^3 \frac{1}{\mathrm{e}^{4t-3}}+\frac{1}{2t-1}\,\mathrm{d}t$ $=\displaystyle\int_1^3 \mathrm{e}^{-4t+3}+\frac{1}{2t-1}\,\mathrm{d}t$ $=\left[-\dfrac{1}{4}\mathrm{e}^{-4t+3}+\dfrac{1}{2}\ln(2t-1)\right]_1^3$ $=-\dfrac{1}{4}\mathrm{e}^{-9}+\dfrac{1}{2}\ln(5)-\left(-\dfrac{1}{4}\mathrm{e}^{-1}+\dfrac{1}{2}\ln(1)\right)$ $=-\dfrac{1}{4}\left(\mathrm{e}^{-9}-\mathrm{e}^{-1}\right)+\dfrac{1}{2}\ln 5$

3

(i)

$$3-10e^{2x}-8e^{4x}=0$$

Let $u=e^{2x}$. Then

$\Rightarrow \quad 3-10u-8u^2=0$

$\Rightarrow \quad (2u+3)(1-4u)=0$

$\Rightarrow \quad u=-\frac{3}{2} \quad \text{or} \quad u=\frac{1}{4}$

$\Rightarrow \quad e^{2x}=-\frac{3}{2} \quad \text{or} \quad e^{2x}=\frac{1}{4}$

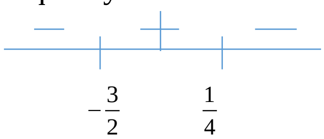
(rej $\because e^{2x}>0$)

$\therefore \quad 2x=\ln\frac{1}{4} \Rightarrow x=\frac{1}{2}\ln\frac{1}{4}=-\ln 2$

$a=-1$ and $b=2$

(ii)

For the inequality $3-10e^{2x}-8e^{4x}\le 0$,

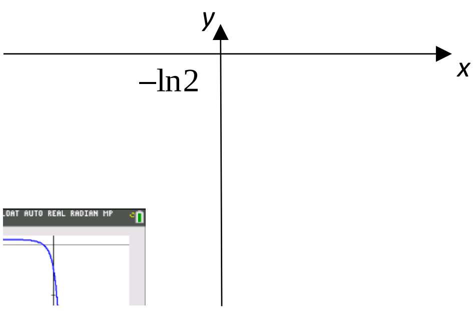


$e^{2x}\le -\frac{3}{2} \quad \text{or} \quad e^{2x}\ge \frac{1}{4}$

(rej $\because e^{2x}>0$)

$\therefore \quad x\ge -\ln 2$

Or use graphical method

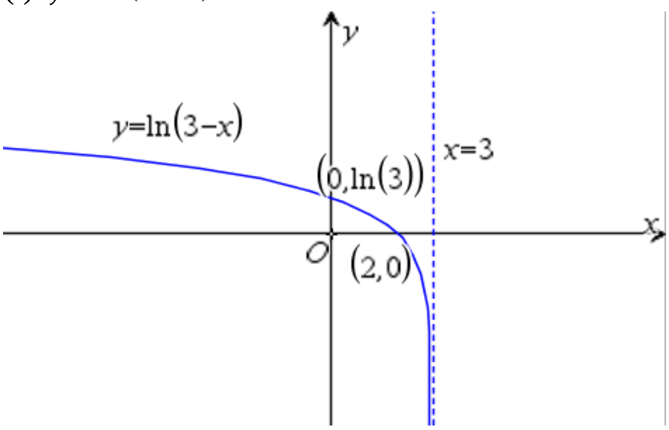


Therefore,

$\therefore \quad x\ge -\ln 2$

4

(i) $y=\ln(3-x)$



(ii)

$$\frac{dy}{dx}=-\frac{1}{3-x}$$

At point P, $x=1$; $\frac{dy}{dx}=-\frac{1}{2}$; $y=\ln(2)$

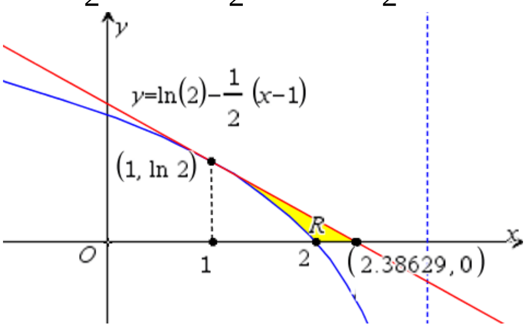
Eqn of tangent at P: $y-\ln(2)=-\frac{1}{2}(x-1)$

$$y=\ln(2)-\frac{1}{2}(x-1)$$

$$y=-\frac{1}{2}x+\ln 2+\frac{1}{2}$$

(iii)

