ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS **HIGHER 2** **9758/01**

Monday **30 August 2021** **3 hrs**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:________________________________ (_____) **C.G.:** __________

TUTOR'S NAME: ________________________________

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: ____________________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Question	1	2	3	4	5	6	7	8	9	10	TOTAL
Marks											
	9	7	11	8	9	10	13	8	13	12	100

This document consists of **6** printed pages including this page.

[Turn Over]

1 **(i)** The curve $y = \dfrac{ax^2 - 2ax + 4}{x - b}$ has an oblique asymptote $y = x$ and a vertical asymptote $x = 2$. State the value of b and find the value of a. **[2]**

(ii) By using an algebraic method, show that y cannot take values between -2 and 6. **[4]**
Using the information found in part **(i)** and part **(ii)**.

(iii) sketch the graph of $y = \dfrac{ax^2 - 2ax + 4}{x - b}$, stating clearly any axial intercept(s), asymptotes and the coordinates of the turning point(s). **[3]**

2 **(a)** A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} 1 - 3x & -1 \le x \le 1, \\ -2 - (x-1)^2 & 1 < x \le 2, \\ 2 & \text{otherwise.} \end{cases}$$

Sketch the graph of $y = \mathrm{f}(x)$ for $-1 \le x \le 4$, stating clearly the coordinates of any axial intercepts. State the range of f. **[4]**

(b) A curve undergoes the transformations A, B and C in succession. The transformations A, B and C are given as follows:
A: Translate 1 unit in the positive x-direction;
B: Scale parallel to x-axis by a scale factor of 5;
C: Reflect in the y-axis.
The equation of the resulting curve is $y = e^{x-2} - x$. Determine the equation of the original curve. **[3]**

3 **(a)** A sequence of numbers u_1, u_2, u_3, ... has a sum S_n where $S_n = \sum_{r=1}^{n} u_r$. It is given that $S_n = A(2^n) + Bn^2 + C$, where A, B and C are non-zero constants.

(i) Find an expression for u_n in terms of A, B and n. **[2]**

(ii) It is also given that $u_1 = 7$ and $S_2 = 25$. Find A, B and C. **[4]**

(b) Show that $r(r+1)(r+2) - (r-2)(r-1)r = kr^2$, where k is a constant to be determined.
Use this result to deduce that $\sum_{r=1}^{n} r^2 = \dfrac{n(n+1)(2n+1)}{6}$. **[5]**

[Turn Over]

4 Sketch the graphs of $y=1+\frac{a-2}{x-a}$, and $y=-\frac{1}{a}x+\frac{2}{a}$ on a single diagram, where a is a positive constant and $1<a<2$, showing all asymptotes and axial intercepts clearly. **[4]**

(i) Using the graphs, solve, in terms of a, $1+\frac{a-2}{x-a}>-\frac{1}{a}x+\frac{2}{a}$. **[1]**

(ii) Hence, solve $1+\frac{ax-2x}{1-ax}>-\frac{1}{ax}+\frac{2}{a}$. **[3]**

5 **(a)** The sum of the first n terms of a sequence is denoted by S_n. Given that $S_n=e^n-1$, prove that the sequence is a geometric progression. **[3]**

(b) The area of island S is $2880\ \text{km}^2$ at the end of 2019. Due to the rise in sea level, the area of the island decreases gradually every year. At the end of 2020, the area decreases by $64\ \text{km}^2$. Two companies, A and B, are engaged to study the trend in the decrease in the area of Island S.

(i) According to Company A, the decrease in area in each subsequent year is $3\ \text{km}^2$ less than that of the previous year. The decrease in area continues annually up to and including the year when the decrease is first less than $5\ \text{km}^2$ occurs. Find the area of Island S when the decrease is first less than $5\ \text{km}^2$. **[4]**

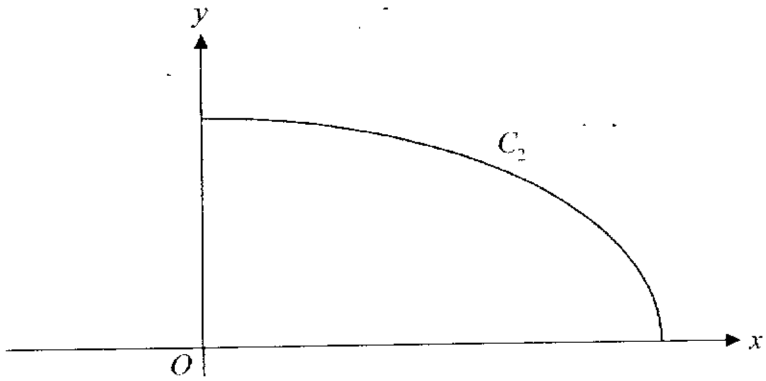
(ii) According to Company B, the decrease in area in each subsequent year is $\frac{5}{6}$ that of the previous year. Determine the theoretical area of Island S in the long run. **[2]**

[Turn Over]

6 A curve C_1 has equation $\dfrac{4-x^2y^2}{x^2+y^2}=\dfrac{1}{2}$, where $y \geq 0$.

(i) Show that $\dfrac{dy}{dx}=-\dfrac{x+2xy^2}{y+2x^2y}$. [3]

(ii) The point P on C_1 has x-coordinate 2. The tangent to C_1 at P cut the x-axis at point Q. Find the exact coordinates of Q. [4]



The diagram above shows a second curve C_2 with parametric equations

$$x = k^2 \sin\theta, \quad y = k\cos\theta,$$

where $0 \leq \theta \leq \dfrac{\pi}{2}$ and k is a positive constant.

(iii) The area bounded by the curve C_2, the x-axis and the y-axis is equal to the area of triangle OPQ, where O is the origin. Find the value of k correct to 2 decimal places. [3]

7 (a) It is given that $z = 1+\sqrt{3}i$ is a root of the equation $3z^3+az^2+bz-8=0$, where a and b are real numbers. Find the exact values of a and b and hence solve the equation completely, giving all the roots in exact form. [5]

(b) (i) Using Euler's formula that $e^{i\theta}=\cos\theta+i\sin\theta$, show that $(\cos\theta+i\sin\theta)^4=\cos 4\theta+i\sin 4\theta$. [1]

(ii) Using result shown in (i), by comparing the real parts, show that $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$.
Obtain an expression for $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.
Hence show that $\tan 4\theta=\dfrac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$. [4]

(iii) Hence, find the possible values of $\tan\theta$ given that $\tan 4\theta = 4$, leaving your answers correct to 3 significant figures. [3]

[Turn Over]

8 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, relative to the origin O. The points O, A and B are not collinear. The point P lies on AB between A and B such that $AP: PB = (1-\lambda):\lambda$, where $0 < \lambda < 1$, $\lambda \in \mathbb{R}$.

(i) Write down the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. **[1]**

It is given that OP bisects $\angle AOB$.

(ii) Show that $\lambda = \dfrac{b}{a+b}$, where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. **[3]**

The point Q also lies on AB between A and B, and is such that $AP = BQ$.

(iii) Find an expression for $\left(OQ^2 - OP^2\right)$ in terms of a and b. **[4]**

9 A scientist is investigating two models for the motion of a falling object of mass 1 kg. At time t seconds, the object has fallen a distance of x metres with velocity v m s^{-1}. The object in both models falls vertically from a building with an initial velocity of 0 m s^{-1}.

(i) In the first model, the motion of the object is modelled by the differential equation $\dfrac{d^2x}{dt^2} = 9.8 - 0.2\left(\dfrac{dx}{dt}\right)^2$. By using the substitution $v = \dfrac{dx}{dt}$, show that the differential equation can be written as $\dfrac{dv}{dt} = 9.8 - 0.2v^2$. Hence, show that $v = \dfrac{7\left(1 - e^{-\frac{14}{5}t}\right)}{1 + e^{-\frac{14}{5}t}}$. **[5]**

(ii) In the second model, the scientist considers that for any falling object, it experiences a downward gravitational acceleration of 9.8 m s^{-2} and an upward acceleration R due to air resistance, where $R \leq 9.8$. The upward acceleration is directly proportional to v, with a constant of proportionality $k > 0$.
The rate of change of v for the falling object is modelled as the difference between the gravitational acceleration and the upward acceleration due to air resistance.
Form a differential equation relating v to t and k, and solve this differential equation to obtain v as a function of t and k. **[3]**

The terminal velocity of an object is the value of v after a long time.

(iii) Given that the objects in both models achieve the same terminal velocity, find the value of k. **[2]**

(iv) Using the value of k in **(iii)**, justify which model predicts that the object reaches 80% of its terminal velocity earlier. **[3]**

[Turn Over]

10

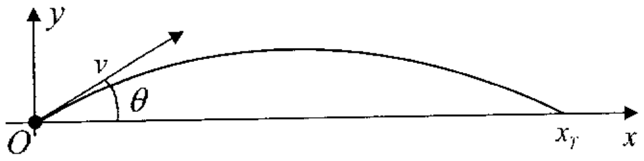


Figure 1

From a point O, a projectile is launched with a fixed velocity v ms^{-1} at a fixed angle of elevation θ from the horizontal, where v is a positive real constant and $0 < \theta < \frac{\pi}{2}$. The horizontal displacement, x metres, and the vertical displacement, y metres, of the projectile at time t seconds may be modelled by the parametric equations

$$x = (v\cos\theta)t, \qquad y = (v\sin\theta)t - \frac{1}{2}gt^2,$$

where g is a constant known as the acceleration due to gravity.

