JC2 PRELIMINARY EXAMINATION 2016

MATHEMATICS
Higher 1

8864/01

Paper 1

Wednesday **14 September 2016** **3 hours**

Additional materials: Answer paper
List of Formula (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT class on the Cover Page and all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.
Do not write anything on the List of Formula (MF15).

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question. At the end of the examination, place the completed cover page on top of your answer scripts and fasten all your work securely together with the string provided.

This question paper consists of 6 printed pages.

Section A : Pure Mathematics [35 marks]

1 Using an algebraic method, find the range of values of y such that the equation $(y-2)x^2+(y+1)x-1=0$ has 2 distinct solutions for all values of $x \in \square$. [4]

2 (i) Sketch the graph of $y=\ln(x-1)^2$, stating clearly the coordinates of any points of intersection with the axes and the equation(s) of any asymptote(s). [2]

(ii) By considering another curve on the same diagram, solve $\ln(x-1)^2-1\le e^{x-5}$. [3]

3 (i) Find $\int e^{4-3x}\,dx$. [2]

(ii) Find the exact value of $\int_2^3 \left(1+\frac{1}{2x}\right)^2 dx$. [4]

4 (i) On a single diagram, sketch the graphs of $y=e^{2x+1}$ and $y=\sqrt[3]{2x+3}$. Find the x-coordinates of the points of intersection of $y=e^{2x+1}$ and $y=\sqrt[3]{2x+3}$. [4]

(ii) Find the area of the finite region bounded by the curves $y=e^{2x+1}$ and $y=\sqrt[3]{2x+3}$. [2]

5 Cylindrical cans are manufactured in bulk for storing paint. Each cylindrical can is to hold $\frac{5\pi}{128}$ m^3 of paint. The material used for the top and base of the can costs \$$a$ per m^2 and the material for the curved surface costs $\$\left(\frac{4a}{5}\right)$ per m^2. Assuming that the thickness of the cylindrical can is negligible, find the radius r (in metres) and height h (in metres) of the can that costs the least to manufacture. [6]

[The curved surface area and the volume of a cylinder with radius r and height h are given by $2\pi rh$ and πr^2h respectively.]

6 The curve has equation $y=\frac{1}{6}(2x-3)^3-4x$.

(i) Find the gradient of the normal at the point P where the curve intersects the y-axis. [2]

(ii) Find the equation of the normal at P in the form $ax+by+c=0$, where a and b are integers. The normal at P meets the x-axis at N. Find the area of triangle OPN. [4]

(iii) Find the set of values of x for which $y=\frac{1}{6}(2x-3)^3-4x$ is an increasing function of x. [2]

Section B : Statistics [60 marks]

7 A secondary school has 600 pupils. It is intended to obtain a sample of 20 pupils to attend a workshop.

(i) Explain how a systematic sample of size 20 might be obtained. [2]

(ii) Suggest an alternative method of sampling and describe how it could be carried out. [2]

8 The random variables X has the distribution N(220 , 25). Given that X_1 and X_2 are independent observations of X, find $\mathrm{P}(X_1 > X_2 + 20)$. [2]

The random variable Y has the distribution N(100 , 16) and it is given that Y is related to X by the formula $Y = bX + c$, where b and c are constants with $b > 0$. Find the values of b and c. [4]

9 A playground has three play areas X, Y and Z that offer different play structures. The play areas X, Y and Z have play structures like swings (S), roundabouts (R), climbing frames (C) and play-houses (P) for the kids to play with. The numbers of play structure in each of the three play areas are as follows.

Play areas	Climbing frames(C)	Roundabouts (R)	Swings(S)	Play-houses(P)
X	0	2	3	4
Y	1	3	6	2
Z	4	3	8	1

Each time Mary takes her child to one of the play areas. The probabilities that she chooses play area X is $\frac{1}{4}$ and play area Y is $\frac{1}{3}$. When she goes to a play area, she chooses one play structure at random.

