6

millennia institute

CANDIDATE NAME

CLASS

ADMISSION NUMBER

2021 Preliminary Examination
Pre-University 3

MATHEMATICS **9758/01**

Paper 1

14 September 2021

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	*	Total
Score													
Max Score	5	7	7	8	8	9	8	10	12	12	14		100

This document consists of **27** printed pages and **1** blank pages.

1 **(i)** Find the derivative of $\frac{1}{4-x^2}$. [1]

(ii) Hence find $\int \frac{x^2}{\left(4-x^2\right)^2}\,\mathrm{d}x$. [4]

2 **(a)** The curve $y=\mathrm{f}(x)$ cuts the axes at $(a,0)$ and $(0,b)$. State, if it is possible to do so, the coordinates of the points where the following curves cut the axes.

(i) $y=\mathrm{f}(x)-b$ [1]

(ii) $y=\mathrm{f}(ax)$ [1]

(b) The functions g and h are defined by

$$\mathrm{g}: x \mapsto 3-\frac{1}{x-1}, \quad x\in\mathbb{R},\ x\neq 1,$$
$$\mathrm{h}: x \mapsto 2-x, \quad x\in\mathbb{R}.$$

(i) Show that the composite function hg exists. [2]

(ii) Find an expression for hg(x) and hence find $(\mathrm{hg})^{-1}(3)$. [3]

3 **(i)** It is given that $x^2\frac{\mathrm{d}y}{\mathrm{d}x}-3xy+4=0$. Using the substitution $y=ux^3$, show that the differential equation can be transformed to $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{c}{x^5}$, where c is a constant to be determined. [3]

(ii) Hence given that $y=3$ when $x=1$, solve the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x}-3xy+4=0$ to find y in terms of x. [4]

4 With respect to the origin O, the position vectors of the points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Point C lies on AB such that $AC:CB=1:2$. It is given that $\mathbf{a}$ is a unit vector and the length of OB is 2 units.

(i) Give a geometrical interpretation of $|\mathbf{a}\cdot\mathbf{c}|$. [1]

(ii) It is given that the angle AOB is $60°$. By considering $(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b})$, find $|2\mathbf{a}-\mathbf{b}|$. [3]

(iii) Find $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [1]

(iv) Hence by considering cosine of angle AOC and cosine of angle COB, determine if the line segment OC bisects the angle AOB. [3]

5 **(i)** By using the substitution $x = \tan\theta$, show that $\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$ can be written as $\int \sec\theta\,\mathrm{d}\theta$. Hence find $\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x$. [4]

(ii) The finite region R is bounded by the curve $y=\sqrt{\frac{1}{(x-1)^2}-1}$, the line $y=\frac{1}{\sqrt{3}}$ and the y-axis. By referring to your answer in part **(i)**, find the exact volume of the solid generated when R is rotated through 2π radians about the y-axis. [4]

6 **(i)** Sketch the curve with equation $y=\left|\frac{1}{a-x}\right|$, where a is a positive constant. State, in terms of a, the equations of the asymptotes and coordinates of any intersections with the x-axis and y-axis. On the same diagram, sketch the line with equation $y=b(x-a)$, where b is a positive constant. [4]

(ii) Find, in terms of a and b, the root of the equation $\left|\frac{1}{a-x}\right|=b(x-a)$. [3]

(iii) Hence solve the inequality $\left|\frac{1}{a-x}\right|>b(x-a)$. [2]

7 A sequence $u_1, u_2, u_3, \ldots$ is such that $u_n = a^{n+1} - (n+1)^a$, where a is a constant and $n \geq 1$.

(i) Given that $u_1 = 0$, find u_3. [2]

For the rest of this question, let $a = 2$. It is given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.

(ii) Find $\sum_{r=1}^{n} u_r$ in terms of n. (You need not simplify your answer.) [4]

(iii) Another sequence $v_1, v_2, v_3, \ldots$ is such that $\sum_{r=1}^{9}(v_{r+1} - v_r) = \sum_{r=1}^{10} u_r$. Given that $v_1 = u_1$, find the value of v_{10}. [2]

8 A curve C has parametric equations

$$x = \cot t + 2, \qquad y = \sec t, \qquad -\frac{\pi}{2} < t < 0.$$

(i) Sketch the graph of C, indicating the equations of any asymptotes. [2]

(ii) Show that $\frac{dy}{dx} = -\frac{\sin^3 t}{\cos^2 t}$. Hence explain why C is increasing for $-\frac{\pi}{2} < t < 0$. [3]

(iii) Find the equation of normal to C when $t = -\frac{\pi}{4}$. [2]

(iv) The point P on C has coordinates $(\cot p + 2, \ \sec p)$. Given that the point R is the midpoint of P and the point with coordinates $(-2, \ 0)$, find the cartesian equation of the curve traced by R as p varies. [3]

9 **Do not use a calculator in answering this question.**

(a) Two complex numbers are $w = 1 - \sqrt{3}\,\mathrm{i}$ and $z = \sqrt{2}\left(\cos\frac{3}{4}\pi - \mathrm{i}\sin\frac{3}{4}\pi\right)$.

(i) Find $w^2 z^*$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(ii) Show that there is no integer value of n for which the real part of w^n is zero. [2]

(b) **(i)** One of the roots of the equation $3z^3 + 13z^2 + 20z + 14 = 0$ is $-1 + \mathrm{i}$. Find the other roots of the equation in cartesian form, $p + \mathrm{i}q$, showing your working. [4]

(ii) Hence find the roots of the equation $w^3 + 13w^2 + 60w + 126 = 0$. [2]

10 A RC series circuit comprises a power source of V volts in series with a resistor of R ohms and capacitor of C farads. When the power is switched on and power is supplied to the capacitor, the charge builds up in the capacitor. At t seconds after the power is switched on, the charge on the capacitor is q coloumbs and the current in the circuit is I amps. It is given that R and C are constants.

A differential equation for a RC series circuit is $RI + \frac{q}{C} = V$, where $I = \frac{\mathrm{d}q}{\mathrm{d}t}$.

(i) Find the maximum value of q in terms of C and V. (You do not need to prove that it is a maximum.) [2]

(ii) Show that, under certain conditions on V which should be stated, $R\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{I}{C} = 0$. [2]

(iii) In a particular circuit, $I = \frac{V}{R}$ when $t = 0$. Solve the differential equation in part **(ii)** and find I in terms of R, C, V and t. [5]

(iv) Sketch the graph of I against t. [2]

(v) Describe what happens to the current in the circuit after a long time. [1]

11 Bank A offers a study loan to students enrolled in an undergraduate course of study. The key features of the loan are:

- interest-free during the course of study,
- fixed monthly interest of 0.3% upon graduation,
- minimum monthly repayment of $100.

Ali decides to take a study loan of $50000 from Bank A on 1 January 2021 at the start of his 3-year undergraduate course.

(a) During his course of study, Ali pays $200 at the end of January 2021 and on the last day of each subsequent month, he pays $10 more than in the previous month. Thus on 28 February 2021, he pays $210 and on 31 March 2021, he pays $220, and so on. How much does Ali owe the bank at the end of his course of study? [2]

(b) Bank A charges interest on any outstanding amount of the loan on the first day of each month, starting on the month right after a student graduates.

(i) Upon graduation, Ali immediately found a job and decides to pay $900 to bank A on the last day of each month, starting on the month right after he graduated. Show that Ali owes the bank $34121 (to the nearest dollar) on the last day of the 3rd month after he graduated. [2]

(ii) Use the formula for the sum of a geometric progression to find an expression for the amount owed by Ali on the last day of the nth month after he graduated. Hence find in which month Ali pays off his study loan. [5]

(iii) Find the total amount of interest that Ali paid. [2]

(iv) If Ali decides to pay off his study loan within 3 years upon graduation. i.e. at the end of December 2026, what is the minimum amount, to the nearest dollar, that he needs to pay per month after graduation? [3]

End of Paper

2021 Preliminary Examination
PU3 MATHEMATICS 9758/01
Solutions

Qn	Solution
1(i) [1]	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{4-x^2}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(4-x^2\right)^{-1}$ $=\frac{2x}{\left(4-x^2\right)^2}$
1(ii) [4]	$\int \frac{x^2}{\left(4-x^2\right)^2}\,\mathrm{d}x$ Let $u=\frac{1}{2}x$, $\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{2x}{\left(4-x^2\right)^2}$ $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{2}$, $v=\frac{1}{4-x^2}$ $=\int\left(\frac{1}{2}x\right)\left[\frac{2x}{\left(4-x^2\right)^2}\right]\mathrm{d}x$ $=\left(\frac{1}{2}x\right)\left(\frac{1}{4-x^2}\right)-\int\left(\frac{1}{2}\right)\left(\frac{1}{4-x^2}\right)\mathrm{d}x$ $=\frac{x}{2\left(4-x^2\right)}-\frac{1}{2}\int\frac{1}{4-x^2}\,\mathrm{d}x$ $=\frac{x}{2\left(4-x^2\right)}-\frac{1}{2}\left(\frac{1}{2(2)}\right)\ln\left\|\frac{2+x}{2-x}\right\|+c$ $=\frac{x}{2\left(x^2+4\right)}-\frac{1}{8}\ln\left\|\frac{2+x}{2-x}\right\|+c$

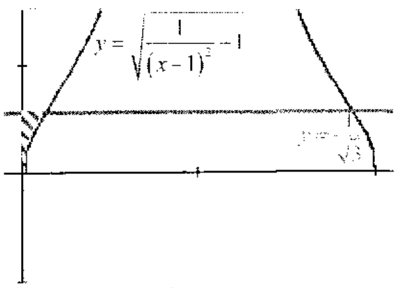
Qn	Solution
2(a) (i) [1]	$(0,b)\xrightarrow{\text{translate } b \text{ units in negative } y \text{ direction}}(0,0)$ Note: $(a,0)\to(a,-b)$ (not required)
2(ii) [1]	$(a,0)\xrightarrow{\text{scale parallel to } x \text{ axis by scale factor } \frac{1}{a}}(1,0)$ $(0,b)\xrightarrow{\text{scale parallel to } x \text{ axis by scale factor } \frac{1}{a}}(0,b)$
2(b) (i) [2]	Range of g $=(-\infty,3)\cup(3,\infty)$ or $\mathbb{R}\setminus\{3\}$ Domain of h $=(-\infty,\infty)$ Since range of g $\subseteq$ domain of h $\Rightarrow$ hg exists
2(b) (ii) [3]	$\mathrm{hg}(x)=2-\left(3-\frac{1}{x-1}\right)=-1+\frac{1}{x-1}$