(i) Show that the time taken by the projectile to hit x_T is given by $\dfrac{2v\sin\theta}{g}$. **[2]**

(ii) A is the area enclosed by the path of the projectile as shown in **Figure 1** and the x-axis. **Without expressing y in terms of x,** show that $A = \dfrac{2v^4\sin^3\theta\cos\theta}{3g^2}$. **[4]**

(iii) Hence find the exact maximum value of A as θ varies in terms of v and g. **[6]**

End of Paper

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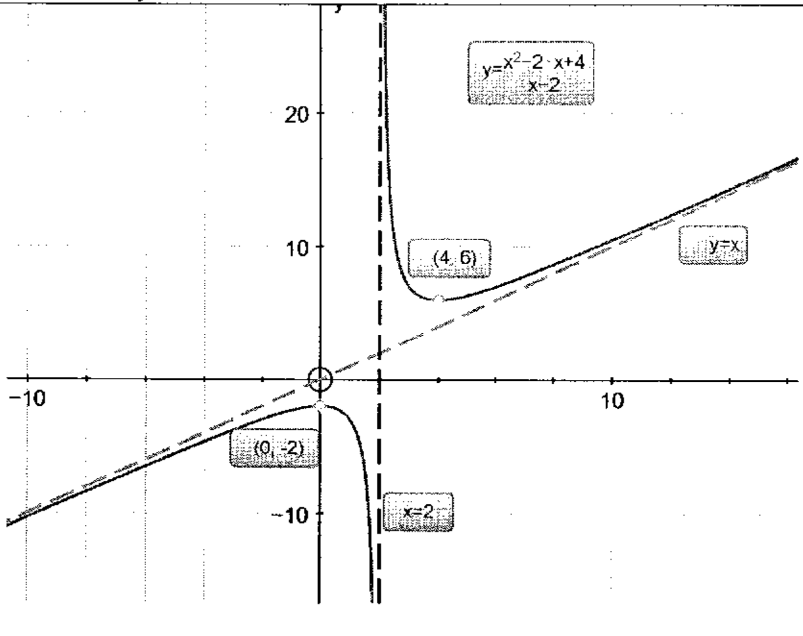
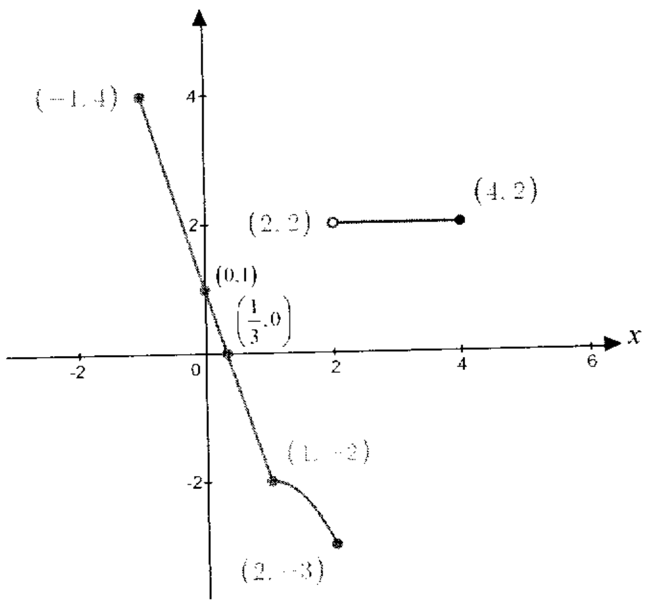
2021 H2 Math Prelim Exam Paper 1 Solutions

Qn	Suggested Solutions
1(i)	$y=\dfrac{ax^2-2ax+4}{x-b}$ Equation of vertical asymptote: $x-b=0 \Rightarrow x=b$ Thus $b=2$ $y=\dfrac{ax^2-2ax+4}{x-2}=ax+\dfrac{4}{x-2}$ Equation of horizontal asymptote: $y=ax$ Thus $a=1$
1(ii)	$y=\dfrac{x^2-2x+4}{x-2}$ $y(x-2)=x^2-2x+4$ $x^2+(-2-y)x+(4+2y)=0---(*)$ For y not able to take any values, the equation (*) will have no real roots, therefore discriminant < 0 $b^2-4ac<0$ $(-2-y)^2-4(1)(4+2y)<0$ $4+4y+y^2-16-8y<0$ $y^2-4y-12<0$ $(y-6)(y+2)<0$ $-2<y<6$ Therefore y cannot take values between -2 and 6.
1(iii)	

St Andrew's Junior College

2021 H2 Math Prelim Exam Paper 1 Solutions

<table>
<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td>2(a)</td><td>

Range of f = [−3, 4]</td></tr>
<tr><td>(b)</td><td>C': Reflect about the y-axis
B': Scale parallel to x-axis by scale factor of $\frac{1}{5}$
A': Translate 1 unit in the negative x-direction

$y = e^{x-2} - x$
$\downarrow C'$: Replace x by $-x$
$y = e^{-x-2} + x$
$y = x + e^{-x-2}$
$\downarrow B'$: Replace x by $5x$
$y = 5x + e^{-5x-2}$
$\downarrow A'$: Replace x by $x+1$
$y = 5(x+1) + e^{-5(x+1)-2}$
$y = 5x + 5 + e^{-5x-7}$</td></tr>
<tr><td>3(a)(i)</td><td>$u_n = S_n - S_{n-1}$
$= A(2^n) + Bn^2 + C - \left[A(2^{n-1}) + B(n-1)^2 + C\right]$
$= A\left(2^n - 2^{n-1}\right) + B\left(n^2 - (n-1)^2\right)$
$= A\left(2^{n-1}\right)(2-1) + B\left(2n-1\right)$
$= A\left(2^{n-1}\right) + B\left(2n-1\right)$</td></tr>
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<table>
<tr><th>Qn</th><th>Suggested Solutions</th></tr>
<tr><td>(a) (ii)</td><td>
<u>Method 1 (Form 3 equations to solve)</u>
$u_1 = A\left(2^{1-1}\right) + B\left(2(1)-1\right) = 7$
$u_2 = A\left(2^{2-1}\right) + B\left(2(2)-1\right) = 18$
$S_2 = A(2^2) + B(2)^2 + C = 25$ (or $S_1 = A\left(2^1\right) + B\left(1\right)^2 + C = 7$)

By GC, $A = 3, B = 4, C = -3$

Method 2 (Use 2 equations to find A and B first)
$u_1 = A\left(2^{1-1}\right) + B\left(2(1)-1\right) = 7$
$u_2 = A\left(2^{2-1}\right) + B\left(2(2)-1\right) = 18$
By GC, $A = 3, B = 4$
<table>
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$S_n = \sum_{r=1}^{n}\left[A\left(2^{r-1}\right) + B\left(2r-1\right)\right]$
$= A\sum_{r=1}^{n} 2^{r-1} + 2B\sum_{r=1}^{n} r - B\sum_{r=1}^{n} 1$
$= A\frac{1(2^n-1)}{2-1} + 2B\frac{n(n+1)}{2} - Bn$
$= A\left(2^n\right) - A + Bn^2$
$\therefore C = -A$
Therefore, $C = -3$
</td><td>
$S_2 = 3(2^2) + 4(2)^2 + C = 25$
(or $S_1 = A\left(2^1\right) + B\left(1\right)^2 + C = 7$)

Therefore, $C = -3$
</td></tr>
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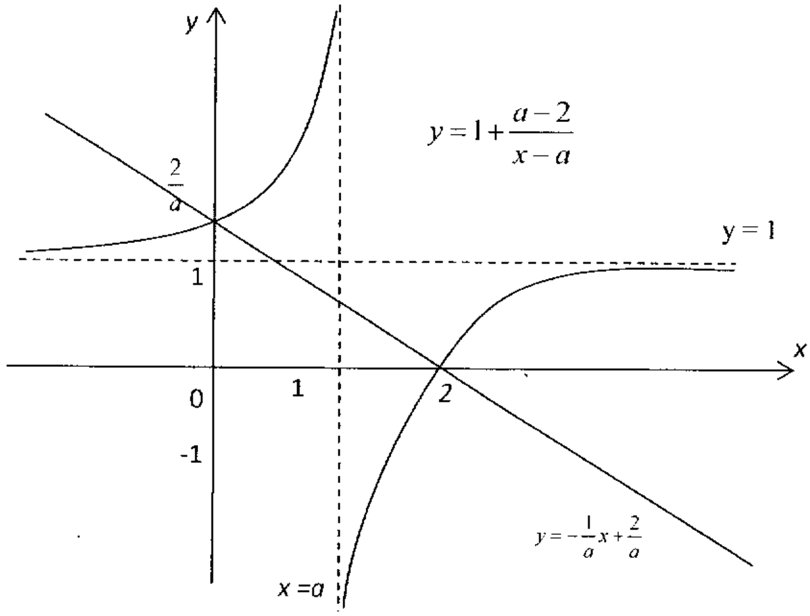
<tr><td>(b)</td><td>
$r(r+1)(r+2) - (r-2)(r-1)r$
$= r[(r+1)(r+2) - (r-2)(r-1)]$
$= r[r^2 + 3r + 2 - (r^2 - 3r + 2)]$
$= r[6r]$
$= 6r^2$
$\therefore k = 6$
</td></tr>
</table>

Qn	**Suggested Solutions**
	$\sum_{r=1}^{n} 6r^2 = \sum_{r=1}^{n}\left[r(r+1)(r+2)-(r-2)(r-1)r\right]$ $\sum_{r=1}^{n} r^2 = \frac{1}{6}\sum_{r=1}^{n}\left[r(r+1)(r+2)-(r-2)(r-1)r\right]$ $= \frac{1}{6}[(1)(2)(3) \quad -(-1)(0)(1)$ $+[(2)(3)(4) \quad -(0)(1)(2)$ $+[(3)(4)(5) \quad -(1)(2)(3)$ $+[(4)(5)(6) \quad -(2)(3)(4)$ $+$ $\vdots$ $+(n-3)(n-2)(n-1)-(n-5)(n-4)(n-3)$ $+(n-2)(n-1)(n) \quad -(n-4)(n-3)(n-2)$ $+(n-1)(n)(n+1) \quad -(n-3)(n-2)(n-1)$ $+(n)(n+1)(n+2) \quad -(n-2)(n-1)(n)]$ $= \frac{1}{6}\left[-(-1)(0)(1)-(0)(1)(2)+(n-1)(n)(n+1)+(n)(n+1)(n+2)\right]$ $= \frac{(n-1)(n)(n+1)+(n)(n+1)(n+2)}{6}$ $= \frac{n(n+1)\left[(n-1)+(n+2)\right]}{6}$ $= \frac{n(n+1)(2n+1)}{6}$ (Shown)
4 (i)	$y = 1+\frac{a-2}{x-a}, \quad x \neq a$ Asymptotes: $x=a, y=1$ When $x=0$, $y=\frac{2}{a}$. When $y=0, x=2$ $y = -\frac{1}{a}x+\frac{2}{a}$ When $x=0$, $y=\frac{2}{a}$ When $y=0, x=2$

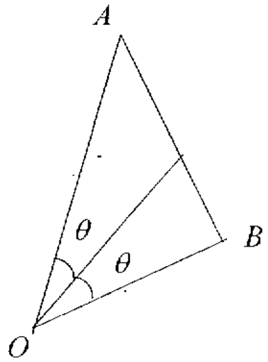
Qn	Suggested Solutions
	
(i)	From the graph, $1+\frac{a-2}{x-a}>-\frac{1}{a}x+\frac{2}{a}$. Ans: $0<x<a$ or $x>2$
(ii)	$1+\frac{ax-2x}{1-ax}>-\frac{1}{ax}+\frac{2}{a}$. $1+\frac{x(a-2)}{x(\frac{1}{x}-a)}>-\frac{1}{a}\left(\frac{1}{x}\right)+\frac{2}{a}$ $1+\frac{a-2}{\frac{1}{x}-a}>-\frac{1}{a}\left(\frac{1}{x}\right)+\frac{2}{a}$ Let $y=\frac{1}{x}$, $1+\frac{a-2}{y-a}>-\frac{1}{a}y+\frac{2}{a}$. From (i), $0<y<a$ or $y>2$ $0<\frac{1}{x}<a$ or $\frac{1}{x}>2$ $\therefore x>\frac{1}{a}$ or $0<x<\frac{1}{2}$

