NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

CANDIDATE NAME

CT CLASS | 2 | 0 | | |

Centre Number/ Index Number | | | | | / | | | |

MATHEMATICS

9758/01

Paper 1

31st August 2021

3 Hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	**Mark**
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

1 The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ respectively, which are non-zero and non-parallel. The points P and Q are fixed and R, with position vector $\mathbf{r}$, varies.

(a) Given that $(2\mathbf{r}-\mathbf{p})\times\mathbf{p}=\mathbf{0}$, describe geometrically the set of all possible position vectors of $\mathbf{r}$. [2]

(b) If $\mathbf{r}=\mathbf{q}+\lambda\mathbf{p}$, where $\lambda<0$, show that the area of triangle PQR is $k\,|\mathbf{p}\times\mathbf{q}|$ where k is a constant to be determined in terms of λ. [3]

2 **(i)** On the same axes, sketch the curves with equations $y=\left|\dfrac{ax-3a+2}{3-x}\right|$ and $y=\dfrac{a}{3}x$, where $a>1$, giving the equations of the asymptotes and the coordinates of the points where the curves meet the axes. [3]

(ii) Hence, solve the inequality $\left|\dfrac{ax-3a+2}{3-x}\right|>\dfrac{a}{3}x$, giving your answer in terms of a. [3]

3 **(i)** By first expressing $\dfrac{1}{r!(r+2)}$ in the form $\dfrac{A}{(r+2)!}+\dfrac{B}{(r+1)!}$ where A and B are constants, find $\displaystyle\sum_{r=1}^{n}\frac{1}{r!(r+2)}$. [3]

(ii) Hence, explain why the series $\displaystyle\sum_{r=1}^{\infty}\frac{1+\left(\frac{1}{3}\right)^{r}r!(r+2)}{r!(r+2)}$ converges and find the exact sum to infinity of this series. [4]

4 A curve C has cartesian equation

$$y=\frac{x}{1-x},\quad 0\le x<1.$$

(i) The distance between a general point (x,y) on C and the fixed point $(1,0)$ is denoted by s. Show that $s^2=(x-1)^2+y^2$. [1]

(ii) Use differentiation to determine the coordinates of the point on C which has the minimum distance from the point $(1,0)$, giving both coordinates correct to 4 decimal places. [5]

[You need not prove that this distance is a minimum]

(iii) Find this minimum distance. [1]

5 **(a)** The curve $y = \mathrm{f}(x)$ has a horizontal asymptote $y = k$ and cuts the axes at $(a, 0)$ and $(0, b)$, where a, b and k are non-zero constants. It is given that $\mathrm{f}^{-1}(x)$ exists. State, if possible, the coordinates of the points where the following curves cut the axes and the equations of their asymptotes.

(i) $y = \mathrm{f}(2x-3)$

(ii) $y = \mathrm{f}^{-1}(x)$ [2]

(b) The function g is given by $\mathrm{g} : x \mapsto \ln\left(\dfrac{\mathrm{e}^x + 5}{\mathrm{e}^x - 1}\right)$, for $x \in \mathbb{R}, x > 0$.

(i) Find $\mathrm{g}^{-1}(x)$ and state its domain. [3]

The function h is defined by

$$\mathrm{h} : x \mapsto 1 + \sqrt{9-(x-2)^2}\text{, for } x \in \mathbb{R}, -1 \le x \le 5.$$

(ii) Find the exact solutions of $\mathrm{g}^{2021}\mathrm{h}(x) = \ln 2$, giving your answer in its simplest form. [3]

6 **(a)** Find the exact value of $\displaystyle\int_{-3}^{-1} \frac{|x+2|}{x^2+4x+5}\,\mathrm{d}x$. [4]

(b) **(i)** Write down $\displaystyle\int \frac{\sin(\ln x)}{x}\,\mathrm{d}x$, where $x > 0$. [1]

(ii) Hence find $\int x\sin(\ln x)\,\mathrm{d}x$. [4]

7 It is given that $y = \ln(1+\mathrm{e}^x)$.

(i) Show that $(1+\mathrm{e}^x)\dfrac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^x = 0$. [1]

(ii) By further differentiation of the result in **(i)**, find the Maclaurin series for y, up to and including the term in x^2. Hence find the series for $\dfrac{\mathrm{e}^x}{1+\mathrm{e}^x}$ up to and including x. [4]

(iii) Using appropriate expansion from the List of Formulae (MF26), verify the correctness of the series $\dfrac{\mathrm{e}^x}{1+\mathrm{e}^x}$ found in **(ii)**. [3]

8 A curve C has parametric equations

$$x = a\sin^5 t, \;\; y = a\cos^5 t$$

where a is a positive constant.

The tangent to C at a general point with parameter t cuts the coordinate axes at the points A and B.

(i) Denoting the origin by O, show that $OA^{\frac{2}{3}} + OB^{\frac{2}{3}} = a^{\frac{2}{3}}$. [5]

(ii) As t varies, the midpoint of the line segment AB traces out a curve. Find the cartesian equation of this curve in its simplest form. [3]

(iii) Find, in terms of a, the area enclosed by the curve C, giving your answer in the form ka^2 where k is to be determined correct to 2 decimal places. [3]

9 The complex numbers z and w where $w \neq 0$ satisfy the relation

$$2z = |w| + 1.$$

(i) It is given that a is a real number and that a and z satisfy the equation $2z^3 - 5z^2 + 2z + (a+3)\mathrm{i} = 0$. Explain, with justification, why $a = -3$ and that the only possible value of z is 2. [6]

It is given that $\arg w = \dfrac{\pi}{3}$.

(ii) Express the complex number w in the form $p + q\mathrm{i}$ where p and q are in non-trigonometric form. [2]

(iii) Find the least integer n such that $|w^n| > 20212021$. [2]

(iv) Find the least positive integer k such that w^k is a positive real number. [2]

10 One day, Eddie came home from a birthday party and brought back a helium filled balloon. After playing with it, he accidentally released the balloon at the point (1, 2, 3) and it floated vertically upwards at a speed of 1 unit per second. Shortly after t seconds, a sudden gust of wind caused the balloon to move in the direction of $\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$.

You may assume that $z = 0$ refers to the horizontal ground.

(i) Find the angle in which the balloon has changed in direction after the gust of wind blew it away. [3]

(ii) Find the Cartesian equation of the plane that the balloon is moving along. [3]

(iii) Given that the balloon eventually stayed at the point (2, 6, 12) on the ceiling, find the time t when the gust of wind blew the balloon away. [3]

Eddie decides to shoot the balloon down with his catapult.

(iv) Assume he was holding his catapult at (3, 2, 1) initially and he walked along the path parallel to $2\mathbf{i} + \mathbf{j}$. Find the position vector of the point where he should place his catapult so that the distance between his catapult and the balloon is at its minimum. Hence find this distance. [4]

11 In economics, a supply and demand chart is made up of two curves: the supply curve and demand curve. The supply curve is a function that shows how the price of a product, P, is related to the quantity, q, supplied during a given period of time. The demand curve is a function that shows how the price of the same product is related to the quantity demanded during a given period of time. Due to the nature of the curves, they will intersect at a point which is known as the *equilibrium point*.

For a particular product, the demand curve is given by the equation $D(q) = 75 - 1.22^q$ where $0 \le q \le 21$ and the supply curve is given by the equation $S(q) = 2(1.22)^q - 1$ using the same domain.

(i) Sketch both curves on the same diagram. Your sketch should indicate the axial intercepts of both curves. [3]

(ii) Find the coordinates of the equilibrium point. [1]

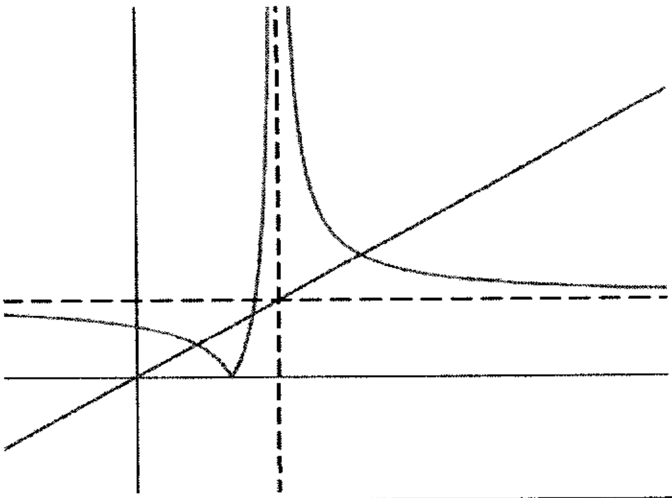
Let p_e be the price of the product at equilibrium point. The area between the demand curve, the line $P = p_e$ and the line $q = 0$ is defined as the consumer surplus. In a similar fashion, the area between the supply curve, the line $P = p_e$ and the line $q = 0$ is defined as the producer surplus.