$y=-\frac{1}{2}x+\ln 2+\frac{1}{2}$ $y=\ln 2-\frac{1}{2}(x-1)$



The area of R

= Area of triangle – area under the curve

$$=\frac{1}{2}(2.38629-1)(\ln 2)-\int_1^2 \ln(3-x)\,dx$$

$$=0.0941586528$$

$$\approx 0.0942 \text{ units}^2$$

5

(i)

$x = t^3 - 13t^2 + 40t + 35$

$\frac{dx}{dt} = 3t^2 - 26t + 40$

For min value of x, $\frac{dx}{dt} = 0$

$3t^2 - 26t + 40 = 0$

$(3t - 20)(t - 2) = 0$

$\Rightarrow t = 2 \ \ or \ \ t = \frac{20}{3}$

t	2^-	2	2^+
Sign of $\frac{dx}{dt}$	+	0	−
slope	/	—	\

t	$\left(\frac{20}{3}\right)^-$	$\frac{20}{3}$	$\left(\frac{20}{3}\right)^+$
Sign of $\frac{dx}{dt}$	−	0	+
slope	\	—	/

Therefore, x is a minimum when $t = \frac{20}{3}$.

Or

$\frac{d^2x}{dt^2} = 6t - 26$

When $t = 2$, $\frac{d^2x}{dt^2} = 6(2) - 26 = -14 < 0$

When $t = \frac{20}{3}$, $\frac{d^2x}{dt^2} = 6\left(\frac{20}{3}\right) - 26 = 14 > 0$

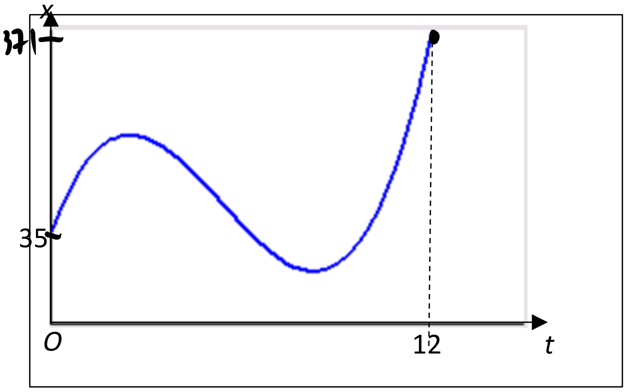
Therefore, x is a minimum when $t = \frac{20}{3}$.

$x = \left(\frac{20}{3}\right)^3 - 13\left(\frac{20}{3}\right)^2 + 40\left(\frac{20}{3}\right) + 35$

$= 20.185$

$x = 20.2 \quad \text{or} \quad \frac{545}{27}$

(ii)



(iii)

Area of the region

$= \int_0^{12} t^3 - 13t^2 + 40t + 35 \ \text{d}t$

$= \left[\frac{t^4}{4} - 13\frac{t^3}{3} + 20t^2 + 35t \right]_0^{12}$

$= \frac{1}{4}(12)^4 - \frac{13}{3}(12)^3 + 20(12)^2 + 35(20) - 0$

$= 996$

(or using GC to solve)

The total manufacturing cost to manufacture the smart phones for a period of 12 months is \$996 million.

(iv)

$P = 45 + 20\ln(3x+4)$

$\frac{dP}{dx} = \frac{60}{3x+4}$

When t = 8, x = 35

$\frac{\text{d}P}{\text{d}x} = \frac{60}{3(35)+4} = \frac{60}{109}$

(v)

$\frac{\text{d}P}{\text{d}t} = \frac{\text{d}P}{\text{d}x} \times \frac{\text{d}x}{\text{d}t}$

When x = 8,

$\frac{\text{d}P}{\text{d}t} = \frac{60}{109} \times [3(8)^2 - 26(8) + 40]$

$= \frac{1440}{109}$ or 13.2

The rate of increase in profit when t = 8 is \$13.2 million per month.