Qn	Suggested Solutions
5 (a)	$u_n = S_n - S_{n-1}$ $= (e^n - 1) - (e^{n-1} - 1)$ $= e^n - e^{n-1}$ $\dfrac{u_n}{u_{n-1}} = \dfrac{e^n - e^{n-1}}{e^{n-1} - e^{n-2}}$ $= \dfrac{e^{n-1}(e-1)}{e^{n-2}(e-1)}$ $= e$ that is a constant independent of n. Hence the sequence is a geometric progression.
(b)(i)	Let T_n be the decrease in area that has happened for the nth year. $T_n = 64 + (n-1)(-3)$ $= 67 - 3n$ The decrease in area of island S forms an AP with common difference – 3. Since the decrease in area stops when the decrease is less than 5 km^2, $67 - 3n < 5$ $n > 20\frac{2}{3}$ Least $n = 21$. Hence the total decrease in area from the first year to the 21st year is $S_{21} = \dfrac{21}{2}[2(64) + (21-1)(-3)]$ $= 714$ Hence the area of Island S when the decrease is less than 5 km^2 is $2880 - 714 = 2166 \text{km}^2$
(b)	According to Company B. the decrease in area now follows a Geometric Progression with first term 64 and common ratio $\dfrac{5}{6}$. Hence the long run/theoretical decrease in area of Island B is $S_\infty = \dfrac{64}{1 - \frac{5}{6}}$ $= 384$ Hence the theoretical area in Island B in the long run is Area $= 2880 - 384$ $= 2496 \text{km}^2$

Qn	Suggested Solutions
6 (i)	$\dfrac{4-x^2y^2}{x^2+y^2}=\dfrac{1}{2}$ $8-2x^2y^2=x^2+y^2$ Differentiating with respect to x, $-2\left(2xy^2+x^2\left(2y\dfrac{dy}{dx}\right)\right)=2x+2y\dfrac{dy}{dx}$ $-2xy^2-2x^2y\dfrac{dy}{dx}=x+y\dfrac{dy}{dx}$ $-\left(y+2x^2y\right)\dfrac{dy}{dx}=x+2xy^2$ $\dfrac{dy}{dx}=-\dfrac{x+2xy^2}{y+2x^2y}$ (shown)
(ii)	When $x=2$, $\dfrac{4-(2)^2y^2}{(2)^2+y^2}=\dfrac{1}{2}$ $8-8y^2=4+y^2$ $y^2=\dfrac{4}{9}$ $y=\dfrac{2}{3}$ $(y\geq 0)$ $\dfrac{dy}{dx}=-\dfrac{2+2(2)\left(\frac{2}{3}\right)^2}{\left(\frac{2}{3}\right)+2(2)^2\left(\frac{2}{3}\right)}=-\dfrac{17}{27}$ **Equation of PQ:** $y-\dfrac{2}{3}=-\dfrac{17}{27}(x-2)$ when $y=0, x=\dfrac{52}{17}$ $Q\left(\dfrac{52}{17},0\right)$

Qn	Suggested Solutions
(iii)	when $y=0$, $\theta=\frac{\pi}{2}, x=k^2$ $\frac{dx}{d\theta}=k^2\cos\theta$ Bounded area $=$ Area of triangle OPQ $\int_0^{k^2} y\,dx = \frac{1}{2}\left(\frac{52}{17}\right)\left(\frac{2}{3}\right)$ $\int_0^{\frac{\pi}{2}} k\cos\theta\frac{dx}{d\theta}\,d\theta = \frac{1}{2}\left(\frac{52}{17}\right)\left(\frac{2}{3}\right)$ $\int_0^{\frac{\pi}{2}} k^3\cos^2\theta\,d\theta = \frac{52}{51}$ $k^3 = \frac{52}{51\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta} = 1.298205$ $k=1.09$
7(a)	Let $z=1+\sqrt{3}i$ is a root of the equation $3z^3+az^2+bz-8=0$. **Since a and b are real numbers, the coefficients of all the terms of the equation are real, hence complex roots exist in conjugate pairs. This implies that since $z=1+\sqrt{3}i$ is a root then its conjugate $z^*=1-\sqrt{3}i$ is also a root of the equation.** We can form a quadratic factor $(z-(1+\sqrt{3}i)(z-(1-\sqrt{3}i)=z^2-2z+4$. $\therefore 3z^3+az^2+bz-8=(3z-2)(z^2-2z+4)$ Comparing the coefficient of $z^2: a=-6-2$ $\Rightarrow a=-8$ Comparing the coefficient of $z: b=12+4$ $\Rightarrow b=16$ The remaining root is $z=\frac{2}{3}$.
7(b) (i)	Since $e^{i\theta}=\cos\theta+i\sin\theta$, L.H.S. $=(\cos\theta+i\sin\theta)^4$ $=\left(e^{i\theta}\right)^4$ $=e^{i(4\theta)}$ $=\cos 4\theta+i\sin 4\theta$ (Shown)

Qn	Suggested Solutions
(ii)	$\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta}$ $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ $\cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 +$ $4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4 = \cos 4\theta + i(\sin 4\theta)$ Comparing real parts, $\cos 4\theta = \cos^4\theta - 6\cos^2\theta(\sin^2\theta) + (\sin^4\theta)$ Comparing imaginary parts, $\sin 4\theta = 4\cos^3\theta(\sin\theta) - 4\cos\theta(\sin^3\theta)$ $\therefore \tan 4\theta = \dfrac{4\cos^3\theta(\sin\theta) - 4\cos\theta(\sin^3\theta)}{\cos^4\theta - 6\cos^2\theta(\sin^2\theta) + (\sin^4\theta)}$ $= \dfrac{\cos^4\theta(4\tan\theta - 4\tan^3\theta)}{\cos^4\theta(1 - 6\tan^2\theta + \tan^4\theta)}$ $= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ (Shown)
(iii)	$\tan 4\theta = 4$ $\Rightarrow \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} = 4$ $\Rightarrow \tan\theta - \tan^3\theta = 1 - 6\tan^2\theta + \tan^4\theta$ $\Rightarrow \tan^4\theta + \tan^3\theta - 6\tan^2\theta - \tan\theta + 1 = 0 - - - (*)$ Using GC The possible values of $\tan\theta$ are $-2.91,\ 2.05,\ -0.488$ and 0.344
8 (i)	Since the point P lines on AB between A and B such that $AP : PB = (1-\lambda) : \lambda$, by ratio theorem, $\overrightarrow{OP} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB}$, $0 < \lambda < 1, \lambda \in \mathbb{R}$

Qn	Suggested Solutions
(ii)	 Let the $\angle AOP = \angle BOP = \theta$. $\cos\theta = \dfrac{\underset{\sim}{p}\bullet\underset{\sim}{a}}{ap} = \dfrac{\underset{\sim}{p}\bullet\underset{\sim}{b}}{bp}$ --- (*) $\Rightarrow \dfrac{\left[\lambda\underset{\sim}{a}+(1-\lambda)\underset{\sim}{b}\right]\bullet\underset{\sim}{a}}{a} = \dfrac{\left[\lambda\underset{\sim}{a}+(1-\lambda)\underset{\sim}{b}\right]\bullet\underset{\sim}{b}}{b}$ $\Rightarrow \dfrac{\lambda a^2+(1-\lambda)\left(\underset{\sim}{a}\bullet\underset{\sim}{b}\right)}{a} = \dfrac{\lambda\underset{\sim}{a}\bullet\underset{\sim}{b}+(1-\lambda)b^2}{b}$ $\Rightarrow \left(\underset{\sim}{a}\bullet\underset{\sim}{b}\right)\left[\dfrac{(1-\lambda)}{a}-\dfrac{\lambda}{b}\right] = (1-\lambda)b-\lambda a$ --- (#) **Method 1a** $\Rightarrow \left(\underset{\sim}{a}\bullet\underset{\sim}{b}\right)[b(1-\lambda)-\lambda a] = ab[(1-\lambda)b-\lambda a]$ $\Rightarrow \left(\underset{\sim}{a}\bullet\underset{\sim}{b}\right)[\lambda(a+b)-b] = ab[\lambda(a+b)-b]$ $\Rightarrow [\lambda(a+b)-b]\left(\underset{\sim}{a}\bullet\underset{\sim}{b}-ab\right) = 0$ Since O, A and B are not collinear, $\underset{\sim}{a}\bullet\underset{\sim}{b} \neq ab$ --- (@) $\therefore \lambda(a+b)-b=0$ $\Rightarrow \lambda = \dfrac{b}{a+b}$ (Shown) **Method 1b** $\Rightarrow \left(ab\cos 2\theta\right)\left[\dfrac{(1-\lambda)}{a}-\dfrac{\lambda}{b}\right] = (1-\lambda)b-\lambda a$ Since (#) is true for all θ, we compare terms independent of θ $\Rightarrow (1-\lambda)b-\lambda a = 0$ $\Rightarrow \lambda = \dfrac{b}{a+b}$ (Shown)
(iii)	Since $AP = BQ$

Qn	Suggested Solutions
	$\Rightarrow AQ:QB=\lambda:(1-\lambda)$ $\Rightarrow \overrightarrow{OQ}=(1-\lambda)\underset{\sim}{a}+\lambda\underset{\sim}{b}$ $OQ^2=\left((1-\lambda)\underset{\sim}{a}+\lambda\underset{\sim}{b}\right)\cdot\left((1-\lambda)\underset{\sim}{a}+\lambda\underset{\sim}{b}\right)$ $=(1-\lambda)^2a^2+\lambda^2b^2+2\lambda(1-\lambda)\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)$ $OP^2=\left(\lambda\underset{\sim}{a}+(1-\lambda)\underset{\sim}{b}\right)\cdot\left(\lambda\underset{\sim}{a}+(1-\lambda)\underset{\sim}{b}\right)$ $=\lambda^2a^2+(1-\lambda)^2b^2+2\lambda(1-\lambda)\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)$ $\therefore OQ^2-OP^2=(2\lambda-1)(b^2-a^2)$ $=\left(2\left(\frac{b}{a+b}\right)-1\right)(b^2-a^2)$ $=\left(\frac{b-a}{a+b}\right)(b-a)(b+a)$ $=(b-a)^2$
9(i)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=9.8-0.2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$ $v=\frac{\mathrm{d}x}{\mathrm{d}t}\Rightarrow\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ Hence, $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=9.8-0.2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\Rightarrow\frac{\mathrm{d}v}{\mathrm{d}t}=9.8-0.2v^2$ (shown) To solve $\frac{\mathrm{d}v}{\mathrm{d}t}=9.8-0.2v^2$: $\frac{\mathrm{d}v}{\mathrm{d}t}=9.8-0.2v^2$ $\int\frac{1}{9.8-0.2v^2}\,\mathrm{d}v=\int 1\,\mathrm{d}t$ $\frac{1}{0.2}\int\frac{1}{49-v^2}\,\mathrm{d}v=t+C$, where C is an arbitrary constant $\frac{1}{0.2}\left(\frac{1}{2(7)}\right)\ln\left\|\frac{7+v}{7-v}\right\|=t+C$ $\frac{5}{14}\ln\left\|\frac{7+v}{7-v}\right\|=t+C$ $\ln\left\|\frac{7+v}{7-v}\right\|=\frac{14}{5}t+D$, where $D=\frac{14}{5}C$