(iii) Without using a graphing calculator, determine the consumer surplus and producer surplus, leaving both answer to 2 decimal places. [4]

A global increase in production lead to a shift of the supply curve to the right. The equation of the new curve is given by $S(q-a)$ where a is a positive constant. The price of the product at the new equilibrium point is denoted by p_e^*.

(iv) Show that $\dfrac{p_e^* + 1}{p_e + 1} = \dfrac{3}{(1.22)^a + 2}$. [4]

(v) For the case when $a = 4$, find the increase in the consumer surplus, leaving your answer to the nearest whole number. [2]

Qn	
1(a)	$(2\mathbf{r}-\mathbf{p})\times\mathbf{p}=\mathbf{0}$ means $2\mathbf{r}-\mathbf{p}$ is parallel to $\mathbf{p}$ $2\mathbf{r}-\mathbf{p}=k\mathbf{p}$ $2\mathbf{r}=(k+1)\mathbf{p}$ $\mathbf{r}=\frac{k+1}{2}\mathbf{p}$ **r** represents the set of position vectors of points that lies on the line through the origin and parallel to **p.**
1(b)	Area ΔPQR $=\frac{1}{2}\lvert\overrightarrow{QR}\times\overrightarrow{QP}\rvert$ $=\frac{1}{2}\lvert\lambda\mathbf{p}\times(\mathbf{p}-\mathbf{q})\rvert$ $=\frac{1}{2}\lvert\lambda(\mathbf{p}\times\mathbf{q})\rvert$ $=\frac{-\lambda}{2}\lvert(\mathbf{p}\times\mathbf{q})\rvert$ $k=-\lambda/2$
2(i)	

2(ii)	The reflected part of $y=\left\|\frac{ax-3a+2}{3-x}\right\|$ is $-\frac{ax-3a+2}{3-x}$. Solving $-\frac{ax-3a+2}{3-x}=\frac{a}{3}x$: $ax-3a+2=\frac{a}{3}x^2-ax$ $ax^2-6ax+9a-6=0$ Method 1: $x=\frac{6a\pm\sqrt{(-6a)^2-4(a)(9a-6)}}{2a}$ $=\frac{6a\pm\sqrt{24a}}{2a}=3\pm\sqrt{\frac{6}{a}}$ Method 2: $x^2-6x+9-\frac{6}{a}=0$, $(x-3)^2=\frac{6}{a}$, $x=3\pm\sqrt{\frac{6}{a}}$ Solving $\frac{ax-3a+2}{3-x}=\frac{a}{3}x$: $\frac{a}{3}x^2-3a+2=0$ $x^2=\frac{9a-6}{a}$ $x=\sqrt{9-\frac{6}{a}}$ (Reject $x=-\sqrt{9-\frac{6}{a}}$) $\therefore \left\|\frac{ax-3a+5}{3-x}\right\|>\frac{a}{3}x$ $\Rightarrow x<3-\sqrt{\frac{6}{a}}$ or $\sqrt{9-\frac{6}{a}}<x<3$ or $3<x<3+\sqrt{\frac{6}{a}}$
3(i)	$\frac{1}{r!(r+2)}=\frac{r+1}{(r+2)!}=\frac{r+2-1}{(r+2)!}=\frac{1}{(r+1)!}-\frac{1}{(r+2)!}$ <u>Alternative:</u> $\frac{1}{r!(r+2)}=\frac{r+1}{(r+2)!}$ $\frac{A}{(r+2)!}+\frac{B}{(r+1)!}=\frac{A+B(r+2)}{(r+2)!}$ $\therefore r+1=A+B(r+2)$ By comparing coeff: $A=-1,\ B=1$

	$\sum_{r=1}^{n}\frac{1}{r!(r+2)}=\sum_{r=1}^{n}\left[\frac{1}{(r+1)!}-\frac{1}{(r+2)!}\right]$ $=\frac{1}{2!}-\frac{1}{3!}$ $+\frac{1}{3!}-\frac{1}{4!}$ $\vdots$ $+\frac{1}{n!}-\frac{1}{(n-1)!}$ $-\frac{1}{(n+1)!}-\frac{1}{(n-2)!}$ $=\frac{1}{2}-\frac{1}{(n+2)!}$
3(ii)	$\sum_{r=1}^{\infty}\frac{1+\left(\frac{1}{3}\right)^{r}r!(r+2)}{r!(r+2)}=\sum_{r=1}^{\infty}\frac{1}{r!(r+2)}+\sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^{r}$ As $n\to\infty$, $\frac{1}{(n+2)!}\to\infty$, thus $\sum_{r=1}^{\infty}\frac{1}{r!(r+2)}$ converges. Since $\left\lvert\frac{1}{3}\right\rvert<1$, thus $\sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^{r}$ converges. Therefore, $\sum_{r=1}^{\infty}\frac{1+\left(\frac{1}{3}\right)^{r}r!(r+2)}{r!(r+2)}$ converges. $\sum_{r=1}^{\infty}\frac{1+\left(\frac{1}{3}\right)^{r}r!(r+2)}{r!(r+2)}=\frac{1}{2}+\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{2}+\frac{1}{2}=1$
4(i)	Take an arbitrary point (x,y) on C. Distance between (x,y) and $(1,0)$, $s=\sqrt{(x-1)^2+y^2}$ $s^2=(x-1)^2+y^2$
4(ii)	Differentiating w.r.t. x gives $2s\frac{\mathrm{d}s}{\mathrm{d}x}=2(x-1)+2y\frac{\mathrm{d}y}{\mathrm{d}x}$. When s takes a minimum value, $\frac{\mathrm{d}s}{\mathrm{d}x}=0$. Thus

	$2(x-1)+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $x-1+\left(\frac{x}{1-x}\right)\left[\frac{1-x-x(-1)}{(1-x)^2}\right]=0$ $\frac{x}{(1-x)^3}=1-x$ $x=(1-x)^4$ $x^4-4x^3+6x^2-5x+1=0$ By GC (Poly Root Finder), $x\approx 0.2755$ (to 4 d.p.) $y=\frac{0.2755}{1-0.2755}\approx 0.3803$ (to 4 d.p.) So the point on C having a minimum distance from the point $(1,0)$ is $(0.2755,0.3803)$.
4(iii)	The minimum distance $=\sqrt{(0.2755-1)^2+0.3803^2}$ ≈ 0.818
5(a)	(i) $(a,0)\to(\frac{a+3}{2},0)$; $(0,b)\to(\frac{3}{2},b)$ $(\frac{a+3}{2},0)$ is the only point. Asymptote: $y=k$ (ii) $(a,0)\to(0,a)$ $(0,b)\to(b,0)$ Asymptote: $x=k$
5(b) (i)	$y=\ln\left(\frac{e^x+5}{e^x-1}\right)\Rightarrow e^y=\frac{e^x+5}{e^x-1}\Rightarrow(e^y-1)e^x=e^y+5$ $\Rightarrow e^x=\frac{e^y+5}{e^y-1}\Rightarrow x=\ln\left(\frac{e^y+5}{e^y-1}\right)$ $\therefore g^{-1}(x)=\ln\left(\frac{e^x+5}{e^x-1}\right)$ $D_{g^{-1}}=R_g=(0,\infty)$

(b)(ii)

Note: g is a self-inverse function

$g^{2021}h(x)=\ln 2$

$gh(x)=\ln 2$

<table>
<tr><td>

Method 1:

$h(x)=g^{-1}(\ln 2)=\ln 7$

$1+\sqrt{9-(x-2)^2}=\ln 7$

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Method 2(not recommended):

$gh(x)=\ln 2$

$\ln\left(\dfrac{e^{1+\sqrt{9-(x-2)^2}}+5}{e^{1+\sqrt{9-(x-2)^2}}-1}\right)=\ln 2$