6	(i) Number of ways $= 4\times3\times2\times2$ $= 48$ Required probability $= \dfrac{48}{{}^5P_4}$ $= \dfrac{48}{120}$ $= 0.4$ (ii) Case 1 : 1st digit is '4' Number of ways $= 1\times3\times1\times2$ $= 6$ Case 2 : 1st digit is '3' or '5' Number of ways $= 2\times3\times2\times2$ $= 24$ Total number of ways $= 24+6=30$ Required probability $= \dfrac{30}{48} = \dfrac{5}{8}$ **Alternative method** $\text{P}\left(\text{greater than 3000} \mid \text{number is even}\right)$ $= \dfrac{\text{P}\left(\text{greater than 3000} \cap \text{number is even}\right)}{\text{P}\left(\text{number is even}\right)}$ $= \dfrac{\frac{24+6}{120}}{0.4}$ $= \dfrac{5}{8}$

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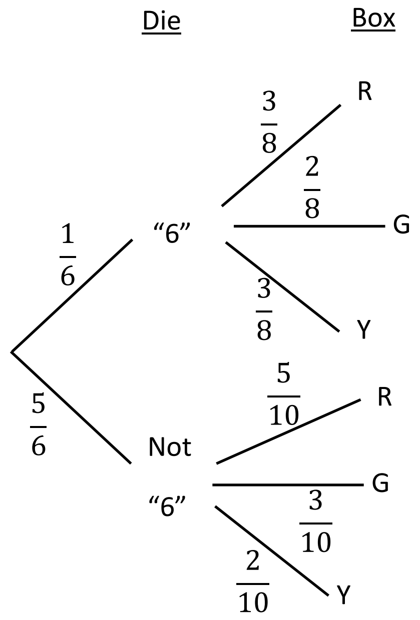

(a)(ii)

$$P\left(\text{wins a prize}\right)=\left(\frac{1}{6}\times\frac{3}{8}\right)+\left(\frac{5}{6}\times\frac{2}{10}\right)$$

$$=\frac{11}{48}$$

(a)(iii)

$$P\left(\text{from box A}\mid\text{wins the prize}\right)=\frac{\frac{1}{6}\times\frac{3}{8}}{\frac{11}{48}}$$

$$=\frac{3}{11}$$

(b)

$$P\left(\text{wins the grand prize}\right)=\left(\frac{1}{6}\times\frac{3}{8}\times\frac{2}{7}\right)+\left(\frac{5}{6}\times\frac{2}{10}\times\frac{1}{9}\right)$$

$$=\frac{55}{1512}$$

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<td>8</td>
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(i) The assumption is that the event of a resident using ShareBike or not is independent of any other residents in the neighbourhood.

(ii) The assumption may not hold as usually families may use ShareBike together as they are going for the activity together.

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(iii)

$X \sim \mathrm{B}(n, 0.09)$

$\mathrm{P}(X \geq 1) < 0.99$

$\mathrm{P}(X = 0) > 0.01$

$\binom{n}{0}(0.09)^0(1-0.09)^n > 0.01$

$(0.91)^n > 0.01$

$n < \dfrac{\ln 0.01}{\ln 0.91}$

$n < 48.830$

$\therefore$ Greatest value of n is 48.

<u>Alternative method</u>

Using GC,

When $n = 47$, $\mathrm{P}(X = 0) = 0.0119$ (> 0.01)

When $n = 48$, $\mathrm{P}(X = 0) = 0.0108$ (> 0.01)

When $n = 49$, $\mathrm{P}(X = 0) = 0.0098$ (<0.01)

$\therefore$ Greatest value of n is 48.

(iv)

$X \sim \mathrm{B}(20, 0.09)$

$\mathrm{P}(2 < X \leq 5) = \mathrm{P}(X \leq 5) - \mathrm{P}(X \leq 2)$

$= 0.260$ (3 s.f.)

(v)

$\mathrm{E}(X) = 20 \times 0.09 = 1.8$

$\mathrm{Var}(X) = 20 \times 0.09 \times (1 - 0.09) = 1.638$

Sample mean $\bar{X} = \dfrac{X_1 + X_2 + \ldots X_{40}}{40}$

Since $n = 40$ is sufficiently large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(1.8, \dfrac{1.638}{40}\right)$ approximately.