Qn	Suggested Solutions
	$\frac{7+v}{7-v} = Ae^{\frac{14}{5}t}$, where $A = \pm e^{D}$ When $t = 0, v = 0$ $\frac{7+0}{7-0} = Ae^{0} \Rightarrow A = 1$ $\therefore \frac{7+v}{7-v} = e^{\frac{14}{5}t} \Rightarrow \frac{7-v}{7+v} = e^{-\frac{14}{5}t}$ $7 - v = 7e^{-\frac{14}{5}t} + e^{-\frac{14}{5}t}v$ $7\left(1 - e^{-\frac{14}{5}t}\right) = v\left(1 + e^{-\frac{14}{5}t}\right)$ $\therefore v = \frac{7\left(1 - e^{-\frac{14}{5}t}\right)}{1 + e^{-\frac{14}{5}t}}$ (shown)
(ii)	$\frac{dv}{dt} = 9.8 - kv, \quad k > 0$ $\int \frac{1}{9.8 - kv}\, dv = \int 1\, dt$ $-\frac{1}{k}\int \frac{-k}{9.8 - kv}\, dv = t + C$, where C is an arbitrary constant $-\frac{1}{k}\ln\lvert 9.8 - kv\rvert = t + C$ $\ln\lvert 9.8 - kv\rvert = -kt + D$, where $D = -Ck$ $9.8 - kv = Ae^{-kt}$, where $A = \pm e^{D}$ When $t = 0, v = 0$ $9.8 - k(0) = Ae^{0} \Rightarrow A = 9.8$ $\therefore 9.8 - kv = 9.8e^{-kt}$ $v = \frac{9.8(1 - e^{-kt})}{k}$

Qn	Suggested Solutions
(iii)	$v=\dfrac{7\left(1-e^{-\frac{14}{5}t}\right)}{1+e^{-\frac{14}{5}t}}$ As $t\to\infty$, $e^{-\frac{14}{5}t}\to 0$ $\therefore \dfrac{1-e^{-\frac{14}{5}t}}{1+e^{-\frac{14}{5}t}}\to 1$ $\lim\limits_{t\to\infty}\dfrac{7\left(1-e^{-\frac{14}{5}t}\right)}{1+e^{-\frac{14}{5}t}}=7$ $\therefore$ Terminal velocity for object in Model 1 is 7 $\lim\limits_{t\to\infty}\dfrac{9.8(1-e^{-kt})}{k}=7$ As $t\to\infty$, $e^{-kt}\to 0$ $\therefore \dfrac{9.8}{k}=7\Rightarrow k=1.4$
(iv)	For model 1: When $v=7\times 0.8=5.6$ $5.6=\dfrac{7\left(1-e^{-\frac{14}{5}t}\right)}{1+e^{-\frac{14}{5}t}}$ Using GC, $t=0.785$ (3 s.f.) For model 2: When $v=7\times 0.8=5.6$ $5.6=\dfrac{9.8(1-e^{-(1.4)t})}{1.4}$ Using GC, $t=1.15$ s Object in Model 1 reaches 80% of its terminal velocity earlier.

Qn	Suggested Solutions
11(i)	When the projectile to hits x_T, $y=0$ $(v\sin\theta)t-\frac{1}{2}gt^2=0$ $t\left(v\sin\theta-\frac{1}{2}gt\right)=0$ $t=0$ (rejected as $t>0$) or $v\sin\theta-\frac{1}{2}gt=0$ $t=\frac{2v\sin\theta}{g}$ (shown)
(ii)	$A=\int_0^{x_T} y\,\mathrm{d}x$ $=\int_0^{\frac{2v\sin\theta}{g}} y\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)\mathrm{d}t$ $=\int_0^{\frac{2v\sin\theta}{g}}\left((v\sin\theta)t-\frac{1}{2}gt^2\right)(v\cos\theta)\,\mathrm{d}t$ $=(v\cos\theta)\int_0^{\frac{2v\sin\theta}{g}}\left((v\sin\theta)t-\frac{1}{2}gt^2\right)\mathrm{d}t$ $=(v\cos\theta)\left[(v\sin\theta)\frac{t^2}{2}-\frac{1}{2}g\left(\frac{t^3}{3}\right)\right]_0^{\frac{2v\sin\theta}{g}}$ $=(v\cos\theta)\left(\frac{(v\sin\theta)}{2}\left(\frac{2v\sin\theta}{g}\right)^2-\frac{1}{6}g\left(\frac{2v\sin\theta}{g}\right)^3\right)$ $=(v\cos\theta)\left(\frac{2(v\sin\theta)^3}{g^2}-\frac{1}{6g^2}(8)(v\sin\theta)^3\right)$ $=(v\cos\theta)\left(\frac{2}{3}\frac{(v\sin\theta)^3}{g^2}\right)$ $=\frac{2v^4\sin^3\theta\cos\theta}{3g^2}$
(iii)	$A=\frac{2v^4\sin^3\theta\cos\theta}{3g^2}$ $\frac{\mathrm{d}A}{\mathrm{d}\theta}=\frac{2v^4}{3g^2}\left(\sin^3\theta(-\sin\theta)+\cos\theta(3\sin^2\theta\cos\theta)\right)$ $=\frac{2v^4}{3g^2}\left(3\sin^2\theta\cos^2\theta-\sin^4\theta\right)$ For stationary values,

Qn	Suggested Solutions
	$\frac{\mathrm{d}A}{\mathrm{d}\theta}=0$ $\frac{2v^4}{3g^2}\left(3\sin^2\theta\cos^2\theta-\sin^4\theta\right)=0$ $\sin^2\theta\left(3\cos^2\theta-\sin^2\theta\right)=0$ $\sin^2\theta=0$ or $3\cos^2\theta-\sin^2\theta=0$ $\sin\theta=0$ or $\tan^2\theta=3$ $\theta=0$ or $\tan\theta=\pm\sqrt{3}$ (rejected $\theta>0$) $\tan\theta=\sqrt{3}$ or $\tan\theta=-\sqrt{3}$ $\theta=\frac{\pi}{3}$ (rejected as θ is acute, so $\tan\theta>0$) Hence when $\theta=\frac{\pi}{3}$, $A=\frac{2v^4\sin^3\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)}{3g^2}=\frac{2v^4\left(\left(\frac{\sqrt{3}}{2}\right)^3\right)\left(\frac{1}{2}\right)}{3g^2}=\frac{\sqrt{3}}{8}\frac{v^4}{g^2}$ $\frac{\mathrm{d}A}{\mathrm{d}\theta}=\frac{2v^4}{3g^2}\left(3\sin^2\theta\cos^2\theta-\sin^4\theta\right)$ $=\frac{2v^4}{3g^2}\left(\frac{3}{4}(4\sin^2\theta\cos^2\theta)-\sin^4\theta\right)$ $=\frac{2v^4}{3g^2}\left(\frac{3}{4}(\sin 2\theta)^2-\sin^4\theta\right)$ $\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=\frac{2v^4}{3g^2}\left(\frac{3}{4}\left(4\sin 2\theta\cos 2\theta\right)-4\sin^3\theta\cos\theta\right)$ $=\frac{2v^4}{3g^2}\left(\frac{3}{2}\left(\sin 4\theta\right)-4\sin^3\theta\cos\theta\right)$ When $\theta=\frac{\pi}{3}$

Qn	Suggested Solutions
	$\frac{\mathrm{d}^2A}{\mathrm{d}\theta^2}=\frac{2v^4}{3g^2}\left(\frac{3}{2}(\sin 4\theta)-4\sin^3\theta\cos\theta\right)$ $=\frac{2v^4}{3g^2}\left(\frac{3}{2}\left(-\frac{\sqrt{3}}{2}\right)-4\left(\frac{\sqrt{3}}{2}\right)^3\frac{1}{2}\right)$ $=\frac{2v^4}{3g^2}\left(-\frac{3\sqrt{3}}{4}-\frac{3\sqrt{3}}{4}\right)$ $=-\frac{2v^4}{3g^2}\left(\frac{3\sqrt{3}}{2}\right)$ $=-\frac{\sqrt{3}v^4}{g^2}$ Hence maximum $A=\frac{\sqrt{3}v^4}{8g^2}$

ST ANDREW'S JUNIOR COLLEGE		
PRELIMINARY EXAMINATION		
MATHEMATICS	**HIGHER 2**	**9758/02**
Tuesday	**14 September 2021**	**3 hrs**

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

NAME:______________________________ (____) **C.G.:** ________

TUTOR'S NAME: ______________________________

SCIENTIFIC / GRAPHIC CALCULATOR MODEL: ____________________

READ THESE INSTRUCTIONS FIRST

Write your name, civics group, index number and calculator models on the cover page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions. Total marks : **100**

Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

Question	1	2	3	4	5	6	7	8	9	10	11	TOTAL
Marks												
	9	11	10	10	7	7	8	9	5	12	12	100

This document consists of **7** printed pages including this page.

[Turn Over]

Section A: Pure Mathematics [40 marks]

1 A function f is said to self-inverse if $f(x) = f^{-1}(x)$ for all x in the domain of f.
The functions f and g are defined by

$$f : x \mapsto \frac{ax + 7a^2}{x - a}, \; x \in \mathbb{R}, x \neq a, \; 0 < a < 1$$
$$g : x \mapsto (x - 2)(x + 1), \; x \in \mathbb{R}, x > 3.$$

(i) Explain why f^{-1} exists and show that f is self-inverse. [4]

(ii) Hence, evaluate $f^{241}\left(\frac{1}{a}\right)$ in terms of a, showing your working clearly. [2]

(iii) Show that fg exists. Find the exact range of fg in terms of a. [3]

2 (a) By using the substitution $x = \frac{1}{2}\sin u$, find the exact area of the region enclosed by the curve $y = \sqrt{1 - 4x^2}$ and the x-axis. [4]

(b) The region A is bounded by the curve C with equation $y = e^x \sin x$, the x-axis and the lines $x = -\pi$ and $x = \frac{\pi}{2}$.