$\dfrac{e^{1+\sqrt{9-(x-2)^2}}+5}{e^{1+\sqrt{9-(x-2)^2}}-1}=2$

$e^{1+\sqrt{9-(x-2)^2}}=7$

$1+\sqrt{9-(x-2)^2}=\ln 7$

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$9-(x-2)^2=(\ln 7-1)^2$

$(x-2)^2=9-(\ln 7-1)^2=(2+\ln 7)(4-\ln 7)$

$x=2\pm\sqrt{(2+\ln 7)(4-\ln 7)}$

6(a)

$$\int_{-3}^{-1}\frac{|x+2|}{x^2+4x+5}\,dx=-\int_{-3}^{-2}\frac{x+2}{x^2+4x+5}\,dx+\int_{-2}^{-1}\frac{x+2}{x^2+4x+5}\,dx$$

$$=-\frac{1}{2}\int_{-3}^{-2}\frac{2x+4}{x^2+4x+5}\,dx+\frac{1}{2}\int_{-2}^{-1}\frac{2x+4}{x^2+4x+5}\,dx$$

$$=\frac{1}{2}\left(-\Big[\ln\left|x^2+4x+5\right|\Big]_{-3}^{-2}+\Big[\ln\left|x^2+4x+5\right|\Big]_{-2}^{-1}\right)$$

$$=\frac{1}{2}[(-\ln 1+\ln 2)+(\ln 2-\ln 1)]=\ln 2$$

6(b)(i)

$$\int\frac{\sin(\ln x)}{x}\,dx=\int\frac{1}{x}\sin(\ln x)\,dx=-\cos(\ln x)+c$$

6(b)(ii)

$$\int x\sin(\ln x)\,dx=\int x\left(\frac{x}{x}\right)\sin(\ln x)\,dx$$

$$=\int x^2\left(\frac{1}{x}\right)\sin(\ln x)\,dx$$

$$=x^2(-\cos(\ln x))-\int 2x(-\cos(\ln x))\,dx$$

$$=-x^2\cos(\ln x)+2\int x\left(\frac{x}{x}\right)\cos(\ln x)\,dx$$

$$=-x^2\cos(\ln x)+2\left[x^2\sin(\ln x)-\int 2x\sin(\ln x)\,dx\right]$$

$$5\int x\sin(\ln x)\,dx=-x^2\cos(\ln x)+2x^2\sin(\ln x)$$

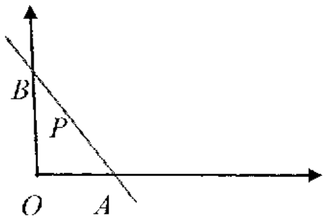
$$\int x\sin(\ln x)\,dx=\frac{1}{5}(-x^2\cos(\ln x)+2x^2\sin(\ln x))+c$$

7(i)	$y=\ln(1+e^x)$ $\frac{dy}{dx}=\frac{e^x}{1+e^x}$ $(1+e^x)\frac{dy}{dx}=e^x$ $(1+e^x)\frac{dy}{dx}-e^x=0$ (shown)
7(ii)	$(1+e^x)\frac{d^2y}{dx^2}+e^x\frac{dy}{dx}-e^x=0$ $(1+e^x)\frac{d^3y}{dx^3}+e^x\frac{d^2y}{dx^2}+e^x\frac{d^2y}{dx^2}+e^x\frac{dy}{dx}-e^x=0$ $(1+e^x)\frac{d^3y}{dx^3}+2e^x\frac{d^2y}{dx^2}+e^x\frac{dy}{dx}-e^x=0$ When $x=0$, $y=\ln 2$ $\frac{dy}{dx}=\frac{e^0}{1+e^0}=\frac{1}{2}$ $(1+e^0)\frac{d^2y}{dx^2}+e^0(\frac{1}{2})-e^0=0$ $\frac{d^2y}{dx^2}=\frac{1}{4}$ By Maclaurin's series, $y=\ln 2+\frac{1}{2}x+\frac{\frac{1}{4}}{2!}x^2+\cdots$ $y=\ln 2+\frac{1}{2}x+\frac{1}{8}x^2+\cdots$ Since $y=\ln(1+e^x)$, $\frac{dy}{dx}=\frac{e^x}{1+e^x}$ $\frac{e^x}{1+e^x}=\frac{1}{2}+\frac{1}{4}x+\ldots$
7(iii)	Using MF26, $\frac{e^x}{1+e^x}=e^x(1+e^x)^{-1}$ $=(1+x+\cdots)(1+1+x+\cdots)^{-1}$ $\approx(1+x)(2^{-1}(1+\frac{x}{2})^{-1})$ $=\frac{1}{2}(1+x)(1+\frac{x}{2}+\cdots)$ $=\frac{1}{2}(1+x+\frac{x}{2}+\cdots)$ $=\frac{1}{2}+\frac{1}{4}x+\ldots$ OR

$$\frac{e^x}{1+e^x}=\frac{1}{1+e^{-x}}=\left(1+e^{-x}\right)^{-1}$$
$$=(1+1-x+\cdots)^{-1}$$
$$=\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}$$
$$=\frac{1}{2}\left(1+\frac{x}{2}+\cdots\right)$$
$$=\frac{1}{2}+\frac{1}{4}x+\cdots$$

$$\frac{e^x}{1+e^x}=1-\frac{1}{1+e^x}=1-(1+e^x)^{-1}$$
$$=1-(1+1+x+\cdots)^{-1}$$
$$=1-\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}$$
$$=1-\frac{1}{2}\left(1-\frac{x}{2}+\cdots\right)$$
$$=\frac{1}{2}+\frac{1}{4}x+\cdots$$

8(i)



Let the tangent to C at the point $P\left(a\sin^5 t, a\cos^5 t\right)$ cut the x and y-axes at A and B respectively.

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{-5a\sin t\cos^4 t}{5a\cos t\sin^4 t}=-\cot^3 t$$

Equation of tangent at P is

$$y-a\cos^5 t=-\cot^3 t\left(x-a\sin^5 t\right)$$

At A, $y=0$. Substituting into equation of tangent gives

$$x=a\sin^5 t+\frac{a\cos^5 t}{\cot^3 t}$$
$$=a\sin^5 t+a\sin^3 t\cos^2 t$$
$$=a\sin^3 t\left(\sin^2 t+\cos^2 t\right)$$
$$=a\sin^3 t$$

At B, $x=0$. Substituting into equation of tangent gives

$$y=a\cos^5 t+a\sin^5 t\cot^3 t$$
$$=a\cos^5 t+a\sin^2 t\cos^3 t$$
$$=a\cos^3 t\left(\cos^2 t+\sin^2 t\right)$$
$$=a\cos^3 t$$

$$OA^{\frac{2}{3}}+OB^{\frac{2}{3}}=\left(a\sin^3 t\right)^{\frac{2}{3}}+\left(a\cos^3 t\right)^{\frac{2}{3}}$$
$$=a^{\frac{2}{3}}\left(\sin^2 t+\cos^2 t\right)=a^{\frac{2}{3}}$$

8(ii)

Midpoint of AB is $M\left(\dfrac{a\sin^3 t}{2},\dfrac{a\cos^3 t}{2}\right)$.

Set $x=\dfrac{a\sin^3 t}{2}$ and $y=\dfrac{a\cos^3 t}{2}$

$\Rightarrow a\sin^3 t=2x$ and $a\cos^3 t=2y$

$\Rightarrow \left(a\sin^3 t\right)^{\frac{2}{3}}=(2x)^{\frac{2}{3}}$ and $\left(a\cos^3 t\right)^{\frac{2}{3}}=(2y)^{\frac{2}{3}}$

$\Rightarrow a^{\frac{2}{3}}\sin^2 t=(2x)^{\frac{2}{3}}$ and $a^{\frac{2}{3}}\cos^2 t=(2y)^{\frac{2}{3}}$

Adding,

$$2^{\frac{2}{3}}\left(x^{\frac{2}{3}}+y^{\frac{2}{3}}\right)=a^{\frac{2}{3}}\left(\sin^2 t+\cos^2 t\right)$$

$$\Rightarrow x^{\frac{2}{3}}+y^{\frac{2}{3}}=\left(\frac{a}{2}\right)^{\frac{2}{3}}$$

8(iii)

The curve C is symmetrical about the x and y axes.

Area enclosed by C

$=4\int_0^a y\ \mathrm{d}x$

$=4\int_0^{\frac{\pi}{2}} a\cos^5 t\left(5a\sin^4 t\cos t\right)\ \mathrm{d}t$

$=20a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cos^6 t\ \mathrm{d}t$

$\approx 0.37a^2$

9(i)

$2z-1=|w|\Rightarrow z=\dfrac{|w|+1}{2}\in\mathbb{R}$, $2z^3-5z^2+2z\in\mathbb{R}$. Thus for $a\in\mathbb{R}$,

$2z^3-5z^2+2z+(a+3)\mathrm{i}=0$

$\Rightarrow 2z^3-5z^2+2z=0$ and $a+3=0$

$\Rightarrow z(2z-1)(z-2)=0$ and $a=-3$

$z(2z-1)(z-2)=0\Rightarrow z=0,\dfrac{1}{2},2$.

If $z=0$, then $|w|=2z-1=-1<0$ which is impossible.

If $z=\dfrac{1}{2}$, then $|w|=0\Rightarrow w=0$ which contradicts $w\neq 0$.

Hence $z=2$.

9(ii)

$|w|=2(2)-1=3$.

$w=3\left(\cos\dfrac{\pi}{3}+\mathrm{i}\sin\dfrac{\pi}{3}\right)=\dfrac{3}{2}+\dfrac{3\sqrt{3}}{2}\mathrm{i}$.