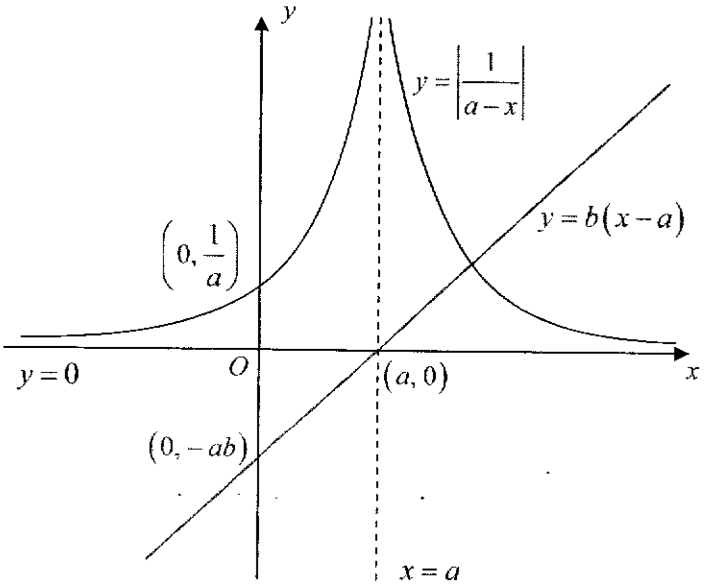
Qn	Solution
	Method 1 $(\text{hg})^{-1}(3)=a$ $\text{hg}\left((\text{hg})^{-1}(3)\right)=\text{hg}(a)$ $3=-1+\dfrac{1}{a-1}$ $\dfrac{1}{a-1}=4$ $a=\dfrac{5}{4}$ ----------------------------------- **Method 2** To find $(\text{hg})^{-1}$ Let $y=-1+\dfrac{1}{x-1}$ $y+1=\dfrac{1}{x-1}$ $x-1=\dfrac{1}{y+1}$ $x=\dfrac{1}{y+1}+1$ $(\text{hg})^{-1}(x)=\dfrac{1}{x+1}+1$ $(\text{hg})^{-1}(3)=\dfrac{1}{3+1}+1=\dfrac{5}{4}$

Qn	Solution
3(i) [3]	$y=ux^3 \Rightarrow \dfrac{dy}{dx}=x^3\dfrac{du}{dx}+3ux^2$ $x^2\left(x^3\dfrac{du}{dx}+3ux^2\right)-3x\left(ux^3\right)+4=0$ $x^5\dfrac{du}{dx}+3ux^4-3ux^4+4=0$ $x^5\dfrac{du}{dx}+4=0$ $\dfrac{du}{dx}=-\dfrac{4}{x^5}$
3(ii) [4]	$\dfrac{du}{dx}=-\dfrac{4}{x^5}$

Qn	Solution
	Integrating with respect to x on both sides of the equation, $u=\int -\frac{4}{x^5}\,dx$ $=\int -4x^{-5}\,dx$ $=x^{-4}+c$ $y=ux^3 \Rightarrow u=\frac{y}{x^3}$ $\frac{y}{x^3}=x^{-4}+c$ $y=x^{-1}+cx^3$ When $x=1$, $y=3$, $c=2$ $\therefore y=x^{-1}+2x^3$

Qn	Solution
4(i) [1]	$\lvert\mathbf{a}\cdot\mathbf{c}\rvert$ is the length of projection of $\mathbf{c}$ onto $\mathbf{a}$ OR $\lvert\mathbf{a}\cdot\mathbf{c}\rvert$ is the length of projection of $\overline{OC}$ onto $\overline{OA}$.
4(ii) [3]	$(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b})=4(\mathbf{a}\cdot\mathbf{a})-2(\mathbf{a}\cdot\mathbf{b})-2(\mathbf{b}\cdot\mathbf{a})+(\mathbf{b}\cdot\mathbf{b})$ $=4\lvert\mathbf{a}\rvert^2-4(\mathbf{a}\cdot\mathbf{b})+\lvert\mathbf{b}\rvert^2 \quad (\because\ \mathbf{b}\cdot\mathbf{a}=\mathbf{a}\cdot\mathbf{b})$ $=4(1)^2-4\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos 60^\circ+(2)^2$ $=4-4(1)(2)\left(\frac{1}{2}\right)+4=4.$ $(2\mathbf{a}-\mathbf{b})\cdot(2\mathbf{a}-\mathbf{b})=\lvert 2\mathbf{a}-\mathbf{b}\rvert^2=4 \Rightarrow \lvert 2\mathbf{a}-\mathbf{b}\rvert=2.$
4(iii) [1]	By Ratio Theorem, $\mathbf{c}=\frac{\mathbf{b}+2\mathbf{a}}{3}.$

4(iv) **[3]**	$\cos\angle AOC = \dfrac{\mathbf{a}\cdot\mathbf{c}}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert}$ $= \dfrac{\mathbf{a}\cdot\left(\dfrac{\mathbf{b}+2\mathbf{a}}{3}\right)}{\lvert\mathbf{a}\rvert\lvert\mathbf{c}\rvert}$ [from **(iii)**] $= \dfrac{1}{3}\left(\dfrac{\mathbf{a}\cdot\mathbf{b}+2(1^2)}{\lvert\mathbf{c}\rvert}\right)$ [since $\mathbf{a}\cdot\mathbf{a} = \lvert\mathbf{a}\rvert^2 = 1^2$] $= \dfrac{1}{3}\left(\dfrac{\mathbf{a}\cdot\mathbf{b}+2}{\lvert\mathbf{c}\rvert}\right)$. $\cos\angle COB = \dfrac{\mathbf{b}\cdot\mathbf{c}}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert}$ $= \dfrac{\mathbf{b}\cdot\left(\dfrac{\mathbf{b}+2\mathbf{a}}{3}\right)}{\lvert\mathbf{b}\rvert\lvert\mathbf{c}\rvert}$ [from **(iii)**] $= \dfrac{1}{3}\left(\dfrac{2^2+2(\mathbf{a}\cdot\mathbf{b})}{2\lvert\mathbf{c}\rvert}\right)$ [since $\mathbf{b}\cdot\mathbf{b} = \lvert\mathbf{b}\rvert^2 = 2^2$] $= \dfrac{1}{3}\left(\dfrac{2+\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{c}\rvert}\right) = \cos\angle AOC$. Therefore line OC bisects the angle AOB.
Qn	**Solution**
5(i) **[4]**	$x = \tan\theta \Rightarrow \dfrac{dx}{d\theta} = \sec^2\theta$ $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{\tan^2\theta+1}}\left(\sec^2\theta\right)\,d\theta$ $\displaystyle= \int \frac{1}{\sec\theta}\left(\sec^2\theta\right)\,d\theta$ $\displaystyle= \int \sec\theta\,d\theta$ $= \ln\lvert\sec\theta+\tan\theta\rvert + c$ ☐ $= \ln\left\lvert\sqrt{x^2+1}+x\right\rvert + c$ Alternatively, $1+\tan^2\theta = \sec^2\theta \Rightarrow \sec\theta = \sqrt{1+x^2}$.
5(ii) **[4]**	$y = \sqrt{\dfrac{1}{(x-1)^2}-1}$ $y^2 = \dfrac{1}{(x-1)^2}-1$ $(x-1)^2 = \dfrac{1}{y^2+1}$ $x-1 = \pm\dfrac{1}{\sqrt{y^2+1}}$ 

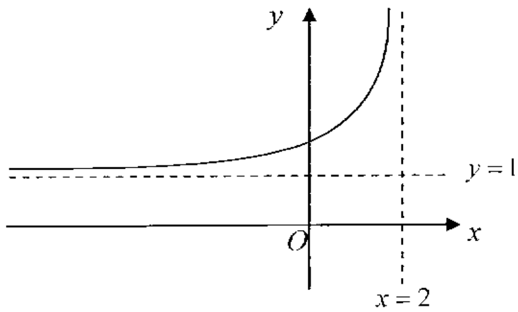
	$x=1-\frac{1}{\sqrt{y^2+1}}$ $(\because x<1)$ Volume of solid $=\pi\int_0^{\frac{1}{\sqrt{3}}}\left(1-\frac{1}{\sqrt{y^2+1}}\right)^2 \mathrm{d}y=\pi\int_0^{\frac{1}{\sqrt{3}}}1-\frac{2}{\sqrt{y^2+1}}+\frac{1}{y^2+1}\mathrm{d}y$ $=\pi\left[y-2\ln\left\|\sqrt{y^2+1}+y\right\|+\tan^{-1}y\right]_0^{\frac{1}{\sqrt{3}}}$ [from part (i)] $=\pi\left(\frac{1}{\sqrt{3}}-2\ln\left(\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+1}+\frac{1}{\sqrt{3}}\right)+\tan^{-1}\frac{1}{\sqrt{3}}\right)-0$ $=\pi\left(\frac{1}{\sqrt{3}}-2\ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)+\frac{\pi}{6}\right)$ $=\pi\left(\frac{1}{\sqrt{3}}-2\ln\sqrt{3}+\frac{\pi}{6}\right)$ OR $\pi\left(\frac{1}{\sqrt{3}}-\ln 3+\frac{\pi}{6}\right)$ units^3.
Qn	**Solution**
6(i) **[4]**	
6(ii) **[3]**	**Method 1**: Simplifying the modulus using the graph From the graph, the root of $\left\|\frac{1}{a-x}\right\|=b(x-a)$ is equal to the root of $-\left(\frac{1}{a-x}\right)=b(x-a)\Rightarrow\frac{1}{b}=(x-a)^2\Rightarrow x=a\pm\frac{1}{\sqrt{b}}$. Since $x>a$, $x=a+\frac{1}{\sqrt{b}}$. --- **Method 2**: Solve by squaring both sides

	$\left\|\frac{1}{a-x}\right\| = b(x-a) \Rightarrow \left(\frac{1}{a-x}\right)^2 = b^2(x-a)^2$ $\Rightarrow \left(\frac{1}{b}\right)^2 = (x-a)^2(a-x)^2 = (x-a)^4$ $\Rightarrow \frac{1}{b} = (x-a)^2$ (since $b > 0$) $\Rightarrow x = a \pm \frac{1}{\sqrt{b}}$. Since $x > a$, $x = a + \frac{1}{\sqrt{b}}$.
6(iii) **[2]**	From the sketch in part **(i)**, $x < a$ or $a < x < a + \frac{1}{\sqrt{b}}$ OR $x < a + \frac{1}{\sqrt{b}}$, $x \neq a$