(i) Find the exact area of A. [5]

(ii) A is now rotated about the x-axis through 2π radians. Find the volume of solid formed. [2]

[Turn Over]

3 The line L has equation $\mathbf{r} = -\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + \lambda(6\mathbf{i} + 3\mathbf{j} - 2\mathbf{k})$, where $\lambda \in \mathbb{R}$.

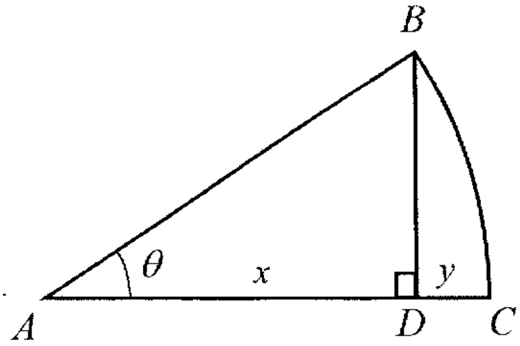
(i) Find the acute angle between L and the x-axis. **[2]**

The point P has position vector $6\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$.

(ii) Find the points on L which are at a distance of $\sqrt{59}$ units from P.
Hence or otherwise find the point on L which is closest to P. **[5]**

(iii) Find a cartesian equation of the plane that includes the line L and the point P. **[3]**

4



ABC is a sector of a circle with radius AB and AC and $\measuredangle BAC = \theta$. D is a point on AC such that $AD = x$ m, $CD = y$ m and $\measuredangle ADB = \frac{\pi}{2}$.

(i) Show that $BD = \sqrt{y^2 + 2xy}$. **[1]**

(ii) Hence show that $\sin\theta \approx (2\alpha)^{\frac{1}{2}}\left(1 - \frac{3}{4}\alpha\right)$, where $\alpha = \frac{y}{x}$, and y is small compared to x. **[4]**

You are now given that $\theta = 10^0$ and $x = 100$.

(iii) Use part **(ii)** to find estimates for α and y correct to 2 decimal places. **[3]**

(iv) Find the actual value of y. **[1]**

(v) Comparing the values of y obtained in parts **(iii)** and **(iv)**, comment on the accuracy of your approximations and explain your answer. **[1]**

[Turn Over]

Section B: Statistics [60 marks]

5 The events X, Y and Z are such that $\mathrm{P}(X) = x$, $\mathrm{P}(Y) = y$, and $\mathrm{P}(Z) = \frac{1}{4}$, where x and y are non-zero. It is given that the events X and Y are independent, events Y and Z are independent, and that events X and Z are independent.

(i) Find an expression for $\mathrm{P}(X \cap Z')$. **[2]**

(ii) Find an expression for $\mathrm{P}(Y' | X')$. **[2]**

It is now given that $\mathrm{P}((X \cup Y \cup Z)') = \frac{2}{5}$, $\mathrm{P}(X \cap Y \cap Z) = 0$, and $y = 5x$

(iii) By drawing a suitable Venn Diagram, find the value of x. **[3]**

6 Shane plays a game by tossing three biased coins. If a coin shows a head, Shane scores 1 point; if a coin shows a tail, he scores -1 point.
The random variable X is the total score obtained when three such coins are tossed.
It is given that the probability of obtaining a head when a biased coin is tossed is p, where $0 < p < 1$.

(i) Tabulate the probability distribution of X. **[3]**

(ii) Find E(X) in terms of p.
Hence show that $\left(\frac{X+3}{6}\right)$ is an unbiased estimator of p. **[4]**

7 A class committee consists of 6 boys and 3 girls. They are to sit at random around a table for a meeting.

(i) Find the number of possible arrangements if not all 3 girls are seated together. **[3]**

(ii) Find the number of possible arrangements with all 3 girls separated. **[2]**

(iii) Find the probability that there are exactly two boys seated between any two girls. **[3]**

[Turn Over]

8 A company that sells potato chips in packets claims that the mean amount of sodium content per packet is 798 mg.

(a) It is given that the standard deviation of sodium content in the packets of potato chips is 6.5 mg. Upon receiving complaints that the potato chips were too salty, the local food agency chooses a large random sample of n packets of potato chips and found that the mean sodium content is 799.5 mg.
Given that the local food agency concludes at the 5% level of significance that the company has understated the amount of sodium content, find the set of values of n. **[4]**

(b) The company decided to try out a new healthier recipe that reduces the amount of sodium content per packet. The company collected data from a random sample of 50 packets and found that the mean is 796.3 mg and standard deviation is 6.2 mg.

(i) Test at 5% level of significance whether the mean amount of sodium in a packet of potato chips has been reduced from the original claim, after the introduction of the new healthier recipe. **[4]**

(ii) Explain, in the context of the question, the meaning of "at 5% level of significance" for the test in **(b)(i)**. **[1]**

9 **(i)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A), (B) and (C) below. In each of the case, your diagram should include 6 points, approximately equally spaced with respect to x, and with all x- and y-values positive. The letters a, b, c, d, e and f represent constants.

(A) $y = a + bx^2$, where a is positive and b is negative.

(B) $y = c + d \ln x$, where c is positive and d is negative.

[2]

A motoring website gives the following information about the distance travelled, y km, by a certain type of car at different speeds, x km h^{-1}, on a fixed amount of fuel.

Speed, x	88	96	104	112	120	128
Distance, y	144	147	144	138	126	107

(ii) Draw a scatter diagram for these values, labelling the axes. **[1]**

(iii) Explain which of the three cases in part **(i)** is the most appropriate for modelling these values, and calculate the product moment correlation coefficient for this case. **[2]**

[Turn Over]

10 During a pandemic, an unknown infectious disease spread through a population. Researchers wanted to study the probability, p, that a person is infected. They collected some data from a random sample of 25 people.

(i) State, in the context of this question, two assumptions needed for the number of infected people to be well modelled by a binomial distribution. **[2]**

Assume now that these assumptions do in fact hold.

(ii) It is given that the probability that at most 2 people are infected is 0.0982. Write an equation in p and solve for p. **[2]**

A total of 30 random samples of 25 people were chosen from this population.

(iii) Find the probability that the total number of samples with at most 2 infected people is at least 5. **[3]**

It is now given that $p = 0.10$.

A diagnostic test for the presence of the disease was carried out on the population. It was found that the percentage of infected people who were tested positive is 85%, while the percentage of non-infected people who were tested negative is 96%.

(iv) Find the probability that a person who tested positive carries the disease. **[3]**

(v) Hence or otherwise, find the probability that a person who tested positive does not carry the disease. **[1]**

(vi) Discuss briefly if the diagnostic test is worthwhile. **[1]**

11(a) Anne, a Bubble Tea (BBT) seller, intends to increase the sales of BBT using a drink vending machine which delivers BBT into a cup when cash payment is made into the machine. The volume of BBT dispensed is normally distributed with mean 210 ml and standard deviation 5 ml. The capacity of a cup is 220 ml and the nominal amount of BBT in a cup is stated as 212 ml.

(i) Find the probability that a cup overflows when BBT is dispensed into the cup. **[1]**

(ii) A customer bought five cups of BBT from the vending machine. Find the probability that at most one cup of BBT will overflow. **[2]**

(iii) Anne received complaints from some customers that there is a high proportion of cups with less than the nominal amount of BBT. It is assumed that the standard deviation of the volume of BBT dispensed is fixed while the mean volume of BBT dispensed could be adjusted. Find the range of the mean volume of BBT dispensed such that not more than 10% of the cups will contain less than the nominal amount of BBT. **[3]**

(iv) Another customer bought n cups of BBT. Find the approximate probability that the total volume of BBT dispensed exceeds $211n$ ml as n becomes very large. **[2]**

(v) Anne wishes to gather feedback about her BBT. She decided to interview 50 customers who bought BBT from the vending machine during lunch time. Give a reason as to whether Anne would obtain a random sample of customers. **[1]**

[Turn Over]

(b) On another occasion, Anne deployed a staff to operate a BBT counter at a wedding reception. The average volume of BBT per serving is 200 ml and the standard deviation of the volume of BBT is given to be 10 ml. Find the probability that the mean volume of 60 servings of BBT prepared by the staff is less than 198 ml.

[3]

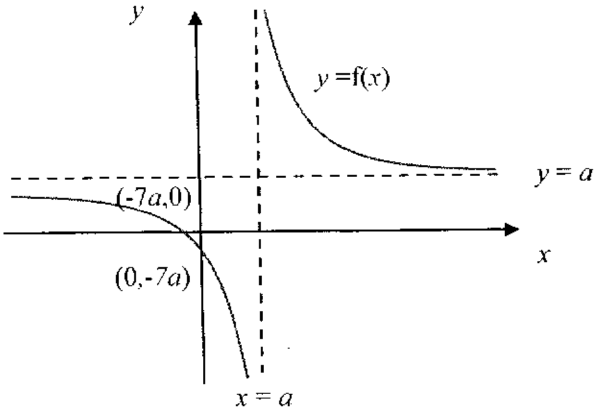
End of Paper

2021 H2 Math Paper 2 Preliminary Exam Solutions

Qn | **Solutions**

1(i)

$$y=\frac{ax+7a^2}{x-a}=a+\frac{8a^2}{x-a}$$



Since **every/all** horizontal line $y=k, k\in\mathbb{R}\setminus\{a\}$, intersects the graph of $y=\mathrm{f}(x)$ **exactly once**, the function f is **one-one**. Hence f^{-1} exists.