9(iii)	$\left\|w^n\right\| > 20212021$ $\left\|w\right\|^n > 20212021$ $3^n > 20212021$ $n > \dfrac{\lg 20212021}{\lg 3} \approx 15.3$ So least n is 16.
9(iv)	For w^k to be a positive real number, $\arg w^k = 2m\pi,\ m = 0,1,2,\ldots$ $\Rightarrow k \arg w = 2m\pi,\ m = 0,1,2,\ldots$ $\Rightarrow \dfrac{k\pi}{3} = 0, 2\pi, 4\pi, \ldots$ $\Rightarrow k = 0, 6, 12, \ldots$ So least positive integer k is 6.
10(i)	$\cos\theta = \dfrac{\begin{pmatrix}0\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\4\\6\end{pmatrix}}{\sqrt{1^2+4^2+6^2}} = \dfrac{6}{\sqrt{53}}$ $\theta = 34.5^\circ$
10(ii)	$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}0\\0\\1\end{pmatrix} + s\begin{pmatrix}1\\4\\6\end{pmatrix}$ $\mathbf{n} = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \begin{pmatrix}1\\4\\6\end{pmatrix} = \begin{pmatrix}-4\\1\\0\end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix}-4\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}-4\\1\\0\end{pmatrix} = -2$ $-4x + y = -2$
10(iii)	$\begin{pmatrix}2\\6\\12\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}0\\0\\1\end{pmatrix} + s\begin{pmatrix}1\\4\\6\end{pmatrix}$ $2 = 1 + s \Rightarrow s = 1$ $12 = 3 + t + 6(1) \Rightarrow t = 3$

10(iv)

$l:\mathbf{r}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix},\lambda\in\mathbb{R}$

$\overrightarrow{OF}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}$

Let the final position vector of balloon be X.

$\overrightarrow{XF}\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}=0$

$\left(\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix}-\begin{pmatrix}2\\6\\12\end{pmatrix}\right)\cdot\begin{pmatrix}2\\1\\0\end{pmatrix}=0$

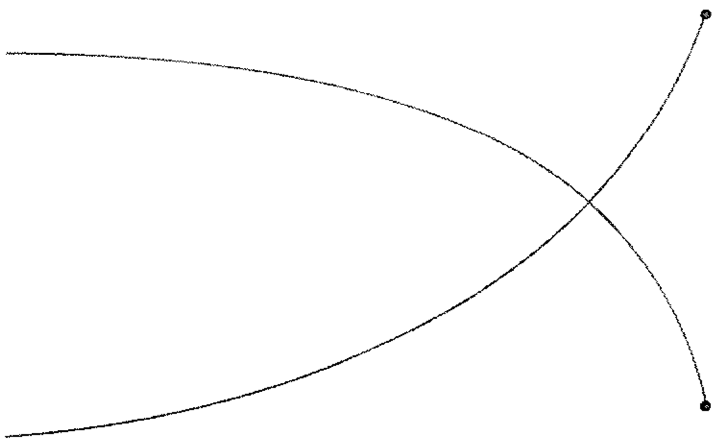
$2-4+\lambda(4+1)=0$

$\lambda=\frac{2}{5}$

$\overrightarrow{OF}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\frac{2}{5}\begin{pmatrix}2\\1\\0\end{pmatrix}=\frac{1}{5}\begin{pmatrix}19\\12\\5\end{pmatrix}$

$XF=\sqrt{(1\frac{4}{5})^2+(-3\frac{3}{5})^2+(-11)^2}=11.7$

11(i)



O
y
x
74
1
21

11(ii)	Using GC, the coordinates of the equilibrium point is (16.3, 49.7).
11(iii)	Let q_e be the quantity at equilibrium point. Thus $\text{C.S.} = \int_0^{q_e} D(q)\,\mathrm{d}q - p_e q_e$ $= \int_0^{q_e} \left(75 - 1.22^q\right)\mathrm{d}q - p_e q_e$ $= \left[75q - \frac{1.22^q}{\ln(1.22)}\right]_0^{q_e} - p_e q_e$ $= 289.40$ $\text{P.S.} = p_e q_e - \int_0^{q_e} S(q)\,\mathrm{d}q$ $= p_e q_e - \int_0^{q_e} \left(2\left(1.22^q\right) - 1\right)\mathrm{d}q$ $= p_e q_e - \left[\frac{2\left(1.22^q\right)}{\ln(1.22)} - q\right]_0^{q_e}$ $= 578.80$
11(iv)	$75 - 1.22^{q_e} = 2(1.22^{q_e}) - 1$ $\Rightarrow 3(1.22)^{q_e} = 76$ Let q_e^* be the quantity at the new equilibrium point. Thus $75 - 1.22^{q_e^*} = 2(1.22^{q_e^* - a}) - 1 = 2(1.22^{-a})(1.22^{q_e^*}) - 1$ $\Rightarrow \left(1 + 2(1.22^{-a})\right)(1.22)^{q_e^*} = 76$ Dividing, we have $(1.22)^{q_e^* - q_e} = \frac{3}{1 + 2(1.22^{-a})}$ Also, we have $p_e = 2(1.22)^{q_e} - 1$ $p_e^* = S(q_e^* - a) = 2(1.22)^{q_e^* - a} - 1 = 2(1.22)^{-a}(1.22)^{q_e^*} - 1$ Thus $\frac{p_e^* + 1}{p_e + 1} = \frac{2(1.22)^{-a}(1.22)^{q_e^*}}{2(1.22)^{q_e}}$ $= (1.22)^{-a}(1.22)^{q_e^* - q_e}$ $= \frac{3(1.22)^{-a}}{1 + 2(1.22^{-a})}$ $= \frac{3}{(1.22)^a + 2}$

11(v)	$p = 75 - 1.22^q$ $\Rightarrow q = \dfrac{\ln(75-p)}{\ln(1.22)}$ Using (iv), $p_e^* = \dfrac{3(p_e+1)}{(1.22)^4+2} - 1 = 35.0588$ Using GC, $Increase = \displaystyle\int_{p_e^*}^{p_e} \dfrac{\ln(75-p)}{\ln(1.22)}\,\mathrm{d}p$ $= 255$ (to nearest whole number)

NANYANG JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

Higher 2

CANDIDATE NAME	

CT CLASS	2	0			

Centre Number/ Index Number					/				

MATHEMATICS 9758/02

Paper 2

13th September 2021

3 Hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

NANYANG JUNIOR COLLEGE
Internal Examinations

[Turn Over

Section A: Pure Mathematics [40 marks]

1 The plane Π_1 passes through $(3,-1,2)$ and is perpendicular to the line $\mathbf{r} = 2\mathbf{i}+4\mathbf{j}-3\mathbf{k}+\lambda(5\mathbf{i}+2\mathbf{j}-\mathbf{k})$. The plane Π_2 contains the points $(2,3,2)$, $(4,1,-1)$ and $(0,-1,2)$.

(i) Show that the acute angle, θ, between the planes Π_1 and Π_2 is such that $\cos\theta = \dfrac{\sqrt{30}}{15}$. [3]

(ii) Show that the line of intersection, L, of the planes Π_1 and Π_2 has vector equation $\mathbf{r} = \begin{pmatrix}0\\9\\7\end{pmatrix} + \mu\begin{pmatrix}-1\\4\\3\end{pmatrix}, \mu \in \mathbb{R}$. [3]

The plane Π_3 has the equation $4(k-2)x+(k+1)y-4k^2z=8$, where k is a constant.

(iii) The three planes Π_1, Π_2 and Π_3 have no points in common. By considering the relationship between the line L and the plane Π_3, find the possible values of k. [2]

(iv) For the positive value of k found in **(iii)**, find the distance between L and Π_3. [2]

2 Many industries use rectangular tanks to handle their water, wastewater and chemical storage and processing needs. Because of their shape, rectangular tanks can offer tremendous cost savings for shipping compared to cylindrical tanks.

In fabricating one such industrial-strength storage tank, a customer requires water to flow into the rectangular tank with a horizontal base area A, at a constant rate of n units of volume per unit time. The water should flow out of the tank through a hole in the bottom, at a rate that is proportional to the square root of the depth of water in the tank. It is also required that when the depth of the water in the tank is h, the level of water in the tank remains constant.

(i) Obtain a differential equation for the depth x at time t. [3]

(ii) It is known that the tank is filled to a depth of $4h$ initially. By using the substitution $x = hu^2$, show that u satisfies the differential equation $\dfrac{2Ah}{n}\dfrac{du}{dt} = -\dfrac{u-1}{u}$. [2]

(iii) By solving this differential equation, find, in terms of A, h and n, the time needed for the depth to reach $\dfrac{16}{9}h$. Describe how x varies with t. [6]

3 **(a)** An arithmetic progression has n terms and a common difference of d. Prove that the difference between the sum of the last k terms and the sum of the first k terms is $(n-k)kd$. [4]

(b) The rth term, u_r, of a series is given by $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$. Express $\sum_{r=1}^{n} u_r$ in the form $A\left(1-\frac{B}{27^n}\right)$, where A and B are constants to be found. Find the sum to infinity of the series. [4]

(c) **(i)** Write down the sum of the geometric series $z + z^2 + \ldots + z^n$. [1]

(ii) By putting $z = e^{i\theta}$ in your result, show that this sum can be written as $\dfrac{\sin\frac{n\theta}{2} e^{i\left(\frac{n+1}{2}\right)\theta}}{\sin\frac{\theta}{2}}$. [2]

(iii) Hence, by using the identity $e^{i\theta} = \cos\theta + i\sin\theta$, or otherwise, show that

$$\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}.$$ [3]

4 Air is pumped into a spherical elastic ball through a tiny hole at a constant rate of 10 cm^3 per second. Assuming that the ball maintains a spherical shape throughout the process, find

(i) the rate of increase of the radius of the ball at the instant when the radius is 5 cm, [2]

(ii) the rate of increase of the surface area of the ball at the instant when the radius of the ball is increasing at the rate of $\frac{5}{72\pi}$ cm per second. [3]

[It is given that the surface area and volume of a sphere of radius r are $4\pi r^2$ and $\frac{4}{3}\pi r^3$ respectively.]