Qn	Solution
7(i) **[2]**	$u_n = a^{n+1} - (n+1)^a$ $u_1 = 0 \Rightarrow a^2 - 2^a = 0 \Rightarrow a^2 = 2^a$ … (*) **Method 1:** $u_3 = a^4 - 4^a = (a^2)^2 - 4^a$ $= (2^a)^2 - 4^a$ [from (*)] $= (2^2)^a - 4^a = 4^a - 4^a = 0$. **Method 2:** $u_3 = a^4 - 4^a = (a^2)^2 - (2^2)^a$ $= (a^2)^2 - (2^a)^2$ $= (a^2 - 2^a)(a^2 + 2^a)$ $= 0$. [since $a^2 - 2^a = 0$ from (*)]
7(ii) **[4]**	$u_n = 2^{n+1} - (n+1)^2$ **Method 1**: Expansion

Qn	Solution
	$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left[2^{r+1}-(r+1)^2\right]$ $=\sum_{r=1}^{n} 2^{r+1}-\sum_{r=1}^{n}(r+1)^2$ $=\sum_{r=1}^{n} 2^{r+1}-\sum_{r=1}^{n}\left(r^2+2r+1\right)$ $=\sum_{r=1}^{n} 2^{r+1}-\left(\sum_{r=1}^{n} r^2+2\sum_{r=1}^{n} r+\sum_{r=1}^{n} 1\right)$ $=\frac{4\left(2^n-1\right)}{2-1}-\left[\frac{n}{6}(n+1)(2n+1)+2\left(\frac{n}{2}(n+1)\right)+n\right]$ $=4\left(2^n-1\right)-\frac{n}{6}(n+1)(2n+1)-n(n+1)-n$ OR $4\left(2^n-1\right)-\frac{n}{6}(n+1)(2n+1)-n^2-2n$ OR $4\left(2^n-1\right)-\frac{n}{6}(n+1)(2n+1)-n(n+2)$ **Method 2**: Change of index $\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left[2^{r+1}-(r+1)^2\right]$ $=\sum_{r=1}^{n} 2^{r+1}-\sum_{r=1}^{n}(r+1)^2$ $=\sum_{r=1}^{n} 2^r\times 2-\sum_{r=2}^{n+1} r^2$ $=2\sum_{r=1}^{n} 2^r-\left[\left(\sum_{r=1}^{n+1} r^2\right)-1^2\right]$ $=2\left[\frac{2\left(2^n-1\right)}{2-1}\right]-\left[\frac{n+1}{6}(n+1+1)\left(2(n+1)+1\right)-1\right]$ $=4\left(2^n-1\right)-\frac{1}{6}(n+1)(n+2)(2n+3)+1$

Qn	Solution
7(iii) [2]	$\sum_{r=1}^{9}(v_{r+1}-v_r) = v_2 - v_1$ $+v_3 - v_2$ $+\ldots$ $+v_9 - v_8$ $+v_{10} - v_9$ $= v_{10} - v_1$ $\sum_{r=1}^{10} u_r = 3587$. [from **(ii)** or from graphing calculator] Hence, $v_{10} - v_1 = 3587$ $\Rightarrow v_{10} = 3587 + v_1 = 3587 + u_1$ [since $v_1 = u_1$ (given)] $\Rightarrow v_{10} = 3587 + [2^{1+1} - (1+1)^2]$ (since $a = 2$) $\Rightarrow v_{10} = 3587 + 0 = 3587$.

Qn	Solution
8(i) [2]	
8(ii) [3]	$x = \cot t + 2 \qquad y = \sec t$ $\frac{dx}{dt} = -\operatorname{cosec}^2 t \qquad \frac{dy}{dt} = \sec t \tan t$ $\frac{dy}{dx} = \frac{\sec t \tan t}{-\operatorname{cosec}^2 t}$ $= \frac{\left(\frac{1}{\cos t}\right)\left(\frac{\sin t}{\cos t}\right)}{\left(-\frac{1}{\sin^2 t}\right)}$ $= -\frac{\sin^3 t}{\cos^2 t}$ Since $-\frac{\pi}{2} < t < 0$, $\sin^3 t < 0$. $\cos^2 t > 0$ Therefore $\frac{dy}{dx} > 0$, C is increasing
8(iii)	**Method 1**

Qn	Solution
[2]	Using GC, At $t=-\frac{\pi}{4}$, $x=1$, $y=1.4142$, $\frac{dy}{dx}=0.70711$, Gradient of normal $\frac{-1}{0.70711}=-1.4142$ Equation of normal $y-1.4142=-1.4142(x-1)$ $y=-1.4142x+2.8284$ $y=-1.41x+2.83$ (3s.f.) **Method 2** At $t=-\frac{\pi}{4}$, $x=1$, $y=\sec\frac{\pi}{4}=\sqrt{2}$, $\frac{dy}{dx}=-\frac{\sin^3\left(-\frac{\pi}{4}\right)}{\cos^2\left(-\frac{\pi}{4}\right)}=\frac{1}{\sqrt{2}}$, Gradient of normal $-\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}=-\sqrt{2}$ Equation of normal $y-\sqrt{2}=-\sqrt{2}(x-1)$ $y=-\sqrt{2}x+2\sqrt{2}$
8(iv) [3]	Midpoint R $=\left(\frac{(\cot p+2)-2}{2},\frac{\sec p}{2}\right)$ $=\left(\frac{\cot p}{2},\frac{\sec p}{2}\right)$ $x=\frac{\cot p}{2}$ $\quad y=\frac{\sec p}{2}$ $\tan p=\frac{1}{2x}$ $\quad \sec p=2y$ **Method 1** Using trigonometric identity, $\tan^2 p+1=\sec^2 p$ $\left(\frac{1}{2x}\right)^2+1=(2y)^2$ $4y^2=\frac{1}{4x^2}+1$ --

Qn	Solution
	Method 2 $\tan p=\dfrac{1}{2x}$ Using the right angle triangle, $\dfrac{1}{\left(\dfrac{2x}{\sqrt{4x^2+1}}\right)}=2y$ ☐ $y=\dfrac{\sqrt{4x^2+1}}{4x}$ --- **Method 3** $\cos p=\dfrac{1}{2y}$ Using the right angle triangle, $\tan p=\dfrac{1}{2x}$ ☐ $\dfrac{1}{\sqrt{4y^2-1}}=2x$ $\dfrac{1}{4y^2-1}=4x^2$

Qn	Solution
9(a) (i) [4]	For $w=1-\sqrt{3}\,\mathrm{i}$: $\lvert w\rvert=\lvert 1-\sqrt{3}\,\mathrm{i}\rvert=\sqrt{1^2+\left(\sqrt{3}\right)^2}=2$ $\arg(w)=\arg\left(1-\sqrt{3}\,\mathrm{i}\right)$ $=-\tan^{-1}\left(\dfrac{\sqrt{3}}{1}\right)=-\dfrac{1}{3}\pi.$ For $z=\sqrt{2}\left(\cos\dfrac{3}{4}\pi-\mathrm{i}\sin\dfrac{3}{4}\pi\right)$: $z=\sqrt{2}\left(\cos\dfrac{3}{4}\pi-\mathrm{i}\sin\dfrac{3}{4}\pi\right)=\sqrt{2}\left(\cos\left(-\dfrac{3}{4}\pi\right)+\mathrm{i}\sin\left(-\dfrac{3}{4}\pi\right)\right)$ Therefore $\lvert z\rvert=\sqrt{2}$ and $\arg(z)=-\dfrac{3}{4}\pi$. **Method 1** $\lvert w^2z^*\rvert=\lvert w\rvert^2\lvert z\rvert=2^2\left(\sqrt{2}\right)=4\sqrt{2}$

Qn	Solution
	$\arg(w^2 z^*) = \arg(w^2) + \arg(z^*)$ $= 2\arg(w) - \arg(z)$ $= 2\left(-\frac{1}{3}\pi\right) - \left(-\frac{3}{4}\pi\right) = \frac{1}{12}\pi.$ Therefore, $w^2 z^* = 4\sqrt{2}\left(\cos\frac{1}{12}\pi + \mathrm{i}\sin\frac{1}{12}\pi\right).$ **Method 2** $w = 2e^{-\frac{\pi}{3}\mathrm{i}},\ z = \sqrt{2}e^{-\frac{3\pi}{4}\mathrm{i}} \Rightarrow z^* = \sqrt{2}e^{\frac{3\pi}{4}\mathrm{i}}$ $w^2 z^* = \left(2e^{-\frac{\pi}{3}\mathrm{i}}\right)^2\left(\sqrt{2}e^{\frac{3\pi}{4}\mathrm{i}}\right)$ $= 4e^{-\frac{2\pi}{3}\mathrm{i}}\left(\sqrt{2}e^{\frac{3\pi}{4}\mathrm{i}}\right)$ $= 4\sqrt{2}\ e^{\frac{\pi}{12}\mathrm{i}}$ $w^2 z^* = 4\sqrt{2}\left(\cos\frac{1}{12}\pi + \mathrm{i}\sin\frac{1}{12}\pi\right)$
9(a) (ii) [2]	**Method 1** $w^n = 2^n\left[\cos\left(-\frac{n\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{n\pi}{3}\right)\right]$ For $\mathrm{Re}(w^n) = 0$, $\mathrm{Re}(w^n) = 2^n\cos\left(-\frac{n\pi}{3}\right) = 0$ $\cos\left(-\frac{n\pi}{3}\right) = 0$ $-\frac{n\pi}{3} = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$ $-\frac{n\pi}{3} = \frac{(2k+1)\pi}{2}$, where $k \in \mathbb{Z}$ $n = -\frac{3(2k+1)}{2}$ Since $3(2k+1)$ is odd for all $k \in \mathbb{Z}$, $\frac{3(2k+1)}{2}$ is never an integer. Thus there is no integer value of n such that the real part of w^n is zero. -- **Method 2** For the real part of w^n to be zero, this means w^n is purely imaginary.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>

$\arg\left(w^n\right) = \ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$= (2k+1)\frac{\pi}{2}$, where $k \in \mathbb{Z}$

Since $\arg\left(w^n\right) = n\arg(w) = -\frac{n}{3}\pi$,

$-\frac{n}{3}\pi = (2k+1)\frac{\pi}{2}$

$-\frac{n}{3} = \frac{2k+1}{2}$

$n = -\frac{3(2k+1)}{2}$

Since $3(2k+1)$ is odd for all $k \in \mathbb{Z}$, $\frac{3(2k+1)}{2}$ is never an integer. Thus there is no integer value of n such that the real part of w^n is zero.

</td></tr>
<tr><td>9(b)
(i)
[4]</td><td>

$3z^3 + 13z^2 + 20z + 14 = 0$

Since all coefficients of the equation are real and $-1+\mathrm{i}$ is a root, $-1-\mathrm{i}$ is another root.