Let $y=\dfrac{ax+7a^2}{x-a}$

$y(x-a)=ax+7a^2$

$x(y-a)=ay+7a^2$

$x=\dfrac{ay+7a^2}{y-a}$

Since $\mathrm{f}^{-1}(x)=\dfrac{ax+7a^2}{x-a},\ x\in\mathbb{R}, x\neq a,$

$D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=(-\infty,a)\cup(a,\infty)=D_{\mathrm{f}}$

$\therefore \mathrm{f}^{-1}=\mathrm{f}$ and f is self-inverse (shown)

(ii)

Note that $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$..

$$\mathrm{f}^{241}\left(\frac{1}{a}\right)=\underbrace{\mathrm{fff...f}}_{241\text{ times}}\left(\frac{1}{a}\right)=\mathrm{f}\left(\underbrace{\mathrm{fff...f}}_{240\text{ times}}\left(\frac{1}{a}\right)\right)=\mathrm{f}\left(\underbrace{\mathrm{ff}^{-1}\mathrm{ff}^{-1}...\mathrm{ff}^{-1}}_{120\text{ times of ff}^{-1}}\left(\frac{1}{a}\right)\right)=\mathrm{f}\left(\frac{1}{a}\right)$$

$$\mathrm{f}\left(\frac{1}{a}\right)=\frac{a\left(\frac{1}{a}\right)+7a^2}{\frac{1}{a}-a}=\frac{a+7a^3}{1-a^2}$$

Alternatively:

$\mathrm{f}(x)=\mathrm{f}^{-1}(x)\Rightarrow \mathrm{f}^2(x)=x$

$\mathrm{f}^3(x)=\mathrm{ff}^2(x)=\mathrm{f}(x)$

$\mathrm{f}^4(x)=\mathrm{ff}^3(x)=\mathrm{ff}(x)=x$

$\vdots$

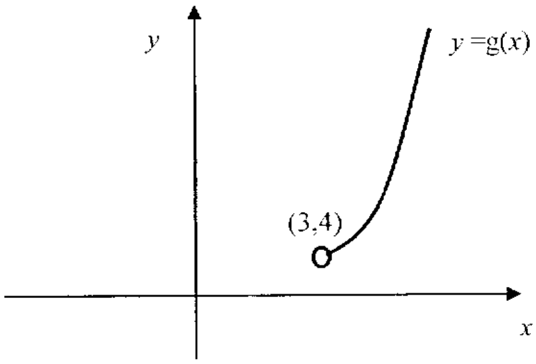
$\mathrm{f}^n(x)=\mathrm{f}(x)$ when n is an odd integer

$\mathrm{f}^n(x)=x$ when n is an even integer

$\therefore \mathrm{f}^{241}\left(\frac{1}{a}\right)=\mathrm{f}\left(\frac{1}{a}\right)$

$$=\frac{a\left(\frac{1}{a}\right)+7a^2}{\frac{1}{a}-a}=\frac{a+7a^3}{1-a^2}$$

(iii)



Since $R_g=(4,\infty)$ and $D_f=(-\infty,\infty)\setminus\{a\}$, where $0<a<1$.

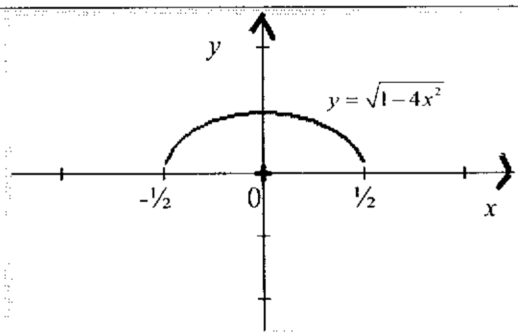
$R_g\subseteq D_f$.

$\therefore$ fg exists.

$$D_g=(3,\infty)\rightarrow(4,\infty)\rightarrow\left(a,\frac{4a+7a^2}{4-a}\right)$$

$$R_{fg}=\left(a,\frac{4a+7a^2}{4-a}\right)$$

2(a)



x-intercepts at $x=\frac{1}{2}$ and $x=-\frac{1}{2}$

$x=-\frac{1}{2},$

$\therefore \frac{1}{2}\sin u=-\frac{1}{2}\Rightarrow u=-\frac{\pi}{2}$

$x=\frac{1}{2},$

$\therefore \frac{1}{2}\sin u=\frac{1}{2}\Rightarrow u=\frac{\pi}{2}$

$x=\frac{1}{2}\sin u$

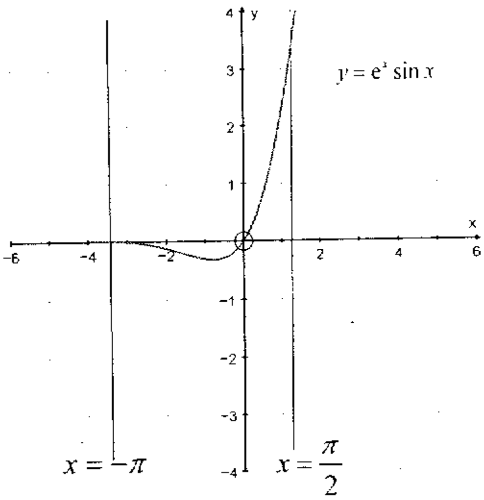
$\frac{\mathrm{d}x}{\mathrm{d}u}=-\frac{1}{2}\cos u$

Area

$=\int_{-\frac{1}{2}}^{\frac{1}{2}}\sqrt{1-4x^2}\,\mathrm{d}x$

$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{1-4\left(\frac{1}{2}\sin u\right)^2}\times\left(\frac{1}{2}\cos u\right)\,\mathrm{d}u$

$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos u\times\cos u\,\mathrm{d}u$

	$=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos u)^2\,du$ $=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2u}{2}\,du$ $=\frac{1}{2}\left[\frac{u}{2}+\frac{\sin 2u}{4}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ $=\frac{1}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\frac{\pi}{4}$ units^2
(bi)	 Area $=\int_0^{\frac{\pi}{2}}e^x\sin x\,dx+\int_{-\pi}^{0}\left(-e^x\sin x\right)\,dx$ ***Note:*** $\int e^x\sin x\,dx=e^x\sin x-\int e^x(\cos x)\,dx$ $=e^x\sin x-\left[e^x\cos x-\int e^x(-\sin x)\,dx\right]$ $2\int e^x\sin x\,dx=e^x\sin x-e^x\cos x$ $\int e^x\sin x\,dx=\frac{1}{2}\left(e^x\sin x-e^x\cos x\right)+c$ where c is an arbitrary constant. **Workings for integration by parts** Let $u_1=\sin x$ $\quad\frac{dv_1}{dx}=e^x$ $\frac{du_1}{dx}=\cos x$ $\quad v_1=e^x$

	Let $u = \cos x$ $\quad \frac{dv}{dx} = e^x$ $\frac{du}{dx} = -\sin x$ $\quad v = e^x$ Thus area of region $= \left[\frac{1}{2}\left(e^x \sin x - e^x \cos x\right)\right]_0^{\frac{\pi}{2}} + \left[-\frac{1}{2}\left(e^x \sin x - e^x \cos x\right)\right]_{-\pi}^{0}$ $= \frac{1}{2}\left[\left(e^{\frac{\pi}{2}} \sin\frac{\pi}{2}\right) - \left(-e^0 \cos 0\right)\right]$ $+\left(-\frac{1}{2}\right)\left[\left(-e^0 \cos 0\right) - \left(-e^{-\pi} \cos(-\pi)\right)\right]$ $= \frac{1}{2}\left[e^{\frac{\pi}{2}} + 1\right] - \frac{1}{2}\left(-1 - e^{-\pi}\right)$ $= \left(\frac{1}{2}e^{\frac{\pi}{2}} + \frac{1}{2}e^{-\pi} + 1\right)$ units^2
(bii)	Volume $= \pi \int_{-\pi}^{\frac{\pi}{2}} \left(e^x \sin x\right)^2 \, dx$ $= 8.6775\pi$ ≈ 27.3 units^3
3(i)	Let the acute angle between L and the x-axis be θ . $\cos\theta = \left(\left\lvert \frac{\begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}{\sqrt{49}\sqrt{1}} \right\rvert\right)$ $\Rightarrow \theta = \cos^{-1}\left(\frac{6}{7}\right)$ $= 31.0^0$.
(ii)	Equation of the line L is $\mathbf{r} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{R}$. Let B be a point on the line L.

Then $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

$$\overrightarrow{PB} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \\ -5 \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -7+6\lambda \\ -1+3\lambda \\ 1-2\lambda \end{pmatrix}$$

$$\left|\overrightarrow{PB}\right| = \sqrt{(-7+6\lambda)^2 + (-1+3\lambda)^2 + (1-2\lambda)^2}$$

$$= \sqrt{49\lambda^2 - 94\lambda + 51} \; ....(1)$$

Given $\left|\overrightarrow{PB}\right| = \sqrt{59}$

$\sqrt{49\lambda^2 - 94\lambda + 51} = \sqrt{59}$

Square both sides and solve for λ,

$49\lambda^2 - 94\lambda + 51 - 59 = 0$

$49\lambda^2 - 94\lambda - 8 = 0$

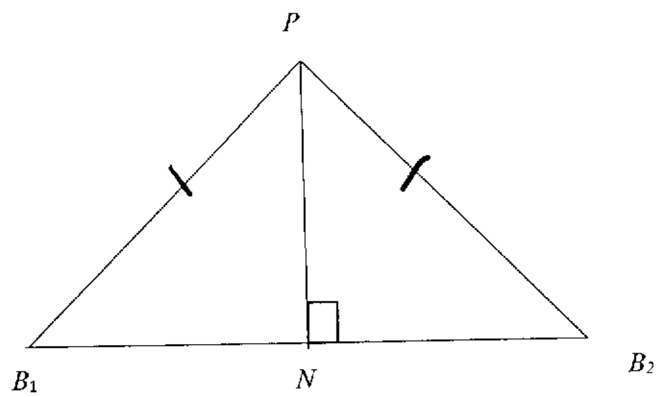
$(49\lambda + 4)(\lambda - 2) = 0$

$\lambda = -\dfrac{4}{49}$ or 2

$\overrightarrow{OB_1} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} - \dfrac{4}{49} \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = \dfrac{1}{49} \begin{pmatrix} -73 \\ 86 \\ -188 \end{pmatrix}$ or $\overrightarrow{OB_2} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 6 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \\ -8 \end{pmatrix}$

The points are (11, 8, −8) or $\left(-\dfrac{73}{49}, \dfrac{86}{49}, -\dfrac{188}{49} \right)$.

Let the point on L closest to P be N.
Since $PB_1 = PB_2 \Rightarrow PB_1B_2$ is an isosceles triangle
Hence the perpendicular PN will bisect the base B_1B_2.
$\therefore$ N is the midpoint of B_1 and B_2.



$$\overrightarrow{ON} = \frac{1}{2}\left[\begin{pmatrix}11\\8\\-8\end{pmatrix} + \frac{1}{49}\begin{pmatrix}-73\\86\\-188\end{pmatrix}\right] = \frac{1}{98}\begin{pmatrix}466\\478\\-580\end{pmatrix} = \frac{1}{49}\begin{pmatrix}233\\239\\-290\end{pmatrix}$$

Hence, the point closest to P is $\left(\frac{233}{49}, \frac{239}{49}, -\frac{290}{49}\right)$.