Section B: Probability and Statistics [60 marks]

5 A school teacher is interested in the amount of time spent for revision per day for his students. He decided to select 5 students from each of his four classes.

(i) Explain why this method may not be appropriate. [1]

(ii) Suggest how the method can be improved to get an appropriate sample and why it should be done this way. [2]

(iii) If it is further known that there are 20 students in each of his four classes, find the number of possible samples that he can have if he selects 5 students from each class. [1]

6 The events A, B and C are such that $\mathrm{P}(A)=0.4$, $\mathrm{P}(B)=0.8$, $\mathrm{P}(C)=0.7$. The events A and B are independent; and the events B and C are independent.

(i) If A, B and C are independent events, find the value of $\mathrm{P}(A\cap B'\cap C')$. [2]

(ii) If $\mathrm{P}(A\cap B\cap C)=0.25$, find the least and greatest value of $\mathrm{P}(A\cap B'\cap C')$. [4]

7 This question is about the arrangement of the nine letters in the word PINEAPPLE.

(i) Find the number of ways of arranging all nine letters of the word such that the letters are **not** in alphabetical order. [2]

(ii) Find the number of different ways of arranging all nine letters of the word PINEAPPLE such that no vowel (A,E, I) is next to another vowel. [2]

It is now given that 4 letters are chosen to form another arrangement.

(iii) Find the probability that it consists of at least 2 identical letters. [4]

8 A factory produces surgical masks. On average, a proportion p of mask fail to meet the requirement of surgical standard. A mask that fails to meet the requirement is considered faulty. The masks are packed in boxes for sale to retail outlets. It should be assumed that the number of faulty surgical masks in a box follows a binomial distribution.

For quality control purposes a random sample of 10 masks from a box is tested.

- If there are no faulty masks, the box is accepted for sale.
- If there are more than 2 faulty masks found in this sample of 10, the box is rejected.
- If there are 1 or 2 faulty masks found in this sample, a further sample of 5 masks is randomly selected. The box will be accepted if this sample of 5 masks has no faulty masks and will be rejected otherwise.

(i) Given that the probability of a box being accepted is 0.923, find p. [3]

(ii) Find the expected number of masks to be sampled for quality control. [2]

The inspection was conducted in batches. In one batch, 60 boxes of masks are chosen to be inspected.

(iii) Find the probability that the 60^{th} box is the 5^{th} box to be rejected if at least 5 boxes of masks are rejected. [3]

9 A bag contains 2 blue counters, 1 white counter and n red counters, where $n > 2$. In a game, John removes counters randomly from the bag, one at a time, until he has taken out 2 red counters. The total number of blue counters John removes from the bag is denoted by C.

(i) Show that $P(C=1) = \frac{4(n-1)}{(n+2)(n+1)}$. Find $P(C=c)$ for all other possible values of c. [4]

(ii) Show that $E(C) = \frac{4}{n+1}$ and $Var(C) = \frac{g(n)}{(n+2)(n+1)^2}$ where $g(n)$ is a quadratic polynomial to be determined. [3]

(iii) John plays the game twice and the number of blue counters obtained from these 2 games are C_1 and C_2 respectively. It is known that $P\left(|C_1 - C_2| > 0\right) < \frac{1}{5}$. Find the least value of n. [3]

10 An office worker, Natalie, has diabetes and has to monitor her blood glucose levels, which vary throughout the day. The results from a sample of 75 readings, x (in mmol/L), taken at random times over a week, are summarised by $\sum x = 511.5$ and $\sum x^2 = 4027.89$.

(i) Calculate unbiased estimates of the population mean and variance for the blood glucose levels. [2]

(ii) Test at 5% significance level whether Natalie's mean blood glucose level, μ (in mmol/L), is greater than 6.0. You should state your hypotheses and give your conclusion in context. [4]

(iii) State, giving a reason, whether the conclusion of the test in part **(ii)** would be valid if the 75 readings were all taken at weekends. [1]

Following a change in her diet, Natalie claims that her mean blood glucose level μ is now less than 6.0. She takes another random sample of 75 readings and notes that the total blood glucose levels is now 420. Using this sample, Natalie concludes that there is no reason to reject her claim at 6% level of significance.

(iv) Find the range of possible values of the variance used in calculating the test statistic. [4]

(v) Explain why there is no need for Natalie to know anything about the population distribution of the glucose blood levels when carrying out the tests in **(ii)** and **(iv)**. [1]

11 **In this question you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

NY Pasta Brava runs a pasta delivery service in the district of Serangoon. Its pasta are first prepared at a central kitchen before being sent out for delivery through its food delivery partner. To ensure freshness of each order, NY Pasta Brava only starts preparing the next customer's order after the previous customer's order has been sent out for delivery.

The time taken for NY Pasta Brava to prepare each customer's order follows a normal distribution with mean μ minutes and standard deviation σ minutes. The time taken for its food delivery partner to send each order also follows a normal distribution with mean 24 minutes and standard deviation 6 minutes.

(i) It is given that the preparation time for an order is equally likely to be faster than 13 minutes and slower than 18 minutes. The boss also recorded that on average, 49.5% of orders were prepared between 14.5 and 16.5 minutes. State the value of μ and show that $\sigma = 1.50$. [2]

(ii) Sketch the distribution for the preparation times between 10.5 minutes and 20.5 minutes. [2]

To improve the efficiency of the pasta preparation process, the chef of NY Pasta Brava purchases new pasta machines and reorganises the kitchen with designated workstations to prevent bottlenecks such that the preparation time for an order is reduced by 15%.

(iii) To maintain customer satisfaction, NY Pasta Brava aims to keep the average time for its orders to reach its customers below 38 minutes for each day. Given that there were 85 randomly chosen orders in total on that day, find the probability that NY Pasta Brava succeeds in maintaining customer satisfaction on that day. [3]

At Merlion Pasta Bar, the time taken to prepare each customer's order has mean 13.5 minutes and standard deviation 7 minutes.

(iv) Explain why the time taken to prepare each customer's order at Merlion Pasta Bar is unlikely to follow a normal distribution with this mean and standard deviation. [1]

(v) For n randomly chosen orders, where n is large, find the least value of n such that with the improved pasta preparation process at NY Pasta Brava, the mean preparation time of NY Pasta Brava for each order is faster than that of Merlion Pasta Bar by more than 80% of the time. [4]

Qn	
1(i)	$\mathbf{n}_2=\begin{pmatrix}2-4\\3-1\\2-(-1)\end{pmatrix}\times\begin{pmatrix}2-0\\3-(-1)\\2-2\end{pmatrix}=\begin{pmatrix}-2\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\4\\0\end{pmatrix}=\begin{pmatrix}-12\\6\\-12\end{pmatrix}=-6\begin{pmatrix}2\\-1\\2\end{pmatrix}$ $\cos\theta=\frac{\left\|\begin{pmatrix}2\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}5\\2\\-1\end{pmatrix}\right\|}{\left\|\begin{pmatrix}2\\-1\\2\end{pmatrix}\right\|\left\|\begin{pmatrix}5\\2\\-1\end{pmatrix}\right\|}=\frac{10-2-2}{3(\sqrt{30})}=\frac{2}{\sqrt{30}}=\frac{2\sqrt{30}}{30}=\frac{\sqrt{30}}{15}$
1(ii)	$\Pi_1:\mathbf{r}\cdot\begin{pmatrix}5\\2\\-1\end{pmatrix}=\begin{pmatrix}3\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}5\\2\\-1\end{pmatrix}=11\quad\Rightarrow\quad 5x+2y-z=11$ $\Pi_2:\mathbf{r}\cdot\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}2\\3\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\2\end{pmatrix}=5\quad\Rightarrow\quad 2x-y+2z=5$ Line of intersection between the 2 planes $2y-z=11$ $-y+2z=5$ When x=0, $\Rightarrow z=7, y=9$ $\mathbf{d}=\begin{pmatrix}5\\2\\-1\end{pmatrix}\times\begin{pmatrix}2\\-1\\2\end{pmatrix}=\begin{pmatrix}3\\-12\\-9\end{pmatrix}=-3\begin{pmatrix}-1\\4\\3\end{pmatrix}$ $\mathbf{r}=\begin{pmatrix}0\\9\\7\end{pmatrix}+\mu\begin{pmatrix}-1\\4\\3\end{pmatrix},\mu\in\mathbb{R}$
1(iii)	Since 3 planes have no points in common, the line of intersection in (i) is parallel and away from the plane $4(k-2)x+(k+1)y-4k^2z=8$ $\begin{pmatrix}-1\\4\\3\end{pmatrix}\cdot\begin{pmatrix}4(k-2)\\k+1\\-4k^2\end{pmatrix}=0$ $-4k+8+4k+4-12k^2=0$ $3k^2-3=0$ $k=\pm1$