$\mathrm{P}\left(\bar{X} > 2\right) = 0.161$ (3 s.f.)

9 (i)

$A \sim \mathrm{N}\left(43, 8^2\right) \quad B \sim \mathrm{N}\left(40, 6^2\right)$

$\mathrm{P}\left(A_1 + A_2 + A_3 < 3B\right) = \mathrm{P}\left(A_1 + A_2 + A_3 - 3B < 0\right)$

Let $S = A_1 + A_2 + A_5 - 3B$

$\mathrm{E}(S) = 3(43) - 3(40) = 9$

$\mathrm{Var}(S) = 3\left(8^2\right) + 3^2\left(6^2\right) = 516$

$\mathrm{P}\left(S < 0\right) = 0.346$ (3 s.f.)

(ii)

$B \sim \mathrm{N}\left(40, 6^2\right)$

$\mathrm{E}\left(B_1 + B_2\right) = 2(40) = 80$

$\mathrm{Var}\left(B_1 + B_2\right) = 2\left(6^2\right) = 72$

$B_1 + B_2 \sim \mathrm{N}(80, 72)$

Required probability $= \mathrm{P}\left(A > 38\right) \times \mathrm{P}\left(B_1 + B_2 > 82\right)$

$= 0.299 \ \ (3 \text{ s.f.})$

(iii)

$\mathrm{E}\left(A + B_1 + B_2\right) = 43 + 2(40) = 123$

$\mathrm{Var}\left(A + B_1 + B_2\right) = 8^2 + 2\left(6^2\right) = 136$

$A + B_1 + B_2 \sim \mathrm{N}(123, 136)$

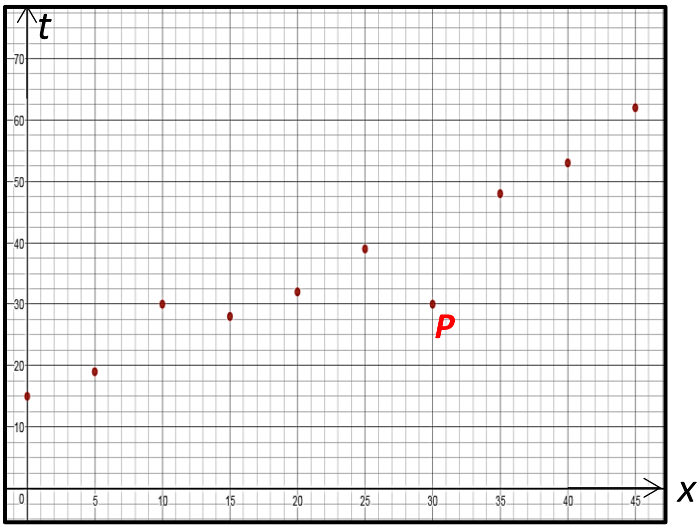
Required probability $= \mathrm{P}\left(A + B_1 + B_2 > 120\right)$

$= 0.602 \ \ (3 \text{ s.f.})$

(iv)

Because the case in (ii) is a proper subset of the case in (iii). For eg, Part iii contains cases whereby the lifespan of component A may not exceed 38 weeks (eg. 36 weeks) but total lifespan of 2 components of B exceeds 82 weeks (eg. 84 weeks), and yet the total lifespan is more than 120 weeks.

10 (i)



(ii)

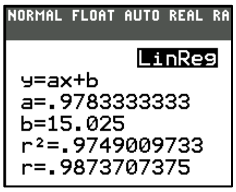
Acceptable reasons:

The traffic condition on the road was good (Lesser cars on the road, no traffic jam) and thus he required much shorter travelling time though he left home only at 7.30am.

It was a public holiday/school holiday/Sunday and yet Mr Lee has to work.

(iii)

$r \approx 0.987$



The pmcc is close to 1, indicating a strong positive linear correlation between *x* and *t*. I.e. the later Mr Lee leaves home after 7 am, the longer the travelling time would take.

(iv) $t = 0.978x + 15.025$

(v)

$a = 0.978$ means that for every additional minute that Mr Lee delays in leaving home after 7am, his travelling time will increase by 0.978 minutes.