<u>Alternative Method 1(Hence)</u>

$$\left|\overrightarrow{PB}\right|^2 = 49\lambda^2 - 94\lambda + 51$$

Differentiate with respect to λ, $\frac{d\left|\overrightarrow{PB}\right|^2}{d\lambda} = 98\lambda - 94$

Minimum distance means $\frac{d\left|\overrightarrow{PB}\right|^2}{d\lambda} = 0$

$98\lambda - 94 = 0$

$\lambda = \frac{47}{49}$

$$\overrightarrow{ON} = \begin{pmatrix}-1\\2\\-4\end{pmatrix} + \frac{47}{49}\begin{pmatrix}6\\3\\-2\end{pmatrix} = \frac{1}{49}\begin{pmatrix}233\\239\\-290\end{pmatrix}$$

Hence, the point closest to P is $\left(\frac{233}{49}, \frac{239}{49}, -\frac{290}{49}\right)$.

<u>Alternative Method 2 (Hence)</u>

$$\left|\overrightarrow{PB}\right|^2 = 49\lambda^2 - 94\lambda + 51 = 49\left(\lambda - \frac{47}{49}\right)^2 + \frac{290}{49}$$

Hence minimum distance is obtained when $\lambda = \frac{47}{49}$.

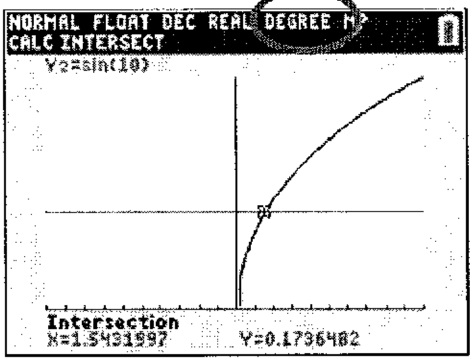
$$\overrightarrow{ON} = \begin{pmatrix}-1\\2\\-4\end{pmatrix} + \frac{47}{49}\begin{pmatrix}6\\3\\-2\end{pmatrix} = \frac{1}{49}\begin{pmatrix}233\\239\\-290\end{pmatrix}$$

Hence, the point closest to P is $\left(\frac{233}{49}, \frac{239}{49}, -\frac{290}{49}\right)$.

<u>Alternative Method 3 (Otherwise)</u>

Let N be the foot of perpendicular from the point P to the line L.

	Then $\overrightarrow{ON}=\begin{pmatrix}-1\\2\\-4\end{pmatrix}+\lambda\begin{pmatrix}6\\3\\-2\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{PN}=\begin{pmatrix}-1\\2\\-4\end{pmatrix}+\lambda\begin{pmatrix}6\\3\\-2\end{pmatrix}-\begin{pmatrix}6\\3\\-5\end{pmatrix}=\begin{pmatrix}-7\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}6\\3\\-2\end{pmatrix}=\begin{pmatrix}-7+6\lambda\\-1+3\lambda\\1-2\lambda\end{pmatrix}$ Since $\overrightarrow{PN}$ is perpendicular to line L, $\overrightarrow{PN}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=0$ $\begin{pmatrix}-7+6\lambda\\-1+3\lambda\\1-2\lambda\end{pmatrix}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=0$ $\lambda=\frac{47}{49}$ $\overrightarrow{ON}=\begin{pmatrix}-1\\2\\-4\end{pmatrix}+\frac{47}{49}\begin{pmatrix}6\\3\\-2\end{pmatrix}=\frac{1}{49}\begin{pmatrix}233\\239\\-290\end{pmatrix}$ Hence, the point closest to P is $\left(\frac{233}{49},\frac{239}{49},-\frac{290}{49}\right)$.
(iii)	$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\begin{pmatrix}6\\3\\-5\end{pmatrix}-\begin{pmatrix}-1\\2\\-4\end{pmatrix}=\begin{pmatrix}7\\1\\-1\end{pmatrix}$ A normal to the new plane is given by $\begin{pmatrix}7\\1\\-1\end{pmatrix}\times\begin{pmatrix}6\\3\\-2\end{pmatrix}=\begin{pmatrix}1\\8\\15\end{pmatrix}$ $\mathbf{r}.\begin{pmatrix}1\\8\\15\end{pmatrix}=\begin{pmatrix}-1\\2\\-4\end{pmatrix}\begin{pmatrix}1\\8\\15\end{pmatrix}=-45$ Cartesian equation: $x+8y+15z=-45$
4(i)	Since AB and AC are radii of the circle, $AB=AC$ $\therefore AB=x+y$

	By Pythagoras's Theorem, $BD=\sqrt{AB^2-AD^2}$ $BD=\sqrt{\left(x+y^2\right)-x^2}$ $=\sqrt{y^2+2xy}$ (Shown)
(ii)	$\sin\theta=\frac{BD}{AB}$ $=\frac{\sqrt{y^2+2xy}}{x+y}$ $=\frac{\sqrt{2xy}\sqrt{1+\frac{1}{2}\frac{y}{x}}}{x\left(1+\frac{y}{x}\right)}$ $=\sqrt{2\alpha}\left(1+\frac{1}{2}\alpha\right)^{\frac{1}{2}}\left(1+\alpha\right)^{-1}$ $=\sqrt{2\alpha}\left(1+\frac{1}{4}\alpha+...\right)\left(1-\alpha+...\right)$, since α is small $\approx\sqrt{2\alpha}\left(1-\frac{3}{4}\alpha\right)$ (Shown)
(iii)	$\sin 10°=\sqrt{2\alpha}\left(1-\frac{3}{4}\alpha\right)$ (1) using GC, Draw $y=\sin 10°$ and $y=\sqrt{\frac{2x}{100}}\left(1-\frac{3x}{400}\right)$  $\therefore y=1.54$ (2dp) Since $\alpha=\frac{y}{100}\Rightarrow\alpha=0.02$ (2dp) **Method 2**

	Let $\sqrt{\alpha} = u$ $\Rightarrow \sin 10^\circ = \sqrt{2}u\left(1-\frac{3}{4}u^2\right)$ $\Rightarrow \frac{3\sqrt{2}}{4}u^3 - \sqrt{2}u + \sin 10^\circ = 0$ Using GC, $u = -1.2117907(rej), 1.0875651(rej), 0.12422559$ $\Rightarrow \alpha = 0.015432 = 0.02$ (2dp) $\Rightarrow y = 1.54$ (2dp) **Note to students: The solution closer to the y-axis is chosen because α is small.*
(iv)	The actual value of y using trigonometric methods: $AB = \frac{100}{\cos 10^\circ}$ $y = \frac{100}{\cos 10^\circ} - 100 = 1.54$ (2 d.p.)
(v)	The estimate of y is accurate up to 2 decimal places. The estimate using equation (1) is accurate because equation (1) holds only when α very close to zero (or when y is small compared to x. This is true as we found that $\alpha = 0.0154$ in part (iii)
5(i)	$\mathrm{P}(X \cap Z') = \mathrm{P}(X) - \mathrm{P}(X \cap Z)$ $= x - \frac{1}{4}x$ $= \frac{3}{4}x$ Alternatively, Since X and Z are independent, then X and Z' are independent. $\mathrm{P}(X \cap Z') = \mathrm{P}(X)\mathrm{P}(Z')$ $= x\left(1-\frac{1}{4}\right)$ $= \frac{3}{4}x$
(ii)	Since Y and X are independent, then Y' and X' are independent (*)

$$P(Y'|X') = \frac{P(Y' \cap X')}{P(X')}$$

$$= \frac{P(Y')P(X')}{P(X')}$$

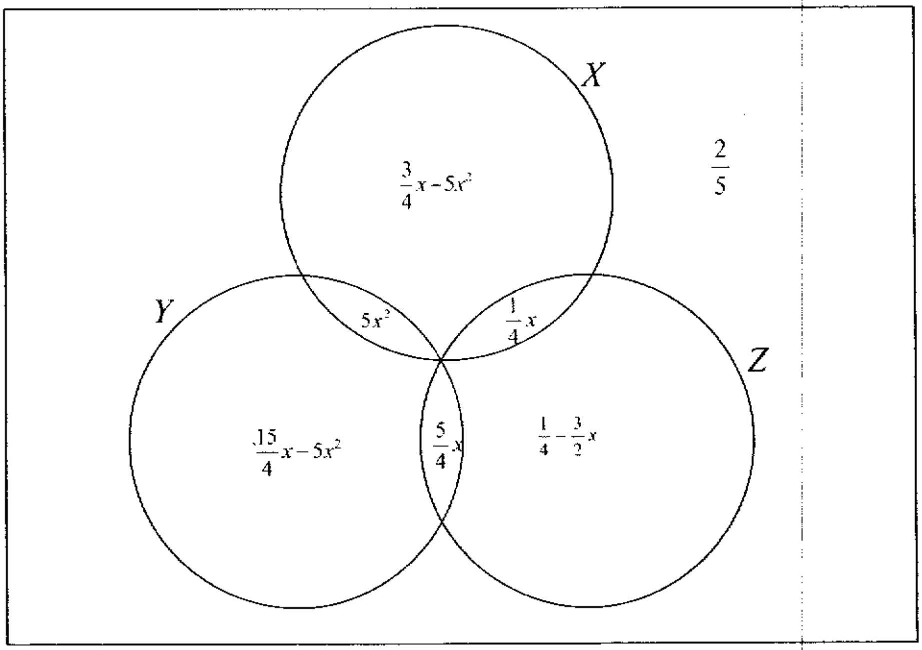
$$= \frac{(1-y)(1-x)}{1-x}$$

$$= 1 - y$$

Alternatively, due to (*),

$P(Y'|X') = P(Y') = 1 - y$

(iii)



Sum of probabilities = 1

$$5x + x + \frac{1}{4} - 5x^2 - \frac{5}{4}x - \frac{1}{4}x + \frac{2}{5} = 1$$

$$-5x^2 + \frac{9}{2}x - \frac{7}{20} = 0$$

$x = 0.086$ or $x = 0.814$

However, since P(Z) = 0.25 and $P((X \cup Y \cup Z)') = \frac{2}{5}$,

Then $x \leq 0.65$ (reason for rejecting $x = 0.814$)

$\therefore x = 0.086.$

6(i) Possible outcomes:

Cases	Outcomes	Total Score	Probability
1	HHH	3	p^3
2	TTT	−3	$(1-p)^3$

3	HTT	-1	$3p(1-p)^2$
4	HHT	1	$3p^2(1-p)$

x	-3	-1	1	3
$\mathrm{P}(X=x)$	$(1-p)^3$	$3p(1-p)^2$	$3p^2(1-p)$	p^3

(ii)

$$\mathrm{E}(X)=\sum_{all\ x} xP(X=x)$$

$$=(-3)(1-p)^3+(-1)\left(3p(1-p)^2\right)+(1)\left(3p^2(1-p)\right)+(3)\,p^3$$

$$=-3(1-p)\left[(1-p)^2+p(1-p)-p^2\right]+3p^3$$

$$=-3(1-p)\left[1-2p+p^2+p-p^2-p^2\right]+3p^3$$

$$=-3(1-p)\left[1-p-p^2\right]+3p^3$$

$$=-3\left[1-p-p^2-p+p^2+p^3\right]+3p^3$$

$$=-3(1-2p)$$

$$=6p-3$$

$$\mathrm{E}\left(\frac{X+3}{6}\right)=\frac{1}{6}\mathrm{E}(X+3)=\frac{1}{6}\mathrm{E}(X)+\frac{1}{6}\mathrm{E}(3)=\frac{1}{6}\mathrm{E}(X)+\frac{1}{2}$$

$$=\frac{1}{6}(6p-3)+\frac{1}{2}$$

$$=p$$

$\therefore \dfrac{X+3}{6}$ is an unbiased estimator of p. (Shown)