(i) Find the probability that Mary chooses a play-house. [4]

(ii) Find the probability that Mary goes to play area Y given that she chooses a climbing frame. [3]

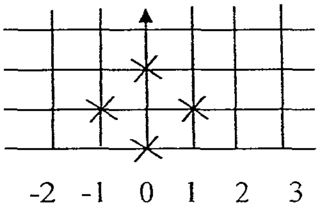
10 Jane works in the service sector and her job requires her to work every weekday, even if the weekday is a public holiday. She takes the bus to work and will only take the taxi should she wake up late. The probability of her waking up late on any day is 0.1.

(i) In a particular week, find the probability that Friday is the third day that she takes a taxi to work. [2]

(ii) The bus fare to work costs \$1.50 while the taxi fare to work costs \$10. Considering the duration of 4 weeks, find the probability that the amount of taxi fare paid by Jane is more than the total bus fare paid if she did not wake up late throughout the 4 weeks (i.e., 20 weekdays). [3]

(iii) Using a suitable approximation, find the probability that Jane has to pay more than \$200 worth of taxi fare in 52 complete weeks. [4]

11 (a) The following scatter diagram shows a sample of four pairs of values.



Give the coordinates of a fifth point such that the linear product moment correlation coefficient for all five points is

(i) negative [1]
(ii) zero [1]

(b) One end H of an elastic string was attached to a horizontal bar and a mass, m grams, was attached to the other end. The mass was suspended freely and allowed to settle vertically below H. The length of the elastic string, l mm was recorded for eight masses as follows:

m	50	75	100	125	150	200	250	300
l	134	150	170	199	240	243	271	300

(i) Find the product moment correlation coefficient for l and m. [1]

(ii) Draw a scatter diagram for the data. [1]

(iii) Using parts (i) and (ii), comment on the suitability of a linear model between l and m. [2]

It was suspected that one of the readings taken was inaccurate.

(iv) Identify the reading which is most likely to be inaccurate. [1]

The inaccurate reading was removed.

(v) Calculate the product moment correlation coefficient using the remaining values of l and m. [1]

(vi) Using the appropriate regression line, estimate the value of m when $l = 250$ mm. [2]

(vii) Comment on the use of the regression line in part (vi) to estimate the original length of the elastic string. [1]

12 A bottled soft drink is labelled as having a volume of 150ml. However, customers complained that they were shortchanged. The manager of the company producing the soft drink decided to take a random sample of 50 bottles and measure the amount of contents in each of the bottles. The results are summarized by

$$\sum(150-x)=35 \quad \text{and} \quad \sum(x-149.3)^2=167.$$

(i) Show that the unbiased estimate of the population mean is 149.3ml and find the unbiased estimate of the population variance. [3]

(ii) Test, at 5% significance level, if the complaints are valid. [3]

(iii) Explain the meaning of "5% significance level" in the context of the question. [1]

(iv) Explain how the conclusion in part **(ii)** would be affected if the manager discovered that

 (a) the sample was not randomly chosen, [1]

 (b) the distribution of the amount of the drinks was not normally distributed. [1]

(v) The manager decided to make changes to the alternative hypothesis. Without redoing the hypothesis testing process, explain if it is necessarily true to conclude that at 5% significance level using the same sample, that there is sufficient evidence that a bottled soft drink does not contain, on average, 150ml of drinks. [2]

13 A supermarket sells two types of oranges, A and B. The masses, in kilograms, of the oranges of each type have independent normal distributions. The means and standard deviations of these distributions, and the selling prices, in \$ per kilogram, are shown in the following table.

	Mean (kg)	Standard deviation (kg)	Selling price (\$ per kg)
Type A	0.18	0.05	7.00
Type B	0.20	0.04	8.00