1(iv)	Method 1 When $k=1$, $\Pi_3 : -4x+2y-4z=8$ Distance $= \dfrac{\left\|\begin{pmatrix}0\\9\\7\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\-4\end{pmatrix}-8\right\|}{\left\|\begin{pmatrix}-4\\2\\-4\end{pmatrix}\right\|} = \dfrac{18}{6} = 3$
	Method 2 Take any point on the plane Π_3, for example $\begin{pmatrix}0\\4\\0\end{pmatrix}$. Let $\overrightarrow{OA}=\begin{pmatrix}0\\9\\7\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}0\\4\\0\end{pmatrix}$. Then distance $= \left\|\overrightarrow{AB}\cdot\hat{\mathbf{n}}\right\| = \dfrac{\left\|\begin{pmatrix}0\\-5\\-7\end{pmatrix}\cdot\begin{pmatrix}-4\\2\\-4\end{pmatrix}\right\|}{6} = 3$
	Method 3 Let F be the foot of perpendicular of $(0,9,7)$ onto $\Pi_3 : -4x+2y-4z=8$. $\overrightarrow{OF}=\begin{pmatrix}0\\9\\7\end{pmatrix}+k\begin{pmatrix}-4\\2\\-4\end{pmatrix}$ for some $k\in\mathbb{R}$ Since F lies on Π_3, $\left[\begin{pmatrix}0\\9\\7\end{pmatrix}+k\begin{pmatrix}-4\\2\\-4\end{pmatrix}\right]\cdot\begin{pmatrix}-4\\2\\-4\end{pmatrix}=8$

	$-10+36k=8$ $k=\frac{18}{36}$ $=\frac{1}{2}$ $\overrightarrow{OF}=\begin{pmatrix}0\\9\\7\end{pmatrix}+\frac{1}{2}\begin{pmatrix}-4\\2\\-4\end{pmatrix}=\begin{pmatrix}-2\\10\\5\end{pmatrix}$ $\therefore$ distance $=\left\lvert\begin{pmatrix}-2\\10\\5\end{pmatrix}-\begin{pmatrix}0\\9\\7\end{pmatrix}\right\rvert=3$ (or just $\left\lvert\frac{1}{2}\begin{pmatrix}-4\\2\\4\end{pmatrix}\right\rvert$)
2(i)	Let the volume of water in the tank be V. $\frac{dV}{dt}=n-k\sqrt{x}$ $\frac{d(Ax)}{dt}=n-k\sqrt{x}$ $A\frac{dx}{dt}=n-k\sqrt{x}$ When $x=h$, $\frac{dx}{dt}=0$, therefore $A\frac{dx}{dt}=n-k\sqrt{h}=0\Rightarrow k=\frac{n}{\sqrt{h}}$. $A\frac{dx}{dt}=n-k\sqrt{x}\Rightarrow\frac{dx}{dt}=\frac{n}{A}\left(1-\sqrt{\frac{x}{h}}\right)$
2(ii)	Using $x=hu^2$, differentiating wrt t, $\frac{dx}{dt}=2hu\frac{du}{dt}$ $A\frac{dx}{dt}=n-k\sqrt{x}\Rightarrow 2huA\frac{du}{dt}=n-\frac{n}{\sqrt{h}}\sqrt{hu^2}$ $2huA\frac{du}{dt}=n-nu=-n(u-1)$ $\frac{2Ah}{n}\frac{du}{dt}=-\frac{u-1}{u}$
2(iii)	$\frac{2Ah}{n}\frac{du}{dt}=-\frac{u-1}{u}$ Separating the variables,

$\int \frac{u}{u-1}\,du = -\frac{n}{2Ah}\int dt$

$\int 1+\frac{1}{u-1}\,du = -\frac{n}{2Ah}t + C$

$u + \ln(u-1) = -\frac{n}{2Ah}t + C$

When $x = 4h,\ u = 2$

When $x = \frac{16}{9}h,\ u = \frac{4}{3}$

When $x = h,\ u = 1$

When $t = 0,\ x = 4h,\ u = 2$, we have $C = 2$

When $x = \frac{16}{9}h,\ u = \frac{4}{3}$, we have $\frac{4}{3} + \ln\left(\frac{1}{3}\right) = -\frac{n}{2Ah}t + 2$

Therefore, $t = \frac{2Ah}{n}\left(\frac{2}{3} + \ln 3\right)$

As t increases, x decreases and approaches a depth of h.

3(a)	Sum of first k terms = $\frac{k}{2}[2a+(k-1)d]$ First term of last k terms is given by $u_{n-k+1}=a+(n-k)d$ Sum of last k terms = $\frac{k}{2}[a+(n-k)d+a+(n-1)d]$ $=\frac{k}{2}[2a+(n-k)d+(n-1)d]$ Difference between the sums = $\frac{k}{2}[2a+(n-k)d+(n-1)d]-\frac{k}{2}[2a+(k-1)d]$ $=\frac{kd}{2}(n-k)+\frac{kd}{2}(n-1)-\frac{kd}{2}(k-1)$ $=\frac{kd}{2}(n-k+n-1-k+1)=\frac{kd}{2}(2n-2k)$ $=kd(n-k)$ *Alternatively:* Sum of first k terms = $\frac{k}{2}[2a+(k-1)d]$ Sum of the last k terms is $S_n-S_{n-k}=\frac{n}{2}[2a+(n-1)d]-\frac{n-k}{2}[2a+(n-k-1)d]$ $=\frac{n}{2}[2a+(n-1)d]-\frac{n}{2}[2a+(n-k-1)d]+\frac{k}{2}[2a+(n-k-1)d]$ $=\frac{n}{2}[kd]+\frac{k}{2}[2a+(n-k-1)d]$ Difference between the sums = Sum of the last k terms - Sum of the first k terms $=\frac{n}{2}[kd]+\frac{k}{2}[2a+(n-k-1)d]-\frac{k}{2}[2a+(k-1)d]$ $=\frac{n}{2}[kd]+\frac{k}{2}[2a+(n-k-1)d-(2a+(k-1)d)]$ $=\frac{n}{2}[kd]+\frac{k}{2}[d(n-k-1-k+1)]=\frac{n}{2}[kd]+\frac{k}{2}[d(n-2k)]$ $=kd[n-k]$ (shown)
3(b)	$u_r=\left(\frac{1}{3}\right)^{3r-2}+\left(\frac{1}{3}\right)^{3r-1}=\left(\frac{1}{3}\right)^{3r}\left[\left(\frac{1}{3}\right)^{-2}+\left(\frac{1}{3}\right)^{-1}\right]=12\left(\frac{1}{3}\right)^{3r}$ $\sum_{r=1}^{n}u_r=\sum_{r=1}^{n}12\left(\frac{1}{3}\right)^{3r}=12\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r}$

$$=12\left[\left(\frac{1}{3}\right)^3+\left(\frac{1}{3}\right)^6+\ldots+\left(\frac{1}{3}\right)^{3n}\right]$$

$$=12\left[\frac{\left(\frac{1}{3}\right)^3\left(1-\left(\frac{1}{3}\right)^{3n}\right)}{1-\left(\frac{1}{3}\right)^3}\right]=\frac{6}{13}\left(1-\frac{1}{27^n}\right)$$

As $n\to\infty, \frac{1}{27^n}\to 0$, therefore, $\sum_{r=1}^{\infty}u_r=\frac{6}{13}$

Alternatively:

$$u_r=\left(\frac{1}{3}\right)^{3r-2}+\left(\frac{1}{3}\right)^{3r-1}=\left(\frac{1}{3}\right)^{3r-1}\left[\left(\frac{1}{3}\right)^{-1}+1\right]=4\left(\frac{1}{3}\right)^{3r-1}$$

$$\sum_{r=1}^{n}u_r=\sum_{r=1}^{n}4\left(\frac{1}{3}\right)^{3r-1}=4\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r-1}$$

$$=4\left[\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^5+\ldots+\left(\frac{1}{3}\right)^{3n-1}\right]$$

$$=4\left[\frac{\left(\frac{1}{3}\right)^2\left(1-\left(\frac{1}{3}\right)^{3n}\right)}{1-\left(\frac{1}{3}\right)^3}\right]=\frac{6}{13}\left(1-\frac{1}{27^n}\right)$$

As $n\to\infty, \frac{1}{27^n}\to 0$, therefore, $\sum_{r=1}^{\infty}u_r=\frac{6}{13}$