A quadratic factor:

$[z-(-1+\mathrm{i})][z-(-1-\mathrm{i})] = [z+1-\mathrm{i}][z+1+\mathrm{i}]$

$= [(z+1)-\mathrm{i}][(z+1)+\mathrm{i}]$

$= (z+1)^2 - \mathrm{i}^2$

$= z^2 + 2z + 1 - (-1)$

$= z^2 + 2z + 2$

$\Rightarrow 3z^3 + 13z^2 + 20z + 14 = (z^2+2z+2)(az+b)$

<table>
<tr><th>Method 1</th><th>Method 2</th></tr>
<tr><td>Comparing z^3 terms:
$a = 3$

Comparing constant terms:
$2b = 14$
$\Rightarrow b = 7$</td><td>$$\begin{array}{r} 3z+7 \\ z^2+2z+2 \overline{)\,3z^3+13z^2+20z+14} \\ -(3z^3+6z^2+6z) \\ \hline 7z^2+14z+14 \\ -(7z^2+14z+14) \\ \hline 0 \end{array}$$</td></tr>
</table>

We have. $3z^3 + 13z^2 + 20z + 14 = (z^2+2z+2)(3z+7)$. $3z+7=0 \Rightarrow z = -\frac{7}{3}$.

Therefore, the other roots are $-1-\mathrm{i}$ and $-\frac{7}{3}$.

</td></tr>
<tr><td>9(b)
(ii)
[2]</td><td>

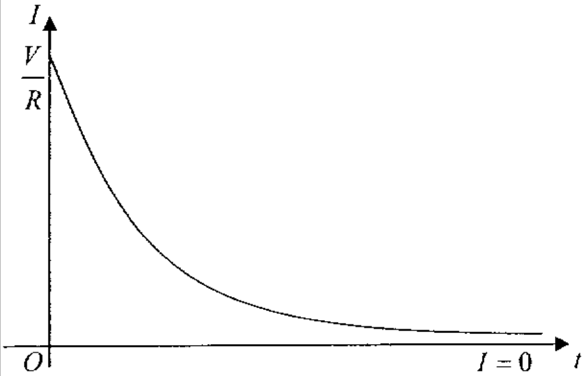
Given $w^3 + 13w^2 + 60w + 126 = 0$

Divide 9 throughout:

</td></tr>
</table>

Qn	Solution
	$\frac{1}{9}w^3+\frac{13}{9}w^2+\frac{60}{9}w+\frac{126}{9}=0$ $\frac{1}{9}w^3+\frac{13}{9}w^2+\frac{20}{3}w+14=0$ $3\left(\frac{1}{3}w\right)^3+13\left(\frac{1}{3}w\right)^2+20\left(\frac{1}{3}w\right)+14=0$ So, z in $3z^3+13z^2+20z+14$ has been replaced with $\frac{1}{3}w$. $z=-1+\mathrm{i}, \quad z=-1-\mathrm{i}, \quad z=-\frac{7}{3}$ $\frac{1}{3}w=-1+\mathrm{i},\ \frac{1}{3}w=-1-\mathrm{i},\ \frac{1}{3}w=-\frac{7}{3}$ Therefore, $w=-3+3\mathrm{i},\ w=-3-3\mathrm{i},\ w=-7$.

Qn	Solution
10(i) **[2]**	To find maximum q, $\frac{\mathrm{d}q}{\mathrm{d}t}=0$, i.e. $I=0$. $0+\frac{q}{C}=V$ $q=VC$
10(ii) **[2]**	$RI+\frac{q}{C}=V$ Differentiate with respect to t, $R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{1}{C}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)=\frac{\mathrm{d}V}{\mathrm{d}t}$ If V is a constant, i.e. $\frac{\mathrm{d}V}{\mathrm{d}t}=0$ $R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{1}{C}(I)=0$ $R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{I}{C}=0$, since $I=\frac{\mathrm{d}q}{\mathrm{d}t}$

Qn	Solution
10(iii) [5]	$R\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{I}{C}=0$ $\frac{\mathrm{d}I}{\mathrm{d}t}=-\frac{I}{RC}$ $\int\frac{1}{I}\,\mathrm{d}I=\int-\frac{1}{RC}\,\mathrm{d}t$ $\ln\lvert I\rvert=-\frac{1}{RC}t+d$ $\lvert I\rvert=\mathrm{e}^{-\frac{1}{RC}t+d}=\mathrm{e}^{-\frac{1}{RC}t}\mathrm{e}^{+d}$ $I=A\mathrm{e}^{-\frac{1}{RC}t}$ When $t=0,\ I=\frac{V}{R}$ $A=\frac{V}{R}$ $\therefore I=\frac{V}{R}\mathrm{e}^{-\frac{1}{RC}t}$
10(iv) [2]	
10(v) [1]	As $t\to\infty,\ I\to 0$. In the long run, the current in the circuit tends to/approaches 0 amp.

Qn	Solution
11(a) [2]	Amount of money Ali paid at the end of 3 years $=\frac{36}{2}\left[2(200)+(36-1)(10)\right]$ $=13500$ Amount Ali owes the bank at the end of 3 years $=50000-13500$ $=36500$
11(b)(i)	At the end of 1 month, amount owed $=1.003(36500)-900$ $=35709.50$

Qn	Solution

[2]

At the end of 2 months, amount owed

$= 1.003(35709.50) - 900$

$= 34916.6285$

At the end of 3 months, amount owed

$= 1.003(34916.6285) - 900$

$= 34121.38$

$= 34121$ (nearest dollar)

11 (b) (ii) [5]

Month	Amount owed at the start of month	Amount owed at the end of month
1	$1.003(36500)$	$1.003(36500)-900$
2	$1.003[1.003(36500)-900]$ $=1.003^2(436500)-1.003(900)$	$1.003^2(36500)-1.003(900)$ -900
3	$1.003[1.003^2(36500)-1.003(900)-900]$ $=1.003^3(36500)-1.003^2(900)-1.003(900)$	$1.003^3(36500)-1.003^2(900)-1.003(900)$ -900
...	...	...
n		$1.003^n(36500)-1.003^{n-1}(900)$ $-1.003^{n-2}(900)-...-900$

On the last day of the nth month, Ali owed

$1.003^n(36500)-1.003^{n-1}(900)-1.003^{n-2}(900)-...-900$

$=1.003^n(36500)-[900+1.003(900)...+1.003^{n-2}(900)+1.003^{n-1}(900)]$

$=1.003^n(36500)-900(1+1.003+...+1.003^{n-2}+1.003^{n-1})$

$=1.003^n(36500)-900\left[\frac{1.003^n-1}{1.003-1}\right]$

$=1.003^n(36500)-300000(1.003^n-1)$

When Ali pays off his study loan,

$1.003^n(36500)-300000(1.003^n-1)\le 0$

Using GC,

n	$1.003^n(36500)-300000(1.003^n-1)$
43	276.54
44	−622.6
45	−1524

Ali takes 44 months to pay off his study loan, i.e. he pays off his study loan in August 2027.

11(b) (iii) [2]

Method 1:

Since Ali pays $\$(1.003\times 276.54)$ in the last month, total interest Ali paid

$=43\times 900+(1.003\times 276.54)-36500$

$= 2477.37$ (nearest cent)

Qn	Solution
	Method 2: Since Ali pays \$276.54 in the last month, total amount he paid $=13500+43\times900+(1.003\times276.54)$ $=52477.37$ Total amount of interest Ali paid $=52477.37-50000$ $=2477.37$ (nearest cent)
11(v) [3]	Let the amount that Ali pays per month upon graduation be $\$x$. $1.003^{36}(36500)-x\left[\frac{1.003^{36}-1}{1.003-1}\right]\leq 0$ $40656.16902-37.956x\leq 0$ $x\geq 1071.14$ Ali needs to pay \$1072 per month.

H

millennia institute

CANDIDATE NAME

CLASS

ADMISSION NUMBER

2021 Preliminary Examination
Pre-University 3

MATHEMATICS **9758/02**

Paper 2

17 September 2021

3 hours

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your admission number, name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Qn No.	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	*	Total
Score													
Max Score	4	6	8	10	12	7	7	9	11	12	14		100

This document consists of **7** printed pages.

Section A: Pure Mathematics [40 marks]

1 Given that θ is a sufficiently small angle, show that

$$\frac{1}{\sin 2\theta + \cos\theta} \approx 1 + a\theta + b\theta^2,$$

where a and b are constants to be determined. [4]

2 It is given that y and x are related by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 2y + 5}{y - 2}.$$

Given that that $y = 1$ when $x = 0$, find the particular solution for the above differential equation. [6]

3 It is given that $\mathrm{f}(x) = \ln(2 + 2\sin x)$.

(i) Show that $\mathrm{f}''(x) = \dfrac{k}{1 + \sin x}$, where k is a constant to be found. [3]

(ii) Hence find the Maclaurin series for $\mathrm{f}(x)$, up to and including the term in x^3. [4]

(iii) Use the series in part **(ii)** to approximate the value of $\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$. [1]

4 The plane p passes through the points with coordinates $(-k, 2, 5)$, $(0, 2, -1)$ and $\left(-\frac{1}{2}, 3, -1\right)$, and the line l has equation $\frac{x+2}{-3} = y-2 = \frac{z-4}{k}$, where k is a constant.

(i) Show that the cartesian equation of the plane is $6x+3y+kz = 6-k$. [2]

(ii) Show that the l cannot be perpendicular to p. [2]

For the rest of this question, let $k = -2$.

(iii) Given that l meets p at point N, find the coordinates of N. [3]

(iv) Another plane π is parallel to the plane p. Given that the distance between p and π is 11 units, find the possible points of intersection between l and π. [3]

5 Parameterisation is the process of finding parametric equations of a curve. The position of a point that moves on a curve in two-dimensional space is determined by the time needed to reach the point when starting from a fixed origin.

For example, if (x, y) are the coordinates of the point, the movements of the x-coordinate and y-coordinate of the point are described by a pair of parametric equation, $x = \mathrm{f}(t)$, $y = \mathrm{g}(t)$ where t is a parameter and denotes the time. For example, to parametrise the equation of the curve $y = x^2 + 1$, let $x = t$, then $y = t^2 + 1$.

A curve D has parametric equations

$$x = t + \frac{1}{t} + 4, \quad y = t - \frac{1}{t} + 1, \quad t \leq -1.$$

A curve E has equation

$$y = x - 2 - \frac{1}{x-3}, \quad x \leq 2.$$

(i) Show that curve D and E intersect only once at $t = -1$ and hence find the coordinates of the point of intersection. [3]

(ii) Sketch the graph of curve D, indicating clearly the point of intersection found in part **(i)**. [2]

(iii) Using $x = t + 3$, parameterise the equation of E. Sketch the graph of E on the same diagram in part **(ii)**. [3]

Given that D intersects the y-axis at $t = -\sqrt{3} - 2$.

(iv) Find the area of the finite region bounded by D, E and the y-axis, giving your answer correct to four decimal places. [4]

Section B: Probability and Statistics [60 marks]

6 The events A and B are such that $\mathrm{P}(A) = 0.6$, $\mathrm{P}(A \cup B) = 0.8$ and $\mathrm{P}(A \cap B') = 0.55$.