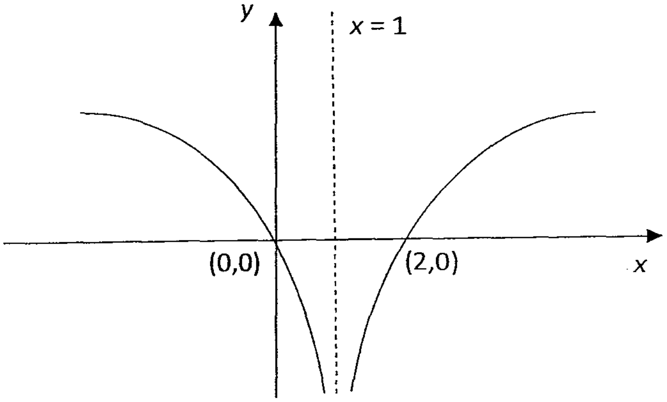
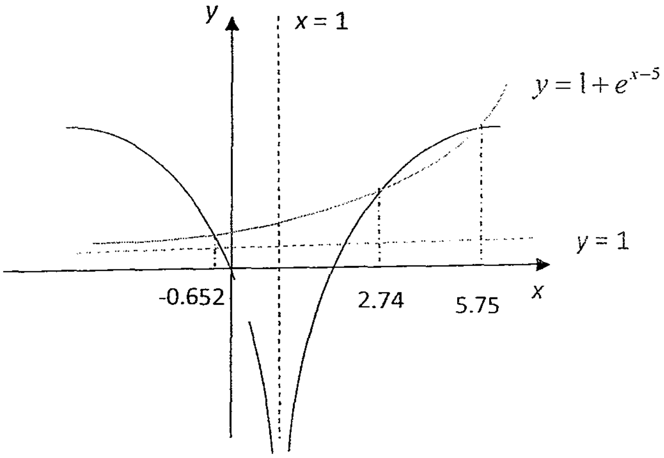
Stating clearly the mean and variance of all distributions that you use, find the probability that

(i) the total mass of 5 randomly chosen oranges of type B is at most 1.1 kg, [3]

(ii) the total mass of 8 randomly chosen oranges of type A is within 0.3 kg of the total mass of 6 randomly chosen oranges of type B. [4]

Mr Lim bought 4 oranges of type A and 4 oranges of type B. Mr Chan bought 6 oranges of type B.

(iii) Find the probability that Mr Lim pays at least \$1.80 more than Mr Chan. [5]

1	$(y-2)x^2+(y+1)x-1=0$ Since there 2 distinct solutions, $b^2-4ac>0$. $\Rightarrow (y+1)^2-4(y-2)(-1)>0 \Rightarrow y^2+2y+1+4y-8>0$ $\Rightarrow y^2+6y-7>0 \Rightarrow (y-1)(y+7)>0$ $\Rightarrow y>1$ or $y<-7, y\neq 2$
2 (i)	
(ii)	$\ln(x-1)^2-1\le e^{x-5}$ $\ln(x-1)^2\le 1+e^{x-5}$ Sketch $y=1+e^{x-5}$.  $-0.652\le x<1$ or $1<x\le 2.74$ or $x\ge 5.75$

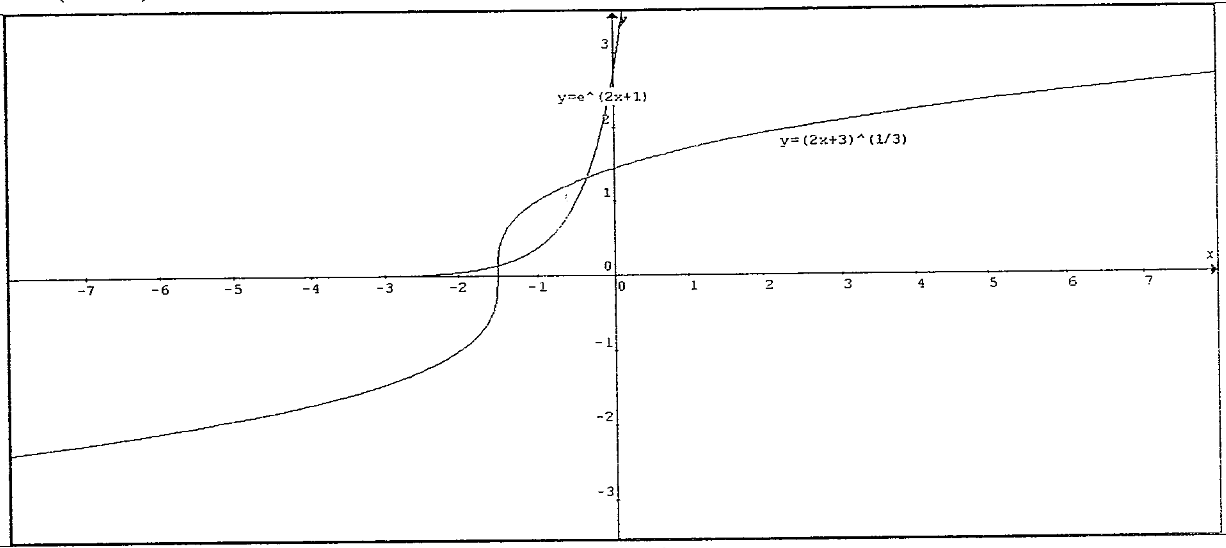
3(i)