7(i)

Number of ways

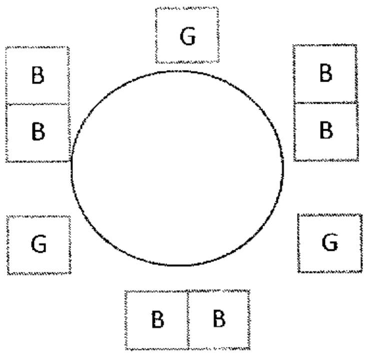
= Total number of ways without restriction - number of ways all 3 girls together

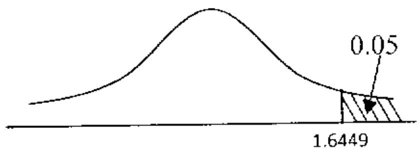
$$=\underbrace{(9-1)!}_{\text{9 people around a circle}}-\left[\underbrace{(7-1)!}_{\text{6 boys and 1 block of 3 girls}}\times\underbrace{3!}_{\text{3 girls}}\right]$$

$=36000$

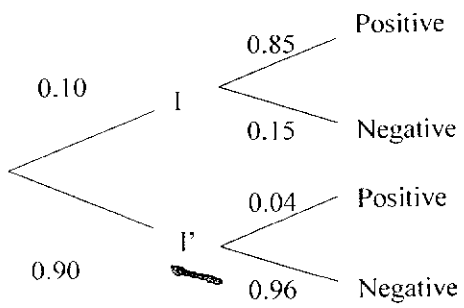
Alternative method

Number of ways

	= number of ways 2 girls together, 1 separated + number of ways all 3 girls separated $=\left(\underbrace{(5-1)!}_{\text{6 boys around a circle}}\times\underbrace{{}^{3}C_{2}}_{\text{3 girls choose 2 to be in a group}}\times\underbrace{{}^{6}P_{2}}_{\text{6 slots permute 2 groups of girls}}\times\underbrace{2!}_{\text{permute within the group of 2 girls}}\right)$ $+\left[\underbrace{(5-1)!}_{\text{6 boys around}}\times\underbrace{{}^{6}P_{3}}_{\text{6 slots permute 3 girls}}\right]$ $=36000$
(ii)	Number of ways $=\underbrace{(6-1)!}_{\text{6 remaining people around a circle}}\times\underbrace{{}^{6}C_{3}}_{\text{6 slots choose 3}}\times\underbrace{3!}_{\text{The 3 girls}}$ $=14400$
(iii)	Note: The arrangement must look like this  Number of ways without restrictions $=(9-1)!$ $=40320$ Number of ways for exactly 2 boys between any 2 girls $=\underbrace{(3-1)!}_{\text{Arranging 3 girls around a circle}}\times\underbrace{6!}_{\text{6 boys permutate within themselves}}$ $=1440$ Required prob. $=\frac{1440}{40320}$ $=\frac{1}{28}$ Alternative Method: Number of ways without restrictions

	$=(9-1)!$ $=40320$ Number of ways for exactly 2 boys between any 2 girls $= \underbrace{(3-1)!}_{\text{Arranging 3 girls around a circle}} \times$ $\left[\underbrace{{}^{6}C_{2}}_{\text{6 boys choose 2 to slot into first slot}} \times \underbrace{{}^{4}C_{2}}_{\text{4 boys choose 2 to slot into second slot}} \times {}^{2}C_{2} \times \underbrace{(2!)^{3}}_{\text{each group of boys permute within themselves}} \right]$ $=1440$ Required prob. $=\frac{1440}{40320}$ $=\frac{1}{28}$
8(a)	Let X be the random variable denoting the amount of sodium in a randomly chosen packet of potato chips in mg and μ be the population mean amount of sodium in a packet of potato chips in mg. Test $H_0 : \mu = 798$ against $H_1 : \mu > 798$ at 5% level of significance Under H_0, since n is large, by Central Limit Theorem, $\overline{X} \sim N\left(798, \frac{6.5^2}{n}\right)$ approximately. Test statistic $Z = \frac{\overline{X} - 798}{\sqrt{\frac{6.5^2}{n}}} \sim N(0,1)$ approximately. Carry out 1-tailed z-test at the 5% level of significance. Since the company has understated the sodium content, we will reject H_0.  For H_0 to be rejected ,

	$z_{calc} \geq 1.6449$ $\frac{799.5-798}{\sqrt{\frac{6.5^2}{n}}} \geq 1.6449$ $\sqrt{\frac{6.5^2}{n}} \leq \frac{1.5}{1.6449}$ $n \geq \frac{6.5^2}{0.83158}$ $\Rightarrow n \geq 50.8$ Thus, set of n is $\{n \in \mathbb{Z}^+ : n \geq 51\}$.
(b)(i)	Let Y be the random variable denoting the amount of sodium in a randomly chosen packet of potato chips of the healthier recipe in mg and μ_1 be the population mean amount of sodium in a packet of potato chips of the healthier recipe in mg. Test $H_0 : \mu_1 = 798$ against $H_1 : \mu_1 < 798$ at 5% level of significance Unbiased estimate of the population variance s^2 $= \frac{n}{n-1} \times \text{sample variance}$ $= \frac{50}{49} \times 6.2^2$ $= 39.224$ Under H_0, since $n = 50$ is large, by Central Limit Theorem, $\overline{Y} \sim N\left(798, \frac{39.224}{50}\right)$ approximately. Test statistic $Z = \frac{\overline{Y}-798}{\sqrt{\frac{39.224}{50}}} \sim N(0,1)$ approximately. Using a 1-tailed z-test, reject H_0 if p-value ≤ 0.05 Using GC, the test statistic value $\overline{x} = 796.3$ and $z_{calc} = -1.9194$ gives p-value = $0.0275 < 0.05$ We reject H_0 and conclude that there is sufficient evidence at the 5% level of significance that the mean amount of sodium in each packet of potato chips is less than 798 mg.
(b)(ii)	The statement means that there is a probability of 0.05 of wrongly concluding that the mean amount of sodium in each packet of potato chips is less than 798 mg when in fact the mean amount of sodium in each packets is 798 mg.
9	COMMON LAST TOPIC
10(i)	1) The probability that a person is infected is constant at p for each person.

	2) The event of a person being infected is independent of any other people.
(ii)	Let X be the random variable " number of people who are infected, out of 25 people" $X \sim B(25, p)$ $P(X=r) = {}^{25}C_r p^r (1-p)^{25-r}$, $r = 0,1,2,...25.$ Since $P(X \le 2) = 0.0982$ $\binom{25}{0} p^0 (1-p)^{25} + \binom{25}{1} p^1 (1-p)^{24} + \binom{25}{2} p^2 (1-p)^{23} = 0.0982$ By GC, $p \approx 0.200.$
(iii)	Let D be the random variable denoting the no. of samples with at most 2 infected people out of 30 samples. $D \sim B(30, 0.0982)$ $P(D \ge 5)$ $= 1 - P(D \le 4)$ $= 0.16639$ $= 0.166$
(iv)	 Legend: I denote Infected $P(\text{tested positive}) = 0.10 \times 0.85 + 0.90 \times 0.04$ $= 0.121$ $P(\text{infected} \mid \text{tested positive})$ $= \dfrac{P(\text{tested positive and infected})}{P(\text{tested positive})}$ $= \dfrac{0.10 \times 0.85}{0.121}$ $= 0.702$

(v)	P(not infected | tested positive) $= 1 - \text{P(infected} \mid \text{tested positive)}$ $= 1 - 0.70248$ $= 0.298$
(vi)	Amongst those who were tested positive, the proportion of population who are actually infected is 70.2%, hence it is worthwhile. [*Early diagnosis allow for timely treatment/Patients identified early can prevent the spread of the virus*] <u>Alternatively,</u> From part (v), the probability of a person not infected but test positive is close to 30%, which is not considered a low proportion. Hence the diagnostic test might not be worthwhile. [*These group of people would undergo unnecessary treatment which is a waste of resources/These group of people would unnecessarily be quarantined with the infected and might eventually be infected*]
11 (a)(i)	Let X be the random variable " volume of BBT dispensed in ml" $X \sim N(210, 5^2)$ $P(X > 220) = 0.02275 = 0.0228$ (3 s.f.) using GC
(ii)	Let Y be the random variable " no. of BBT cups out of 5 with overflow cups" $Y \sim B(5, 0.02275)$ $P(Y \le 1) = 0.995$ (3 s.f.) OR Required probability $= P(X < 220)^5 + \binom{5}{1}\left(P(X < 220)^4\right)\left(P(X > 220)\right)$ ≈ 0.995
(iii)	Let the mean volume of BBT dispensed by μ ml. $X \sim N(\mu, 5^2)$ $P(X < 212) \le 0.10$ $P(Z < \frac{212-\mu}{5}) \le 0.10$ $\Rightarrow \frac{212-\mu}{5} \le -1.2816$ $\Rightarrow \mu \ge 218.4$ Range of mean volume is $218 \le \mu \le 220$ (to 3 s.f.) (Accept $219 \le \mu \le 220$)

(iv)	$X_1 + X_2 + ... + X_n \sim N(210n, 25n)$ $P(X_1 + X_2 + ... + X_n > 211n)$ $= P(Z > \frac{211n - 210n}{5\sqrt{n}})$ $= P(Z > \frac{1}{5}\sqrt{n}) --- (*)$ Since n becomes very large , $P(Z > \frac{1}{5}\sqrt{n}) \to 0$.
(v)	The BBT seller would not obtain a random sample as she only interviewed customers who bought her BBT during lunch time. She would not be able to get feedback from customers who made the purchase during other hours, hence the **not all the customers have an equal chance of being selected.** The sample is not randomly selected as result.
(b)	Let T be the random variable denoting the volume of bubble tea served in ml. $\overline{T} \sim N(200, \frac{100^2}{60})$ approximately by Central Limit Theorem since the sample size $60 > 30$. Using GC, $P(\overline{T} < 198) = 0.0607$ (3s.f)