(i) Find the probability that B occurs. [1]

(ii) Find the probability that neither A nor B occurs. [1]

A third event C is such that B and C are independent and $\mathrm{P}(C) = 0.6$.

(iii) Find $\mathrm{P}(B' \cap C)$. [2]

(iv) Hence find the range of values of $\mathrm{P}(A \cap B' \cap C)$. [3]

7 A company produces packets of almond flour with each packet weighing μ_0 grams. In a quality control inspection, the production manager wishes to check if the mean mass of almond flour per packet is overstated.

(i) Explain why the production manager should take a sample of at least 30 packets of almond flour, and state how these packets should be chosen. [2]

The production manager takes a suitable sample of 40 packets of almond flour and finds that the mean mass of almond flour per packet is 248.5 grams and its standard deviation is 4.3 grams.

(ii) Given that the production manager concludes that the mean mass of almond flour per packet is not overstated at the 1% level of significance, find the range of values of μ_0, giving your answer to 2 decimal places. [4]

(iii) Explain what is meant by "at the 1% level of significance" in the context of the question. [1]

8 A doctor prescribes a specific medication to 12 of his patients who developed a particular type of allergy in his clinic. Past records show that $100p\%$ of adults with such allergy report symptomatic relief after consuming the medication. The number of patients who reported symptomatic relief after consuming the medication is assumed to follow a binomial distribution.

(i) Write down in terms of p, the probability that 10 patients reported symptomatic relief after consuming the medication. [1]

(ii) It is given that the modal number of patients who reported symptomatic relief after consuming the medication is 10. Use this information to find exactly the possible range of values of p. [4]

Suppose now $p = 0.8$.

In another clinic, 15 patients who developed the allergy were also prescribed with the same medication.

(iii) Find the probability that all patients reported symptomatic relief after consuming the medication. [1]

In order to have a sensing of the effectiveness of the medication, the doctor in this clinic checks the number of patients who reported symptomatic relief after consuming the medication. He takes 3 random samples of 15 patients each who had been prescribed with the medication because of the allergy.

(iv) Find the probability that one of the samples has at least 8 patients who reported symptomatic relief after consuming the medication, and the other two samples each has all patients who reported symptomatic relief after consuming the medication. [3]

9 Two notes are drawn, at random and without replacement, from a bag containing n \$1 notes, two \$2 notes and one \$5 note, where $n \geq 2$. It is assumed that all the notes are identical in size. The random variable M is the absolute difference in the amount of money, in dollars, between two notes drawn.

(i) Determine the probability distribution of M, simplifying the probabilities as far as possible. [5]

(ii) Find $\mathrm{E}(M)$ and show that $\mathrm{Var}(M) = \dfrac{36\left[\mathrm{f}(n)\right]}{(n+3)^2(n+2)^2}$, where $\mathrm{f}(n)$ is a cubic polynomial to be determined. [5]

(iii) Given that $\text{Var}(M) = \frac{77}{36}$, find the value of n. [1]

10 **(i)** Sketch a scatter diagram that might be expected when x and y are related approximately as given in each of the cases (A) and (B) below. In each case, your diagram should include 6 points in the first quadrant, approximately equally spaced with respect to x. The letters a, b, c and d represent constants.

(A) $y = ax^2 + b$, where a is negative and b is positive,

(B) $y = c\ln x + d$, where c is negative and d is positive. [2]

A pot-in-pot cooler is an affordable electricity-free evaporative cooling device used to maintain a low temperature inside an inner compartment. In an experiment, water is filled in a particular type of pot-in-pot cooler and the temperature of water is measured by a thermocouple inside the cooler at different time interval. The following table gives details of the temperature of water measured over a period of time.

Time taken from the start of experiment (t hours)	0.25	0.5	0.75	1	1.5	2	3
Temperature of water (W °C)	19.5	18.4	17.1	16.6	16.1	15.7	15.4

(ii) Draw a scatter diagram for these values. Use your diagram to explain whether a linear model is appropriate to model these values. [2]

(iii) Find, correct to 4 decimal places, the product moment correlation coefficient between

(a) t and W,

(b) t^2 and W,

(c) $\ln t$ and W. [3]

(iv) Using your answers to part **(i)**, **(ii)** and **(iii)**, explain which of

$$W = at + b,\ W = ct^2 + d \text{ or } W = e\ln t + f,$$

where a, b, c, d, e and f represent constants, is the most appropriate model. Find the equation of a suitable regression line in this case. [3]

(v) Use your equation to estimate the time taken to reduce the temperature of water in the pot-in-pot cooler to 16 °C and explain whether your estimate is reliable. [2]

11 **In this question, you should assume that L and S follow independent normal distributions. You should also state clearly the mean and variance of all distributions you use.**

An oil company supplies engine oil in cans of two capacities, large and small.

The amount, L millilitres, of oil in a large can is normally distributed, where $L \sim \mathrm{N}(5000,\ \sigma^2)$. The amount, S millilitres, of oil in a small can is normally distributed, where $S \sim \mathrm{N}(1000,\ 25)$.

(i) Find $\mathrm{P}(5000-\sigma < L < 5000+2\sigma)$. [2]

(ii) The probability that a randomly chosen large can has more than 4990 millilitres of oil is 0.943. Find σ^2. [2]

Use $\sigma^2 = 40$ for the rest of the question.

(iii) Find the probability that the amount of oil in a randomly chosen small can is at most 1002 millilitres. [1]

(iv) Twenty small cans of oil are randomly chosen. Find the probability that fewer than ten cans have the amount of oil in the can to be at most 1002 millilitres. [2]

(v) Find the probability that the amount of oil in a randomly chosen large can exceeds five times the amount of oil in a randomly chosen small can by more than 30 millilitres. [3]

The oil company fills the large cans with Type A lubricating oil and small cans with Type B lubricating oil, and sells the Type A industrial lubricating oil at \$0.13 per millilitres and Type B industrial lubricating oil at \$0.05 per millilitres. The oil company supplies 6 large cans of oil and 2 small cans of oil to a manufacturing firm.

(vi) Find the probability that the total cost incurred for the manufacturing firm is at least \$3995. [4]

End of Paper

2021 Preliminary Examination
PU3 MATHEMATICS 9758/02
Solutions

Section A: Pure Mathematics

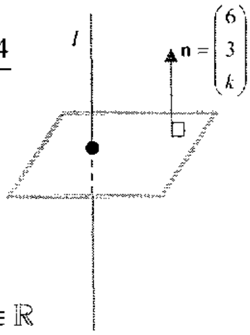
Qn	Solution
1 [4]	Given that θ is a sufficiently small angle, $\frac{1}{\sin 2\theta+\cos\theta}$ $\approx \frac{1}{2\theta+1-\frac{\theta^2}{2}}$ $\approx \left(1+2\theta-\frac{\theta^2}{2}\right)^{-1}$ $=1+(-1)\left(2\theta-\frac{\theta^2}{2}\right)+\frac{(-1)(-2)}{2!}\left(2\theta-\frac{\theta^2}{2}\right)^2+\ldots$ $=1-2\theta+\frac{\theta^2}{2}+(2\theta)^2+\ldots$ $\approx 1-2\theta+\frac{9}{2}\theta^2$ where $a=-2,\ b=\frac{9}{2}$

Qn	Solution
2 [6]	$\frac{dy}{dx}=\frac{y^2-2y+5}{y-2}$ $\frac{y-2}{y^2-2y+5}\left(\frac{dy}{dx}\right)=1$ $\int \frac{y-2}{y^2-2y+5}\,dy=\int 1\,dx$ $\int \frac{1}{2}\left[\frac{2(y-2)}{y^2-2y+5}\right]dy=\int 1\,dx$ $\int \frac{1}{2}\left[\frac{2y-4}{y^2-2y+5}\right]dy=\int 1\,dx$ $\frac{1}{2}\int\left[\frac{2y-2}{y^2-2y+5}-\frac{2}{y^2-2y+5}\right]dy=\int 1\,dx$ $\frac{1}{2}\int\left[\frac{2y-2}{y^2-2y+5}-\frac{2}{(y-1)^2+2^2}\right]dy=\int 1\,dx$

Qn	Solution
	$\frac{1}{2}\left[\ln\left\|y^2-2y+5\right\|-2\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{y-1}{2}\right)\right]=x+c$ $\frac{1}{2}\left[\ln\left(y^2-2y+5\right)-\tan^{-1}\left(\frac{y-1}{2}\right)\right]=x+c$ When $x=0,\ y=1$, $\frac{1}{2}\left[\ln 4-\tan^{-1}(0)\right]=0+c$ $c=\frac{1}{2}\ln 4=\ln 2$ $\frac{1}{2}\left[\ln\left(y^2-2y+5\right)-\tan^{-1}\left(\frac{y-1}{2}\right)\right]=x+\ln 2$

Qn	Solution
3(i) [3]	$\mathrm{f}(x)=\ln(2+2\sin x)$ $\mathrm{f}'(x)=\frac{2\cos x}{2+2\sin x}$ $=\frac{\cos x}{1+\sin x}$ **<u>Method 1: Apply Quotient Rule</u>** $\mathrm{f}''(x)=\frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$ $=\frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$ $=\frac{-\sin x-\left(\sin^2 x+\cos^2 x\right)}{(1+\sin x)^2}$ $=\frac{-\sin x-1}{(1+\sin x)^2}$ $=\frac{-(\sin x+1)}{(1+\sin x)^2}$ $=\frac{-1}{1+\sin x}$ Therefore, $k=-1$. --- **<u>Method 2: Apply Product Rule</u>**

Qn	Solution
	$\mathrm{f}'(x)=\dfrac{\cos x}{1+\sin x}=(\cos x)(1+\sin x)^{-1}$ $\mathrm{f}''(x)=(\cos x)(1+\sin x)^{-1}$ $=(\cos x)\left[-1(1+\sin x)^{-2}(\cos x)\right]+(1+\sin x)^{-1}(-\sin x)$ $=\dfrac{-\cos^2 x}{(1+\sin x)^2}-\dfrac{\sin x}{1+\sin x}$ $=\dfrac{-\cos^2 x-\sin x(1+\sin x)}{(1+\sin x)^2}$ $=\dfrac{-\cos^2 x-\sin x-\sin^2 x}{(1+\sin x)^2}$ $=\dfrac{-\sin x-(\cos^2 x+\sin^2 x)}{(1+\sin x)^2}$ $=\dfrac{-(\sin x+1)}{(1+\sin x)^2}$ $=\dfrac{-1}{1+\sin x}$ Therefore, $k=-1$.
3(ii) **[4]**	$\mathrm{f}''(x)=\dfrac{-1}{1+\sin x}=-(1+\sin x)^{-1}$ $\mathrm{f}'''(x)=(1+\sin x)^{-2}(\cos x)$ When $x=0$, $\mathrm{f}(0)=\ln(2+2\sin 0)=\ln 2$ $\mathrm{f}'(0)=\dfrac{\cos 0}{1+\sin 0}=1$ $\mathrm{f}''(0)=\dfrac{-1}{1+\sin 0}=-1$ $\mathrm{f}'''(0)=(1+\sin 0)^{-2}(\cos 0)=1$ Therefore, $\mathrm{f}(x)=\ln(2+2\sin x)$ $=\ln 2+x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\ldots$
3(iii) **[1]**	$\int_0^2 \mathrm{f}(x)\ \mathrm{d}x\approx\int_0^2\left(\ln 2+x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3\right)\ \mathrm{d}x$ $\approx 2.7196=2.72$ (3 s.f.) (from graphing calculator)