$$\int e^{4-3x}\,dx = \frac{-e^{4-3x}}{3}+C$$

(ii)

$$\int_2^3\left(1+\frac{1}{2x}\right)^2 dx=\int_2^3\left(1+\frac{1}{x}+\frac{1}{4x^2}\right)dx\ \left[x+\ln|x|-\frac{1}{4x}\right]_2^3=\left(3+\ln 3-\frac{1}{12}\right)-\left(2+\ln 2-\frac{1}{8}\right)=\frac{25}{24}+\ln\frac{3}{2}.$$

4(i)



x-coordinates of points of intersection are:
$x=-1.50$ & $x=-0.363$.

(ii)

Area required $=\displaystyle\int_{-1.49875}^{-0.36309}\left(\sqrt[3]{2x+3}-e^{2x+1}\right)dx = 0.531$

5

Volume of cylinder $=\pi r^2 h=\dfrac{5\pi}{128}\ \Rightarrow h=\dfrac{5}{128r^2}$

Let the cost of the cylinder be C. $\quad C= a(2\pi r^2)+\dfrac{4a}{5}(2\pi rh)$

$$C=2a\pi r^2+\frac{8a\pi}{5}r\left(\frac{5}{128r^2}\right)\Rightarrow C=2a\pi r^2+\frac{\pi a}{16r}$$

$$\frac{dC}{dr}=4a\pi r-\frac{\pi a}{16r^2}=0 \qquad \Rightarrow r^3=\frac{1}{64}\Rightarrow r=\frac{1}{4}$$

r	$\left(\frac{1}{4}\right)^-$	$\left(\frac{1}{4}\right)$	$\left(\frac{1}{4}\right)^+$
$\frac{dC}{dr}$	-ve	0	+ve

OR $\dfrac{d^2C}{dr^2}=4a\pi+\dfrac{\pi a}{8r^3}>0$ when $r=\dfrac{1}{4}$.

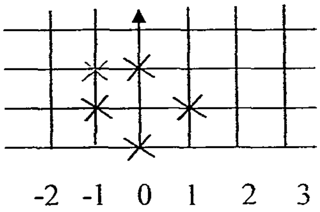
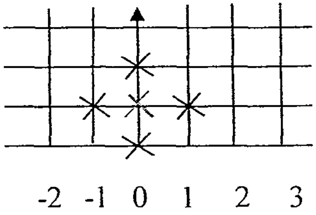
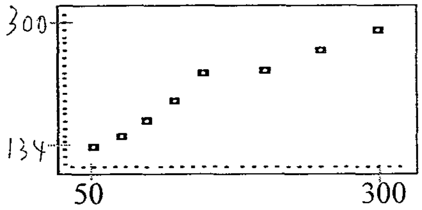
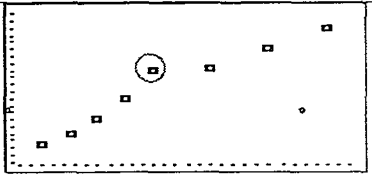
Hence C is minimum when $r=\dfrac{1}{4}$.

$$h=\frac{5}{128\times\frac{1}{16}}=\frac{5}{8}$$

The can that costs the least to manufacture has radius $\frac{1}{4}$ m and height $\frac{5}{8}$ m.