3(c)(i)

$$z+z^2+\ldots+z^n=\frac{z(1-z^n)}{1-z}$$

3(c)(ii)

$$\frac{z(1-z^n)}{1-z}=\frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}}$$

$$=\frac{e^{i\theta}.e^{i\frac{n\theta}{2}}\left(e^{-i\frac{n\theta}{2}}-e^{i\frac{n\theta}{2}}\right)}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)}$$

$$=\frac{e^{i\frac{(n+1)\theta}{2}}\left(-2i\sin\frac{n\theta}{2}\right)}{-2i\sin\frac{\theta}{2}}=\frac{\sin\frac{n\theta}{2}e^{i\frac{(n+1)\theta}{2}}}{\sin\frac{\theta}{2}}$$

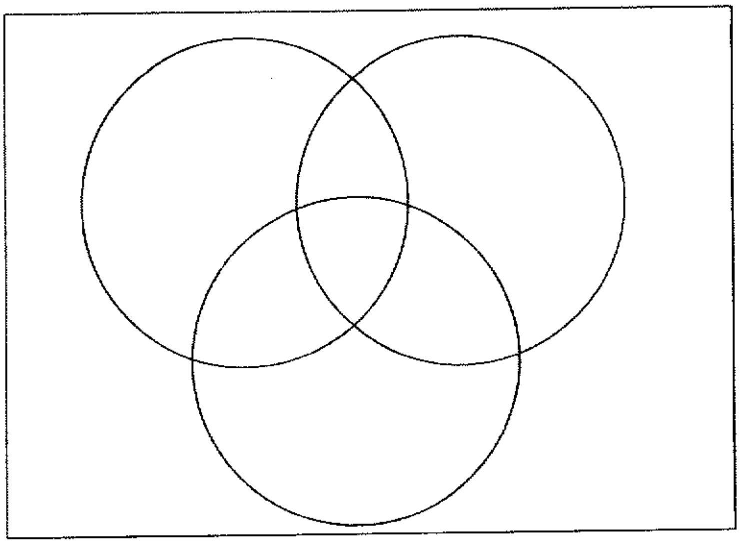
For 'show' question, it is necessary to show the working clearly to reach the answer given to get full credits.

3(c)(iii)

$$z+z^2+\ldots+z^n=\frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}}$$

$$e^{i\theta}+e^{i2\theta}+\ldots+e^{in\theta}=\frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}}$$

	$\cos\theta + \mathrm{i}\sin\theta + \cos 2\theta + \mathrm{i}\sin 2\theta + \ldots + \cos n\theta + \mathrm{i}\sin n\theta = \dfrac{\sin\frac{n\theta}{2}\,\mathrm{e}^{\mathrm{i}\frac{(n+1)\theta}{2}}}{\sin\frac{\theta}{2}}$ $= \dfrac{\sin\frac{n\theta}{2}\left(\cos\frac{(n+1)\theta}{2} + \mathrm{i}\sin\frac{(n+1)\theta}{2}\right)}{\sin\frac{\theta}{2}}$ Comparing the imaginary parts, $\sin\theta + \sin 2\theta + \ldots + \sin n\theta = \dfrac{\sin\frac{(n+1)\theta}{2}\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}$
4	(i) $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$. (1) Substitute $\frac{\mathrm{d}V}{\mathrm{d}t} = 10, r = 5$ into (1) gives $10 = 4\pi(5^2)\frac{\mathrm{d}r}{\mathrm{d}t} \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{10\pi}$. So the radius of the ball is increasing at the rate of $\frac{1}{10\pi}$ cm/s. (ii) $A = 4\pi r^2 \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi r \frac{\mathrm{d}r}{\mathrm{d}t}$. (2) Substitute $\frac{\mathrm{d}V}{\mathrm{d}t} = 10, \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{5}{72\pi}$ into (1) gives $10 = 4\pi r^2\left(\frac{5}{72\pi}\right) \Leftrightarrow r = 6$. Substitute $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{5}{72\pi}, r = 6$ into (2) gives $\frac{\mathrm{d}A}{\mathrm{d}t} = 8\pi(6)\left(\frac{5}{72\pi}\right) = \frac{10}{3}$. So the surface area of the ball is increasing at the rate of $\frac{10}{3}$ cm²/s.

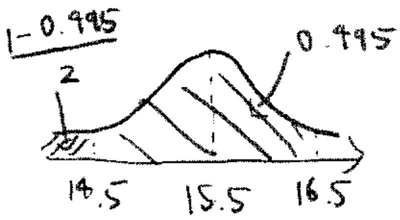
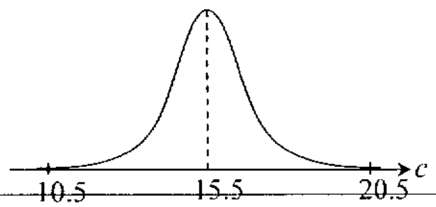
5(i)	The class sizes of the classes may not be the same. Thus the students will not have equal chance of being selected.
5(ii)	The teacher must obtain a random sample in which every student has equal chance of being selected. This will ensure that the sample is not biased.
5(iii)	Number of samples = $\binom{20}{5}^4 = 5.78\times10^{16}$
6(i)	Since A, B and C are independent, A, C and B' are also independent. Thus $P(A\cap C\cap B') = (0.4)(0.7)(0.2) = 0.056$
6(ii)	 A B C y 0.25 0.07 $0.08-y$ 0.17 0.31 $0.14-y$ Let $y = P(A\cap C\cap B')$ From the Venn Diagram, we must have

	$0.08-y\geq 0$ and $0.14-y\geq 0$. Thus $y\leq 0.08$. Using $P(A\cup B\cup C)\leq 1$, we have $1.02-y\leq 1$. Thus $y\geq 0.02$. Thus least value of y is 0.02 and greatest value of y is 0.08
7(i)	Number of ways $=\frac{9!}{3!2!}-1=$ 30239
7(ii)	Number of ways $=\frac{5!}{3!}\times {}^6C_4\times\frac{4!}{2!}=3600$
7(iii)	Number of ways to form 4 letters = n(4 diff letters)+n(2 same, 2 diff)+n(3 same, 1 diff)+ n(2 pairs of same letters) $={}^6C_4(4!)+{}^2C_1\,{}^5C_2(\frac{4!}{2!})+{}^1C_1\,{}^5C_1(\frac{4!}{3!})+\frac{4!}{2!2!}$ Probability required $=\dfrac{{}^2C_1\,{}^5C_2(\frac{4!}{2!})+{}^1C_1\,{}^5C_1(\frac{4!}{3!})+\frac{4!}{2!2!}}{{}^6C_4(4!)+{}^2C_1\,{}^5C_2(\frac{4!}{2!})+{}^1C_1\,{}^5C_1(\frac{4!}{3!})+\frac{4!}{2!2!}}$ $=0.425$ or $\frac{133}{313}$
8(i)	Let X be the number of rejected masks out of 10 masks. $X\sim \mathrm{B}(10,p)$ Let Y be the number of defective masks out of 5 pens. $Y\sim B(5,p)$ P(box is accepted) = 0.923 $\mathrm{P}(X=0)+(\mathrm{P}(X=1)+\mathrm{P}(X=2))\mathrm{P}(Y=0)$ =0.923

	Using GC, $p = 0.043231 = 0.0432$ (correct to 3 s.f)
8(ii)	Let M be the number of masks sampled m | 10 | 15 $P(M=m)$ | $P(X=0)+P(X>2) = 0.650503$ | $P(X=1)+P(X=2) = 0.349497$ Expected number of masks sampled = 10 (0.650503) + 15 (0.349497) = 11.747 = 11.7 Alternative method Let A be the number of additional masks sampled a | 0 | 5 $P(A=a)$ | $P(X=0)+P(X>2) = 0.650503$ | $P(X=1)+P(X=2) = 0.349497$ Expected number of masks sampled = 10 + 5 (0.349497) = 11.747 = 11.7
8(iii)	Let A be the number of rejected boxes out of 60 boxes. $A \sim B(60, 0.077)$ Let C be the number of rejected boxes out of 59 boxes. $C \sim B(59, 0.077)$ P(the 60th box is the 5th box to be rejected | at least 5 boxes of masks are rejected) $= \frac{P(C=4)\times 0.077}{P(A \geq 5)} = 0.030335 = 0.0303$
9(i)	$P(C=1) = 2!P(BRR) + 3!P(BWRR)$ $= 2\frac{2\cdot n\cdot(n-1)}{(n+3)(n+2)(n+1)} + 6\frac{2\cdot 1\cdot n\cdot(n-1)}{(n+3)(n+2)(n-1)(n)}$ $= \frac{4n(n-1)+12(n-1)}{(n+3)(n+2)(n+1)} = \frac{4(n-1)}{(n+2)(n+1)}$ $P(C=0) = P(RR) + 2!P(WRR)$ $= \frac{n\cdot(n-1)}{(n+3)(n+2)} + 2\frac{1\cdot n\cdot(n-1)}{(n+3)(n+2)(n+1)}$ $= \frac{n(n-1)(n+1+2)}{(n+3)(n+2)(n+1)} = \frac{n(n-1)}{(n+2)(n+1)}$ $P(C=2) = 1 - P(C=0) - P(C=1)$ $= 1 - \frac{n(n-1)}{(n+2)(n+1)} - \frac{4(n-1)}{(n+2)(n+1)}$ $= \frac{n^2+3n+2-n^2+n-4n+4}{(n+2)(n+1)} = \frac{6}{(n+2)(n+1)}$