Qn	Solution
4 (i) [2]	Plane p is parallel to $\begin{pmatrix}0\\2\\-1\end{pmatrix}-\begin{pmatrix}-k\\2\\5\end{pmatrix}=\begin{pmatrix}k\\0\\-6\end{pmatrix}$ and $\begin{pmatrix}-\frac{1}{2}\\3\\-1\end{pmatrix}-\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$. Normal of p is parallel to $\begin{pmatrix}k\\0\\-6\end{pmatrix}\times\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}=\begin{pmatrix}6\\3\\k\end{pmatrix}$. Hence equation of p is $\mathbf{r}\cdot\begin{pmatrix}6\\3\\k\end{pmatrix}=\begin{pmatrix}0\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}6\\3\\k\end{pmatrix}=(0)(6)+(2)(3)+(-1)(k)=6-k$. Cartesian equation is $6x+3y+kz=6-k$. (shown)
4 (ii) [2]	l: $\frac{x+2}{-3}=y-2=\frac{z-4}{k}$ Let $\lambda=\frac{x+2}{-3}=y-2=\frac{z-4}{k}$ $x=-2-3\lambda$ $y=2+\lambda$ $z=4+k\lambda$ l: $\mathbf{r}=\begin{pmatrix}-2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\k\end{pmatrix}$, $\lambda\in\mathbb{R}$  **<u>Method 1</u>**: Show l is not parallel to normal of p. Suppose l is perpendicular to p, then l is parallel to the normal of p, i.e. $\begin{pmatrix}-3\\1\\k\end{pmatrix}=t\begin{pmatrix}6\\3\\k\end{pmatrix}$, for some $t\in\mathbb{R}$. $\Rightarrow\begin{cases}-3=6t\\1=3t\\k=tk\end{cases}\Rightarrow\begin{cases}t=-0.5\\t=\frac{1}{3}\\t=1\end{cases}$ Since there is no unique value of t, l is not parallel to the normal of p, i.e. l cannot be perpendicular to p. (shown) **<u>Method 2</u>**: Show l is not perpendicular to a direction parallel to p.

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td></td><td>Suppose l is perpendicular to p, then l is perpendicular to the $\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$. [from (i)]

Since $\begin{pmatrix}-3\\1\\-2\end{pmatrix}\bullet\begin{pmatrix}-0.5\\1\\0\end{pmatrix}=1.5+1=2.5\neq 0,$

l is not perpendicular to the $\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}$
$\Rightarrow$ l cannot be perpendicular to p. (shown)</td></tr>
<tr><td>4(iii)
[3]</td><td>$l:\ \mathbf{r}=\begin{pmatrix}-2\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\1\\-2\end{pmatrix},\ \lambda\in\mathbb{R}$

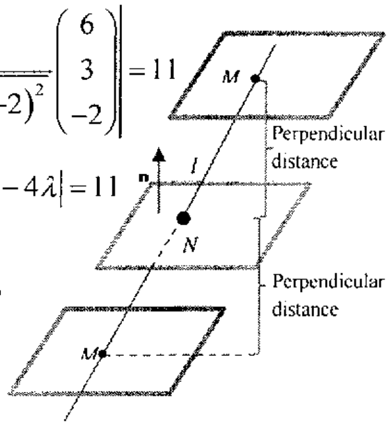
$p:\ \mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=8$

$\begin{pmatrix}-2-3\lambda\\2+\lambda\\4-2\lambda\end{pmatrix}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=8$
$-12-18\lambda+6+3\lambda-8+4\lambda=8$
$-11\lambda=22$
$\lambda=-2$
position vector of $N=\begin{pmatrix}-2-3(-2)\\2+(-2)\\4-2(-2)\end{pmatrix}=\begin{pmatrix}4\\0\\8\end{pmatrix}$
Coordinates of $N(4\ ,0,\ 8)$.</td></tr>
<tr><td>4(iv)
[3]</td><td><u>Method 1</u>: Length of Projection
Let the point of intersection of l and π be M.
Since M lies on l, $\overrightarrow{OM}=\begin{pmatrix}-2-3\lambda\\2+\lambda\\4-2\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$

$\overrightarrow{MN}=\begin{pmatrix}4\\0\\8\end{pmatrix}-\begin{pmatrix}-2-3\lambda\\2+\lambda\\4-2\lambda\end{pmatrix}=\begin{pmatrix}6+3\lambda\\-2-\lambda\\4+2\lambda\end{pmatrix}.$</td></tr>
</table>

Qn	Solution

$$\left|\begin{pmatrix}6+3\lambda\\-2-\lambda\\4+2\lambda\end{pmatrix}\bullet\frac{1}{\sqrt{6^2+3^2+(-2)^2}}\begin{pmatrix}6\\3\\-2\end{pmatrix}\right|=11$$



$$\frac{1}{7}\left|36+18\lambda-6-3\lambda-8-4\lambda\right|=11$$

$$\left|22+11\lambda\right|=77$$

$$22+11\lambda=77 \text{ or } -77$$

$$11\lambda=55 \text{ or } -99$$

$$\lambda=5 \text{ or } -9$$

$$\overrightarrow{OM}=\begin{pmatrix}-2-3(5)\\2+(5)\\4-2(5)\end{pmatrix} \text{ or } \begin{pmatrix}-2-3(-9)\\2+(-9)\\4-2(-9)\end{pmatrix}$$

$$=\begin{pmatrix}-17\\7\\-6\end{pmatrix} \text{ or } \begin{pmatrix}25\\-7\\22\end{pmatrix}$$

Hence, possible points of intersections between l and π are $(-17, 7, -6)$ and $(25, -7, 22)$.

<u>Method 2</u>: Distance between two planes

Let the equation of π be $\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=a$

Distance between p and π $=\left|\frac{8}{\sqrt{6^2+3^2+(-2)^2}}-\frac{a}{\sqrt{6^2+3^2+(-2)^2}}\right|$

$$=\left|\frac{8-a}{7}\right|$$

$$\left|\frac{8-a}{7}\right|=11$$

$$\left|8-a\right|=77$$

$$8-a=77 \text{ or } -77$$

$$a=-69 \text{ or } 85$$

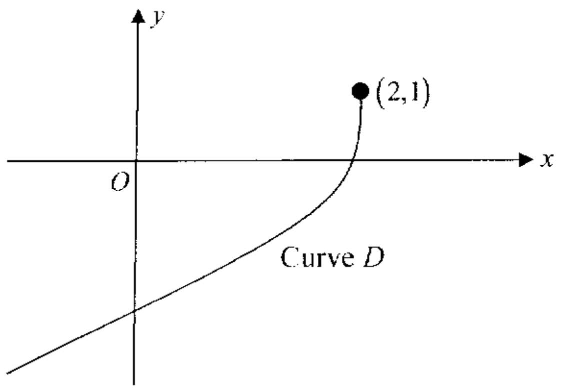
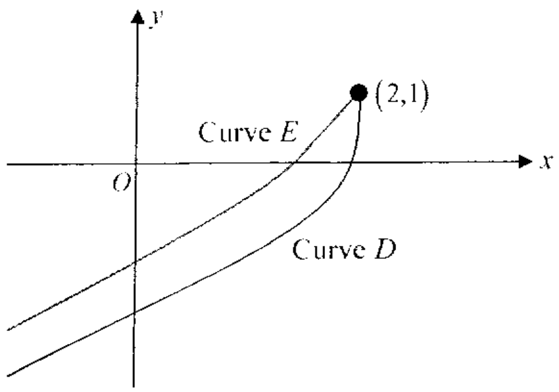
Hence possible equations of π are

$$\mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=-69 \text{ or } \mathbf{r}\bullet\begin{pmatrix}6\\3\\-2\end{pmatrix}=85$$

To find the point of intersection between l and π :

Qn	Solution
	For π: $\mathbf{r}\cdot\begin{pmatrix}6\\3\\-2\end{pmatrix}=-69$, $\begin{pmatrix}-2-3\lambda\\2+\lambda\\4-2\lambda\end{pmatrix}\cdot\begin{pmatrix}6\\3\\-2\end{pmatrix}=-69$ $-12-18\lambda+6+3\lambda-8+4\lambda=-69$ $-11\lambda=-55$ $\lambda=5$ $\overrightarrow{OM}=\begin{pmatrix}-2-3(5)\\2+(5)\\4-2(5)\end{pmatrix}=\begin{pmatrix}-17\\7\\-6\end{pmatrix}$ Similarly, for π: $\mathbf{r}\cdot\begin{pmatrix}6\\3\\-2\end{pmatrix}=85$, $\begin{pmatrix}-2-3\lambda\\2+\lambda\\4-2\lambda\end{pmatrix}\cdot\begin{pmatrix}6\\3\\-2\end{pmatrix}=85$ $-12-18\lambda+6+3\lambda-8+4\lambda=85$ $-11\lambda=99$ $\lambda=-9$ $\overrightarrow{OM}=\begin{pmatrix}-2-3(-9)\\2+(-9)\\4-2(-9)\end{pmatrix}=\begin{pmatrix}25\\-7\\22\end{pmatrix}$. Hence, possible points of intersections between l and π are $(-17, 7, -6)$ and $(25, -7, 22)$.