(vi)

Method 1:

There are 90 minutes from 7 am to 8.30 am.

$$x + t \le 90$$

$$x + (0.97833x + 15.025) \le 90$$

$$1.97833x \le 74.975$$

$$x \le 37.898$$

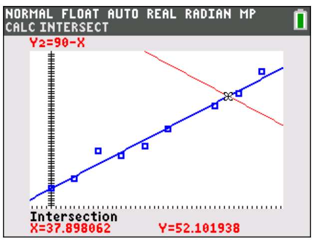
The largest possible value of *x* is 37 (correct to the nearest minute)

The latest time Mr Lee could leave home without being late for work is 7.37 am.

Method 2:

Sketch the line $x + t = 90$ and find *x*-coordinate of the point of intersection with the regression line.

If *x* = 38, Mr Lee will arrive late for work. Thus the latest time he needs to leave home is 7.37am.



Method 3:

By Trial & Error, using GC

From part (vi),

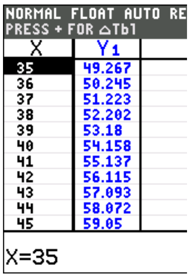
if $x = 40, t = 54.158, x + t > 90$

$x = 39, t = 53.18, \quad x + t > 90$

$x = 38, t = 52.202, \; x + t > 90$

$x = 37, t = 51.223, x + t < 90$

Thus the latest time he needs to leave home is 7.37am.



X	Y1
35	49.267
36	50.245
37	51.223
38	52.202
39	53.18
40	54.158
41	55.137
42	56.115
43	57.093
44	58.072
45	59.05

11

(i) $\sum x = 2526, \qquad \sum (x - \bar{x})^2 = 544$

Unbiased estimates of the population mean μ is

$$\bar{x} = \frac{2526}{30} = 84.2$$

Unbiased estimates of the population variance σ^2 is

$$s^2 = \frac{30}{29}\left[\frac{544}{30}\right] = 18.75862 = 18.8 \; (3 \text{ s.f})$$

(ii)

$H_0: \; \mu = 85$

$H_1: \; \mu < 85$

Test at 10% significance level

Assuming that H_0 is true,

Since $n = 30$ is sufficiently large, by the Central Limit Theorem, $\bar{X} \sim \text{N}\left(85, \frac{s^2}{30}\right)$ approximately.

Test statistic: $Z = \dfrac{\bar{X} - 85}{\sqrt{\dfrac{18.75862}{30}}} \sim \text{N}(0,1)$ approximately.

Using GC, p-value = 0.15584 = 0.156 (3 s.f)

$$\left(\text{or } z = \frac{84.2 - 85}{\sqrt{\dfrac{18.75862}{30}}} = -1.0117 \right)$$

Since p-value = 0.15584 > 0.1 (or $z = -1.0117 > -1.28155$), we do not reject H_0 and conclude that there is insufficient evidence at 10% level, that the mean mass of the Health and Fitness Club members has decreased. (or that the trainer's claim is invalid.)

(iii)
There is 0.15584 probability of drawing **a random sample of 30** Health and Fitness Club members with **sample mean less than 84.2 kg**, assuming that the population mean weight is 85 kg.

(iv)
$H_0: \ \mu = 85$

$H_1: \ \mu < 85$

Test at 10% level significance level.

Assuming that H_0 is true,

Since n is large, by the Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(85, \frac{5^2}{30}\right)$ approximately.

Test statistic: $Z = \frac{\bar{X} - 85}{5/\sqrt{30}} \sim \mathrm{N}(0,1)$ approximately.

Since the null hypothesis is rejected,

$\Rightarrow$ z-value falls inside critical region

$\Rightarrow$ $z\text{-value} < -1.28155$

$$\Rightarrow \frac{m-85}{5/\sqrt{30}} < -1.28155$$

$$m - 85 < -1.1699$$

$$m < 83.83$$

$\therefore \quad 0 < m < 83.83$

Alternate method

Using $\bar{X} \sim \mathrm{N}\left(85, \frac{5^2}{30}\right)$

$\mathrm{P}\left(\bar{X} \le m\right) < 0.1$

$\Rightarrow m < 83.83$

$\therefore \quad 0 < m < 83.83$