9(ii)	$\mathrm{E}(C^2)=\dfrac{4(n-1)}{(n+2)(n+1)}+\dfrac{24}{(n+2)(n+1)}$ $=\dfrac{4(n+5)}{(n+2)(n+1)}$ $\mathrm{Var}(C)=\mathrm{E}(C^2)-[\mathrm{E}(C)]^2$ $=\dfrac{4(n+5)}{(n+2)(n+1)}-\left[\dfrac{4}{n+1}\right]^2$ $=\dfrac{4(n+1)(n+5)-16(n+2)}{(n+2)(n+1)^2}$ $=\dfrac{4n^2+8n-12}{(n+2)(n+1)^2}=\dfrac{4(n+3)(n-1)}{(n+2)(n+1)^2}$
9(iii)	$P(\lvert C_1-C_2\rvert>0)=1-P(\lvert C_1-C_2\rvert=0)$ $=1-P(C_1=C_2)$ $=1-[P(C=0)]^2+[P(C=1)]^2+[P(C=2)]^2$ $=1-\dfrac{n^2(n-1)^2}{(n+2)^2(n+1)^2}-\dfrac{16(n-1)^2}{(n+2)^2(n+1)^2}-\dfrac{36}{(n+2)^2(n+1)^2}$ Using GC, <table><tr><td>n</td><td>$P(\lvert C_1-C_2\rvert>0)$</td></tr><tr><td>33</td><td>0.20093</td></tr><tr><td>34</td><td>0.19605</td></tr></table> Thus least $n=34$.
10(i)	Unbiased estimates for population mean and population variance, $\bar{x}=\dfrac{511.5}{75}=6.82$ $s^2=\dfrac{1}{n-1}\left[\sum x^2-\dfrac{\left(\sum x\right)^2}{n}\right]=\dfrac{1}{74}\left[4027.89-\dfrac{511.5^2}{75}\right]=7.29$
10(ii)	Given μ denote mean blood glucose level, $\mathrm{H}_0:\mu=6.0$ To test : $\mathrm{H}_1:\mu>6.0$ at 5% level of significance

	Under H_0, since sample size of 75 is large, using Central Limit Theorem, $\bar{X} \sim N\left(6, \frac{7.29}{75}\right)$ approximately, and test statistic, $Z = \frac{\bar{X} - 6}{\sqrt{7.29/75}} \sim N(0,1)$ Critical Region: Reject H_0 if p-value ≤ 0.05 Calculations : Using GC, $z_{cal} = 2.630$ and p-value = 0.00427 Conclusion: Since p-value < 0.05, we reject H_0. There is sufficient evidence, at 5% level of significance, that Natalie's average blood glucose level is higher than 6.0.
10(iii)	Readings at weekend may be biased by different life style, so results may not be valid.
10(iv)	$H_0 : \mu = 6.0$ To test $H_1 : \mu < 6.0$ at 10% level of significance Under H_0, since $n = 75$ is large by Central Limit Theorem, $\bar{X} \sim N\left(6, \frac{s^2}{75}\right)$ approximately, and test statistic, $Z = \frac{\bar{X} - 6}{\sqrt{S^2/75}} \sim N(0,1)$ Reject H_0 if $z_{cal} \leq -1.55477$ Calculations : $\bar{x} = \frac{420}{75} = 5.6$ and using $z_{cal} = \frac{5.6 - 6}{s/\sqrt{75}}$ Since H_0 is rejected, $z_{cal} = \frac{5.6 - 6}{s/\sqrt{75}} \leq -1.55477 \Rightarrow s^2 \leq 4.96$ Required range is $s^2 \leq 4.96$

10(v)	Since sample sizes are large enough, Natalie may use Central Limit Theorem to approximate the distribution of the sample mean to be normal. Therefore, no need to know anything about the population distribution of the glucose blood levels.
11(i)	Let C denote the preparation time of a randomly chosen order for NY Pasta Brava. $C \sim \mathrm{N}\left(\mu, \sigma^2\right)$ $\mu = \frac{13+18}{2} = 15.5$ Using symmetry, $\mathrm{P}(14.5 < C < 16.5) = 0.495$ $\Rightarrow \mathrm{P}(C < 14.5) = \frac{1-0.495}{2} = 0.2525$ $\Rightarrow \mathrm{P}(Z < \frac{14.5-15.5}{\sigma}) = 0.2525$ Using InvNorm Left, $-\frac{1}{\sigma} = -0.6666433049$ $\Rightarrow \sigma = 1.5001 \text{ (5 s.f.)} = 1.50 \text{ (3 s.f.)}$  Alternatively, $\mathrm{P}(14.5 < C < 16.5) = 0.495$ $\Rightarrow \mathrm{P}(\frac{14.5-15.5}{\sigma} < Z < \frac{16.5-15.5}{\sigma}) = 0.495$ Using InvNorm Center, $-\frac{1}{\sigma} = -0.6666433049$ & $\frac{1}{\sigma} = -0.6666433049$ $\Rightarrow \sigma = 1.5001 \text{ (5 s.f.)} = 1.50 \text{ (3 s.f.)}$
11(ii)	

11(iii)	Let D denote the delivery time of a randomly chosen order for NY Pasta Brava. $D \sim \mathrm{N}(24, 6^2)$ Let $\overline{T} = \dfrac{0.85(C_1 + C_2 + \ldots + C_{85}) + (D_1 + D_2 + \ldots + D_{85})}{85}$ $\mathrm{E}(\overline{T}) = \dfrac{0.85[85\mathrm{E}(C)] + [85\mathrm{E}(D)]}{85} = 37.175$ $\mathrm{Var}(\overline{T}) = \dfrac{0.85^2[85\mathrm{Var}(C)] + [85\mathrm{Var}(D)]}{85^2} = 0.44625$ (5 s.f.) $\overline{T} \sim \mathrm{N}\left(37.175, \left(\sqrt{0.44625}\right)^2\right)$ $\mathrm{P}(\overline{T} < 38) = 0.89251$ (5 s.f.) $= 0.893$ (3 s.f.)
11(iv)	Let M denote the preparation time of a randomly chosen order for Merlion Pasta Bar. If $M \sim \mathrm{N}(13.5, 7^2)$, $\mathrm{P}(M < 0) = 0.0269$ (3 s.f.) which is not possible OR $\mathrm{P}(13.5 - 3(7) < M < 13.5 + 3(7)) \approx 0.997$ but $13.5 - 3(7) = -7.5 < 0$ which is not possible

11(v)	Let $\overline{C} = \dfrac{0.85(C_1 + C_2 + ... + C_n)}{n}$ & $\overline{M} = \dfrac{M_1 + M_2 + ... + M_n}{n}$ $\mathrm{E}\left(\overline{C}\right) = \dfrac{0.85\left[n\mathrm{E}(C)\right]}{n} = 13.175$ $\mathrm{Var}\left(\overline{C}\right) = \dfrac{0.85^2\left[n\mathrm{Var}(C)\right]}{n^2} = \dfrac{1.625625}{n}$ $\overline{C} \sim \mathrm{N}\left(13.175, \left(\sqrt{\dfrac{1.625625}{n}}\right)^2\right)$ $\mathrm{E}\left(\overline{M}\right) = \mathrm{E}(M) = 13.5$ $\mathrm{Var}\left(\overline{M}\right) = \dfrac{\mathrm{Var}(M)}{n^2} = \dfrac{7^2}{n}$ Since n is large, by CLT, $\overline{M} \sim \mathrm{N}\left(13.5, \left(\dfrac{7}{\sqrt{n}}\right)^2\right)$ approximately $\mathrm{E}\left(\overline{C} - \overline{M}\right) = -0.325$ $\mathrm{Var}\left(\overline{C} - \overline{M}\right) = \dfrac{50.625625}{n}$ $\therefore \overline{C} - \overline{M} \sim \mathrm{N}\left(-0.325, \left(\sqrt{\dfrac{50.625625}{n}}\right)^2\right)$ approximately $\mathrm{P}(\overline{C} < \overline{M}) = \mathrm{P}(\overline{C} - \overline{M} < 0) > 0.8$ $\mathrm{P}\left(Z < \dfrac{0.325}{\sqrt{\dfrac{50.625625}{n}}}\right) > 0.8$ $\mathrm{P}(Z < 0.045677062\sqrt{n}) > 0.8$ $0.045677062\sqrt{n} > 0.8416212335$ $n > 339.4978624$ $\therefore$ Least n is 340.