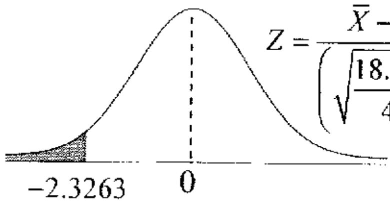
Qn	Solution
5(i) [3]	Substitute $x=t+\frac{1}{t}+4$, $y=t-\frac{1}{t}+1$ into $y=x-2-\frac{1}{x-3}$, We have $t-\frac{1}{t}+1=\left(t+\frac{1}{t}+4\right)-2-\frac{1}{\left(t+\frac{1}{t}+4\right)-3}$ $\Rightarrow t-\frac{1}{t}+1=t+\frac{1}{t}+2-\frac{1}{t+\frac{1}{t}+1}$

	$\Rightarrow -\dfrac{2}{t} = 1 - \dfrac{t}{t^2+t+1} = \dfrac{t^2+1}{t^2+t+1}$ $\Rightarrow -2t^2 - 2t - 2 = t^3 + t$ $\Rightarrow t^3 + 2t^2 + 3t + 2 = 0$ **Method 1**: Use graphing calculator From graphing calculator, $t = -1$ is the only real solution. Hence, curve D and E intersect only once at $t = -1$. **Method 2**: Factorise $t^3 + 2t^2 + 3t + 2 = 0$ $(t+1)(t^2+t+2) = 0$ $t = -1$ or $t^2 + t + 2 = 0$ (no real solution) Hence, curve D and E intersect only once at $t = -1$. When $t = -1$, $x = 2$ and $y = 1$. The coordinate at $t = -1$ is $(2,1)$.
5(ii) **[2]**	
5(iii) **[3]**	Using $x = t+3$, $y = (t+3) - 2 - \dfrac{1}{(t+3)-3} = t + 1 - \dfrac{1}{t}$. $x \le 2 \Rightarrow t + 3 \le 2 \Rightarrow t \le -1$ Curve E: $x = t+3$, $y = t + 1 - \dfrac{1}{t}$, $t \le -1$ 

5(iv) [4]	**Method 1**: Using x-axis (both curve in parametric form) For the Curve D: $x=t+\frac{1}{t}+4 \Rightarrow \frac{dx}{dt}=1-\frac{1}{t^2}$ When $x=0,\ t=-\sqrt{3}-2$ (given) When $x=2,\ t=-1$ For the Curve E: $x=t+3 \Rightarrow \frac{dx}{dt}=1$ When $x=0,\ t=-3$ When $x=2,\ t=-1$ Area $=\int_{-3}^{-1}\left(t-\frac{1}{t}+1\right)(1)\,dt-\int_{-\sqrt{3}-2}^{-1}\left(t-\frac{1}{t}+1\right)\left(1-\frac{1}{t^2}\right)dt$ $=1.3929$ units^2. (4 d.p.) -- **Method 2**: Using x-axis (1 cartesian, 1 parametric) For the Curve D: $x=t+\frac{1}{t}+4 \Rightarrow \frac{dx}{dt}=1-\frac{1}{t^2}$ When $x=0,\ t=-\sqrt{3}-2$ (given) When $x=2,\ t=-1$ Area $=\int_0^2 x-2-\frac{1}{x-3}\,dx-\int_{-\sqrt{3}-2}^{-1}\left(t-\frac{1}{t}+1\right)\left(1-\frac{1}{t^2}\right)dt$ $=1.3929$ units^2. (4 d.p.) -- **Method 3**: Using y-axis For the Curve D: $y=t-\frac{1}{t}+1 \Rightarrow \frac{dy}{dt}=1+\frac{1}{t^2}$ At y-intercept ($x=0$), $t=-\sqrt{3}-2$ (given) When $y=1,\ t=-1$ For the Curve E: $y=t+1-\frac{1}{t} \Rightarrow \frac{dy}{dt}=1+\frac{1}{t^2}$ At y-intercept ($x=0$), $t=-3$ When $y=1,\ t=-1$ Area $=\int_{-\sqrt{3}-2}^{-1}\left(t+\frac{1}{t}+4\right)\left(1+\frac{1}{t^2}\right)dt-\int_{-3}^{-1}(t+3)\left(1+\frac{1}{t^2}\right)dt$ $=1.3929$ units^2. (4 d.p.)

Section B: Probability and Statistics

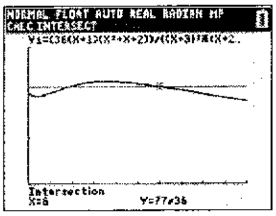
Qn	Solution
6(i) **[1]**	$\text{P}(B)=\text{P}(A\cup B)-\text{P}(A\cap B')$ $=0.8-0.55$ $=0.25$
6(ii) **[1]**	Probability that neither A nor B occurs $=\text{P}(A'\cap B')$ $=1-\text{P}(A\cup B)$ $=1-0.8$ $=0.2$
6(iii) **[2]**	**Method 1** $\text{P}(B'\cap C)$ $=\text{P}(B')\times\text{P}(C)$ $\because$ B and C are independent, B' and C are independent, $=(1-0.25)(0.6)$ $=0.45$ **Method 2** $\text{P}(B'\cap C)$ $=\text{P}(C)-\text{P}(B\cap C)$ $=\text{P}(C)-\text{P}(B)\text{P}(C)$ $\because$ B and C are independent $=0.6-(0.25)(0.6)$ $=0.45$
6(iv) **[3]**	Let $\text{P}(A\cap B'\cap C)$ be x. Since $\text{P}(B'\cap C)=0.45$, then $\text{P}(A'\cap B'\cap C)=0.45-x$. Therefore, $0.45-x\geq 0\Rightarrow x\leq 0.45$ Since $\text{P}(A\cup B)=0.8$, then $0.45-x\leq 0.2$. $\Rightarrow x\geq 0.25$. Therefore, the range of values of $\text{P}(A\cap B'\cap C)$ is $0.25\leq x\leq 0.45$.
7(i) **[2]**	The sample size should be at least 30 so that it is **large enough for the sample mean mass of almond flour per packet to follow a normal distribution approximately**. These packets should be **randomly** selected.
7(ii) **[4]**	Unbiased estimate of the population variance, $s^2=\frac{40}{39}(4.3)^2=18.964$ (5 s.f.)

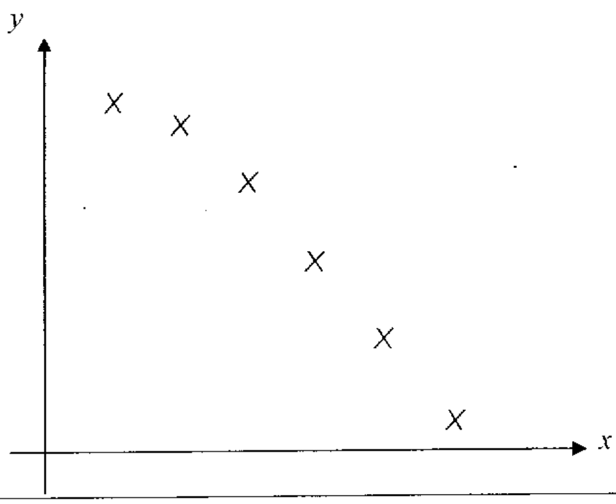
Qn	Solution
	Let X be the mass, in grams of a randomly chosen packet of almond flour, μ be the population mean mass of almond flour per packet $H_0 : \mu = \mu_0$ $H_1 : \mu < \mu_0$ This is the population mean. **Under** H_0, since $n = 40$ **is large**, by **Central Limit Theorem**, $\overline{X} \sim N\left(\mu_0, \frac{18.964}{40}\right)$ **approximately**. Test statistic, $Z = \frac{\overline{X} - \mu_0}{\left(\sqrt{\frac{18.964}{40}}\right)} \sim N(0,1)$. 1-tail z-test is used at $\alpha = 0.01$:  $Z = \frac{\overline{X} - \mu_0}{\left(\sqrt{\frac{18.964}{40}}\right)} \sim N(0,1)$ "Concluded that the mean mass is not overstated" infers that H_0 is not rejected. Since H_0 is **not rejected**, the test statistic value $\frac{248.5 - \mu_0}{\sqrt{\frac{18.964}{40}}}$ does **not** lie inside the rejection (critical) region. $\frac{248.5 - \mu_0}{\sqrt{\frac{18.964}{40}}} > -2.3263$ $248.5 - \mu_0 > -2.3263\left(\sqrt{\frac{18.964}{40}}\right)$ $-\mu_0 > -250.10$ $\mu_0 < 250.10$ (2 d.p.)
7(iii) [1]	"at the 1% level of significance" means 0.01 is the probability of concluding that the **population mean mass** of almond flour per packet is overstated when it is in fact μ_0 grams.

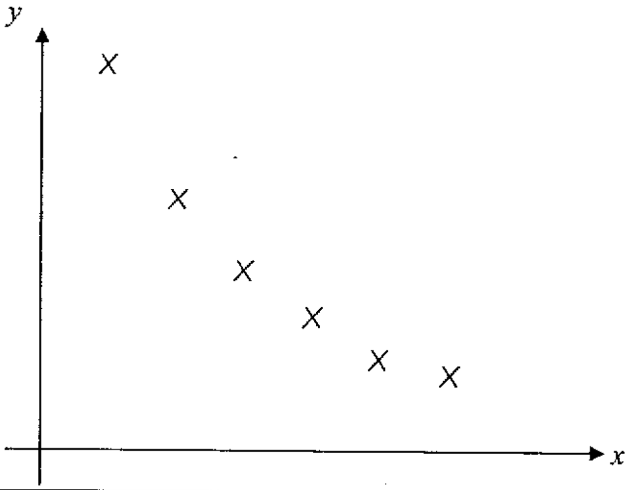
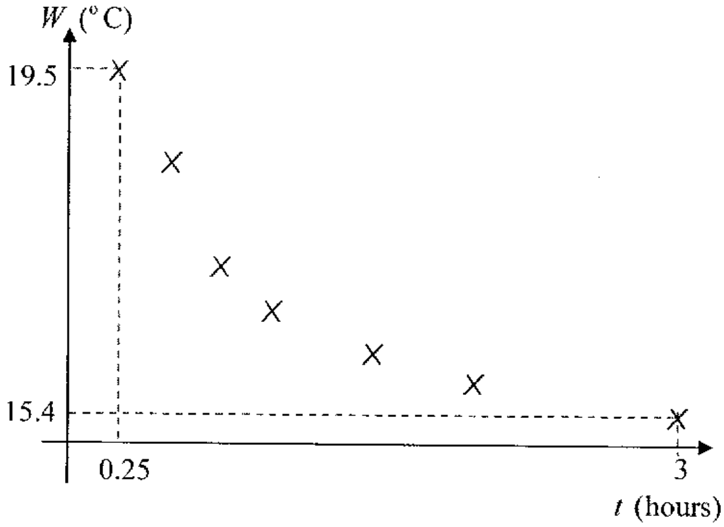
Qn	Solution
8(i) [1]	Let X be the number of patients who reported symptomatic relief after consuming the medication, out of 12, in a clinic. $X \sim B(12, p)$.