6(i)	$y=\frac{1}{6}(2x-3)^3-4x$ $\frac{\mathrm{d}y}{\mathrm{d}x}=(2x-3)^2-4$ $x=0, \frac{\mathrm{d}y}{\mathrm{d}x}=5$ Gradient of the normal $=-\frac{1}{5}$
(ii)	when $x=0, y=-\frac{27}{6}=-\frac{9}{2}$ Equation of the normal $y+\frac{9}{2}=-\frac{1}{5}x$ $10y+45=-2x$ $2x+10y+45=0$ $y=0, x=-\frac{45}{2}$ $\quad N\left(-\frac{45}{2},0\right)$ Area of *OPN* $=\frac{1}{2}\left(\frac{9}{2}\right)\left(\frac{45}{2}\right)=\frac{405}{8}(=50.625)$
(iii)	$(2x-3)^2-4\geq 0$ $(2x-3-2)(2x-3+2)\geq 0$ $(2x-5)(2x-1)\geq 0$ $\Rightarrow\left\{x:\ x\leq\frac{1}{2} \text{ or } x\geq\frac{5}{2}\right\}$
7(i) (ii)	Number all the students in the school from 1 to 600. Compute the sampling interval, k, and obtain $k=\frac{600}{20}=30.$ Randomly select an integer from 1 to 30. Select every 30th pupil thereafter until 20 members are obtained. Another alternative method of sampling is stratified sampling. Divide the pupils into various strata such as their education level (Secondary 1 to Secondary 4) and gender (male and female). The sample size for each stratum is proportionally represented. Pupils are then selected randomly within each stratum. OR Another alternative method of sampling is simple random sampling. Number all the students in the school from 1 to 600. Using a random number generator to obtain 20 random numbers. Those students whose numbers correspond to these 20 numbers were those selected. OR Another alternative method of sampling is quota sampling. Divide 600 pupils into male and female. Select 10 males and 10 females based on the teacher's own discretion.

8	$X \sim \mathrm{N}(220\,,25)$ $X_1 - X_2 \sim \mathrm{N}(0\,,50)$ $\mathrm{P}(X_1 - X_2 > 20) = 0.00234$ $Y = bX + c \;\Rightarrow\; \mathrm{Var}(Y) = b^2\,\mathrm{Var}(X) \;\Rightarrow\; 16 = 25\,b^2 \,\ldots\ldots\ldots (1)$ $b^2 = \dfrac{16}{25} \qquad b > 0 \;\therefore b = \dfrac{4}{5}$ or 0.8 $\mathrm{E}(Y) = \dfrac{4}{5}\mathrm{E}(X) + c \;\Rightarrow\; 100 = \dfrac{4}{5}(220) + c \ldots\ldots\ldots (2) \;\;\Rightarrow\; c = -\,76$
9(i)	$\mathrm{P}\left(\text{Mary goes to playground } X \text{ and she chooses a play-house}\right) = \frac{1}{4} \text{x} \frac{4}{9} = \frac{1}{9}$ $\mathrm{P}\left(\text{Mary goes to playground } Y \text{ and she chooses a play-house}\right) = \frac{1}{3} \text{x} \frac{2}{12} = \frac{1}{18}$ $\mathrm{P}\left(\text{Mary goes to playground Z and she chooses a play-house}\right) = \frac{5}{12} \text{x} \frac{1}{16} = \frac{5}{192}$ Probability required $= \frac{1}{9} + \frac{1}{18} + \frac{5}{192} = \frac{37}{192}$ or 0.193 (3 s.f.)
(ii)	$\mathrm{P}\left(\text{Mary goes to playground} Y \,\middle\vert\, \text{Mary chooes a climbing frame}\right) = \dfrac{\mathrm{P}(Y \cap C)}{\mathrm{P}(C)} = \dfrac{\frac{1}{3} \times \frac{1}{12}}{\frac{1}{3} \times \frac{1}{12} + \frac{5}{12} \times \frac{4}{16}} = \dfrac{4}{19}$ or 0.211 (3 s.f.)
10 (i)	Let X denote the number of days, out of 4 days, Jane woke up late. X □ $\mathrm{B}(4,0.1)$ Required probability $= \left[\mathrm{P}(X = 2)\right](0.1)$ $= 0.00486$ (3 s.f.)
(ii)	4 weeks = 20 weekdays Bus fare for 4 weeks = \$1.50 × 20 = \$30. Thus Jane will have to be late at least 4 times. Let Y denote the number of days, out of 20 days, Jane woke up late. Y □ $\mathrm{B}(20,0.1)$ Required probability $= \mathrm{P}(Y \geq 4)$ $= 1 - \mathrm{P}(Y \leq 3)$ $= 0.133$ (3 s.f.)
(iii)	Let W be no of days, out of 260 days, Jane woke up late. W □ $\mathrm{B}(260, 0.1)$ $n = 260 > 50$ is large. $np = 234 > 5, nq = 26 > 5$ W □ $\mathrm{N}(26, 23.4)$ approx. Required probability $= \mathrm{P}(W > 20)$ $\xrightarrow[cc]{} \mathrm{P}(W > 20.5) = 0.872$ (3 s.f.)