Qn	Solution
	$P(X=10) = {}^{12}C_{10}p^{10}(1-p)^2$ $= 66p^{10}(1-p)^2$.
8(ii) **[4]**	Since the mode is 10, $P(X=10) > P(X=9)$ $66p^{10}(1-p)^2 > {}^{12}C_9 p^9(1-p)^3$ $66p^{10}(1-p)^2 > 220p^9(1-p)^3$ Divide throughout by $p^9(1-p)^3$: $66p > 220(1-p)$ $66p > 220-220p$ $286p > 220$ $p > \frac{10}{13}$ $P(X=10) > P(X=11)$ $66p^{10}(1-p)^2 > {}^{12}C_{11}p^{11}(1-p)^1$ $66p^{10}(1-p)^2 > 12p^{11}(1-p)$ Divide throughout by $p^{10}(1-p)$: $66(1-p) > 12p$ $66-66p > 12p$ $-66p-12p > -66$ $-78p > -66$ $p < \frac{11}{13}$ Therefore, the possible range of values of p is $\frac{10}{13} < p < \frac{11}{13}$.
8(iii) **[1]**	Let Y be the number of patients who reported symptomatic relief after consuming the medication, out of 15, in another clinic. $Y \sim B(15, 0.8)$. $P(Y=15) \approx 0.035184 = 0.0352$. (3 s.f.)
8(iv) **[3]**	Required probability $= P(Y \geq 8) \times (P(Y=15))^2 \times 3$ $= (1-P(Y \leq 7)) \times (0.035184)^2 \times 3$ $= 0.00370$ (to 3 s.f.)

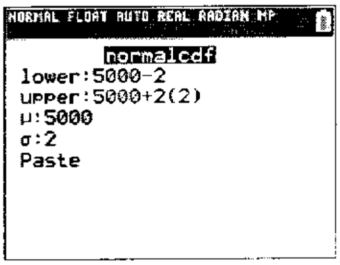
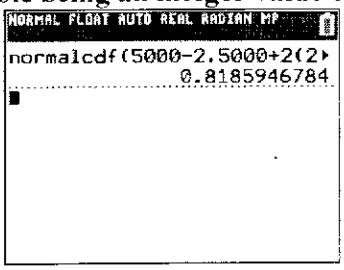
Qn	Solution
9(i) [5]	Possible values of M: 0, 1, 3, 4 Total number of notes: $n+3$ $\mathrm{P}(M=0)=\mathrm{P}(\text{draw 2 \$1 notes})+\mathrm{P}(\text{draw 2 \$2 notes})$ $=\frac{n}{n+3}\left(\frac{n-1}{n+2}\right)+\frac{2}{n+3}\left(\frac{1}{n+2}\right)$ $=\frac{n^2-n+2}{(n+3)(n+2)}$ $\mathrm{P}(M=1)=\mathrm{P}(\text{draw \$1 note and \$2 note})$ $=\frac{n}{n+3}\left(\frac{2}{n+2}\right)\times 2$ $=\frac{4n}{(n+3)(n+2)}$ $\mathrm{P}(M=3)=\mathrm{P}(\text{draw \$2 note and \$5 note})$ $=\frac{2}{n+3}\left(\frac{1}{n+2}\right)\times 2$ $=\frac{4}{(n+3)(n+2)}$ $\mathrm{P}(M=4)=\mathrm{P}(\text{draw \$1 note and \$5 note})$ $=\frac{n}{n+3}\left(\frac{1}{n+2}\right)\times 2$ $=\frac{2n}{(n+3)(n+2)}$
9(ii) [5]	$\mathrm{E}(M)$ $=0+\frac{4n}{(n+3)(n+2)}+3\left(\frac{4}{(n+3)(n+2)}\right)+4\left(\frac{2n}{(n+3)(n+2)}\right)$ $=\frac{12n+12}{(n+3)(n+2)}=\frac{12(n+1)}{(n+3)(n+2)}$. $\mathrm{E}(M^2)$ $=0+1^2\left(\frac{4n}{(n+3)(n+2)}\right)+3^2\left(\frac{4}{(n+3)(n+2)}\right)+4^2\left(\frac{2n}{(n+3)(n+2)}\right)$ $=\frac{4n+36+32n}{(n+3)(n+2)}=\frac{36(n+1)}{(n+3)(n+2)}$.

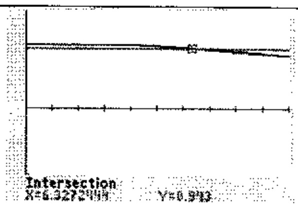
Qn	Solution
	$\text{Var}(M)$ $= \text{E}(M^2) - [\text{E}(M)]^2$ $= \dfrac{36(n+1)}{(n+3)(n+2)} - \left[\dfrac{12(n+1)}{(n+3)(n+2)}\right]^2$ $= \dfrac{36(n+1)}{(n+3)(n+2)}\left[1 - \dfrac{4(n+1)}{(n+3)(n+2)}\right]$ $= \dfrac{36(n+1)}{(n+3)(n+2)}\left[\dfrac{n^2+5n+6-4n-4}{(n+3)(n+2)}\right]$ $= \dfrac{36(n+1)}{(n+3)(n+2)}\left[\dfrac{n^2+n+2}{(n+3)(n+2)}\right]$ $= \dfrac{36(n+1)(n^2+n+2)}{(n+3)^2(n+2)^2}$
9(iii) **[1]**	Given that $\text{Var}(M) = \dfrac{77}{36}$, $\dfrac{36(n+1)(n^2+n+2)}{(n+3)^2(n+2)^2} = \dfrac{77}{36}$  Using GC graph, since $n \geq 2$, $n = 6$

Qn	Solution
10(i) **[2]**	(A) $y = ax^2 + b$, where a is negative and b is positive 

Qn	Solution
	(B) $y = c\ln x + d$, where c is negative and d is positive 
10(ii) **[2]**	 As the points do not lie close to a straight line, a linear model is not appropriate. [Accept: As W decreases at a decreasing rate, a linear model is not appropriate.]
10(iii) **[3]**	**(a)** r-value for t and W is -0.8675 (4 dp) **(b)** r-value for t^2 and W is -0.7294 (4 dp) **(c)** r-value for $\ln t$ and W is -0.9822 (4 dp)
10(iv) **[3]**	Since for the <u>scatter diagram for case (B), y **decreases at a decreasing rate**</u> and <u>$\lvert r\rvert$ **is closest to 1** for the case **(c)**</u>, hence the most appropriate model is $W = e\ln t + f$. Regression line: $W = -1.7295\ln t + 16.929$ $W = -1.73\ln t + 16.9$ (3 s.f.)
10(v) **[2]**	When $W = 16$,

Qn	Solution
	$16 = -1.7295\ln t + 16.929$ $t = 1.7111$ $= 1.71$ (3 s.f.) Time taken is 1.71 hours. Since $W = 16$ **lies within** the given **data range** of W and $\lvert r \rvert$ **is close to 1**, this estimate is **reliable**.

Qn	Solution
11(i) **[2]**	<u>**Method 1: Standardise**</u> $L \sim \text{N}(5000, \sigma^2)$ $\text{P}(5000 - \sigma < L < 5000 + 2\sigma)$ $= \text{P}\left(\frac{5000 - \sigma - 5000}{\sigma} < Z < \frac{5000 + 2\sigma - 5000}{\sigma}\right)$ $= \text{P}(-1 < Z < 2)$ $= 0.81859 = 0.819$ (to 3 s.f.) <u>**Method 2: Use GC**</u> **Note:** By the "68-95-99.7" rule of all normal distributions, it does not matter what the unknown σ value is in order to find $\text{P}(5000 - \sigma < L < 5000 + 2\sigma)$. Hence, we can key in any σ value into the GC to obtain the answer. (In the example below, $\sigma = 2$ is used. You can check that you will obtain the same answer regardless of which σ value you use.) **This only works when the probability we are finding involves the random variable being an integer value of σ away from the mean.**  Using GC, $\text{P}(5000 - \sigma < L < 5000 + 2\sigma) = 0.81859 = 0.819$ (3s.f.)
11(ii) **[2]**	<u>**Method 1:**</u> Use GC (Graph) $L \sim \text{N}(5000, \sigma^2)$ $\text{P}(L > 4990) = 0.943$ <u>Method 1: GC</u>

Qn	Solution
	 From GC: $\sigma = 6.3272$. Therefore, $\sigma^2 = 6.3272^2 = 40.033 = 40.0$ (to 3 s.f.) . **Method 2:** Standardise $\text{P}(L > 4990) = 0.943$ $\text{P}\left(Z > \frac{4990-5000}{\sigma}\right) = 0.943$ $\text{P}\left(Z > \frac{-10}{\sigma}\right) = 0.943$ Using GC inverse norm, $\frac{-10}{\sigma} = -1.5805$ $1.5805\sigma = 10$ $\sigma = \frac{10}{1.5805} = 6.3271$ $\Rightarrow \sigma^2 = 6.3271^2 = 40.032 = 40.0$ (to 3 s.f.)
11(iii) **[1]**	$S \sim \text{N}(1000, 25)$ $\text{P}(S \le 1002)$ $= 0.65542 = 0.655$ (to 3 s.f.)
11(iv) **[2]**	Let A be the number of small cans, out of 20, with the amount of oil in the can to be at most 1002 millilitres. $A \sim \text{B}(20, 0.65542)$ $\text{P}(A < 10)$ $= \text{P}(A \le 9)$ $= 0.047670 = 0.0477$ (to 3 s.f.)
11(v) **[3]**	The required probability is $\text{P}(L > 5S + 30) = \text{P}(L - 5S > 30)$. $\text{E}(L - 5S) = 5000 - 5(1000) = 0$ $\text{Var}(L - 5S) = 40 + 5^2(25) = 665$ Therefore, $L - 5S \sim \text{N}(0, 665)$.

Qn	Solution
	$\mathrm{P}(L-5S>30)$ $=0.12234=0.122$ (3 s.f.)
11(vi) **[4]**	Let $C=0.13\left(L_1+L_2+\ldots+L_6\right)+0.05\left(S_1+S_2\right)$, where C refers to the total cost of 6 large cans of Type A industrial lubricating oil and 2 small cans of Type B industrial lubricating oil. We want to find $\mathrm{P}(C\geq 3995)$. $\mathrm{E}(C)$ $=\mathrm{E}\left(0.13\left(L_1+L_2+\ldots+L_6\right)+0.05\left(S_1+S_2\right)\right)$ $=(0.13)(6)(5000)+(0.05)(2)(1000)$ $=4000$ $\mathrm{Var}(C)$ $=\mathrm{Var}\left(0.13\left(L_1+L_2+\ldots+L_6\right)+0.05\left(S_1+S_2\right)\right)$ $=0.13^2(6)(40)+0.05^2(2)(25)$ $=4.181$ (exact) Therefore, $C\sim \mathrm{N}(4000, 4.181)$. $\mathrm{P}(C\geq 3995)$ $=0.99276=0.993$ (to 3 s.f.)