11(a)	(i)  (ii) 
(b)(i)	$r = 0.974$
(ii)	
(iii)	$r = 0.974 \approx 1$ There is a strong positive linear correlation between m and l, implying that the linear model is suitable. In addition, the scatter diagram also show that the points lie approximately along a straight line, further indicating that a linear model is suitable.
(iv)	 Outlier: $m = 150, l = 240$
(v)	$r = 0.99567 \approx 0.996$
(vi)	Use the regression line of l on m as m is clearly the controlled variable. $l = 0.66763m + 104.659$ When $l = 250$, $250 = 0.66763m + 104.659 \Rightarrow m = 217.6969 \approx 218$
(vii)	It would not be advisable as $m = 0$ falls outside the given data range of $50 \leq m \leq 300$. Extrapolation is generally discouraged.
12(i)	$\sum(x-150) = -35$ $\left(\sum x\right) - 50(150) = -35$ $\sum x = 7465$ $\bar{x} = \frac{7465}{50} = \frac{1493}{10} = 149.3$ $s^2 = \frac{1}{49}(167) = \frac{167}{49} = 3.40816 \approx 3.41(3\text{ sf})$
(ii)	$H_0 : \mu = 150$ against $H_1 : \mu < 150$ Test at 5% significance level. $Z = -2.68$ $p = 0.00367 < 0.05$

	Reject H_0 and conclude that there is sufficient evidence at 5% significance level, the content of the drinks in the bottle is less than what the packaging claimed to be / the complaints are valid.
(iii)	5% significance level means there is a 5% chance that we say that the complaints are valid when the contents are actually not less than 150ml.
(iv) (a)	If the sample was not randomly chosen, the conclusion made can be unreliable. For example, the sample may have been the first 50 bottles that are produced in the same batch and the mean quantity could have changed as production continued.
(iv) (b)	The conclusion is unaffected whether the distribution is normal or not as this is a large sample. By Central Limit Theorem, the mean quantity in the bottled drink is approximately normally distributed.
(v)	$H_0 : \mu = 150$ against $H_1 : \mu \neq 150$ $p = (0.00367)(2) = 0.00734 < 0.05$ Reject H_0 at 5% significance level that there is sufficient evidence that, on average, the bottled soft drink does not contain 150ml of drinks
13(i)	Let A be the mass of a type A orange in kg. $A \sim \mathrm{N}(0.18\,,0.05^2)$. Let B be the mass of a type B orange in kg. $B \sim \mathrm{N}(0.20\,,0.04^2)$. Let $M = B_1 + B_2 + \ldots\ldots + B_5$ $\quad M \sim \mathrm{N}(1\,,0.008)$ $\mathrm{P}(M \leq 1.1) = 0.868$
(ii)	Let $W = (A_1 + A_2 + \ldots\ldots + A_8) - (B_1 + B_2 + \ldots\ldots B_6)$ $W \sim \mathrm{N}(0.24\,,0.0296)$ $\mathrm{P}(\lvert W \rvert < 0.3) = \mathrm{P}(-0.3 < W < 0.3)$ $= 0.636$
(iii)	Let C_L be the amount paid by Mr Lim. $C_L = 7(A_1 + A_2 + A_3 + A_4) + 8(B_1 + B_2 + B_3 + B_4)$ $C_L \sim \mathrm{N}(11.44\,,0.8996)$ Let C_C be the amount paid by Mr Chan. $C_C = 8(B_5 + B_6 + \ldots\ldots + B_{10})$ $\quad C_C \sim \mathrm{N}(9.6\,,0.6144)$ $C_L - C_C \sim N(1.84\,,1.514)$ $\quad \mathrm{P}(C_L - C_C \geq 1.8) = 0.513$