3

1 A particle is moving along the curve with equation $3x^2 + xy = 5$. Given that $\frac{dy}{dt} = 4$ when $x = 1$, find the rate of decrease of x at this instant. [4]

2 A triangle ABC is such that $AC = \sqrt{2}$, $BC = 4$ and angle $ACB = \frac{1}{4}\pi + \theta$. Given that θ is sufficiently small for θ^3 and higher powers of θ to be neglected, show that

$$AB \approx \sqrt{10}\left[1 + a\theta + b\theta^2\right],$$

where a and b are real constants. [5]

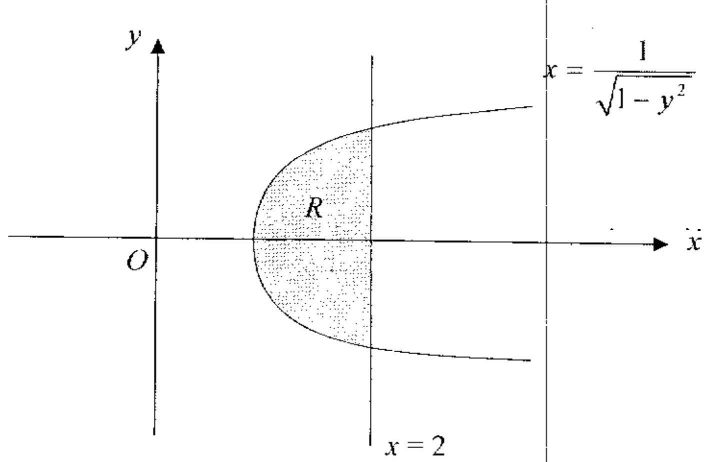
3 The curve C has equation $y = \frac{x^2 + px + 3}{x + q}$, where p and q are non-zero constants and $x \neq -q$. It is given that the asymptotes of C are $y = x + 3$ and $x = 2$.

(i) Find the values of p and q. Hence sketch C, stating the coordinates of any axial intercepts and turning points. [4]

Use the values of p and q that you have found in part (i) to answer part (ii).

(ii) By drawing a suitable curve on the same diagram in part (i), find the range of values of b, where b is a positive constant, such that there are 2 real roots to the equation $(x-5)^2 + b\left(\frac{x^2 + px + 3}{x + q}\right)^2 = 1$. [2]

4 The diagram shows the shaded region R bounded by curve C with equation $x = \frac{1}{\sqrt{1 - y^2}}$ and the line $x = 2$.



(i) Find the exact volume of the solid generated when R is rotated through 2π radians about the y-axis. [5]

(ii) Write down the equation of the curve obtained when C is translated by 2 units in the negative x-direction. Hence, or otherwise, find the volume of the solid generated when R is rotated through 2π radians about the line $x = 2$. [3]

5 The complex numbers z_1 and z_2 are such that

$$z_1 = 1 + b\text{i},\ b > 1,\ \arg(z_1) = \alpha \text{ and}$$

$$z_2 = 1 - c\text{i},\ 0 < c < 1,\ \arg(z_2) = \beta.$$

Let Z_1 and Z_2 be points representing z_1 and z_2 respectively on the Argand diagram.

(i) Indicate Z_1 and Z_2 on an Argand diagram. [2]

Z_3 is the point representing z_3 on an Argand diagram such that $|z_3| = |z_1|$, the origin O is collinear with Z_2 and Z_3, and $z_3 = \text{e}^{k\text{i}} z_1$, where k is a real constant.

(ii) Indicate the two possible positions of Z_3 on the same Argand diagram drawn in part **(i)**. [2]

(iii) Hence determine the two possible values of k, leaving your answers in terms of α and β. [2]

(iv) For value of $k = \frac{1}{2}\pi$, express the area of triangle $Z_1Z_2Z_3$ in terms of $|z_1|$ and $|z_2|$. [2]

6 **Do not use a calculator in answering this question.**

Let $\text{f}(x) = 2x + 1 - 2\ln(\sec x + \tan x)$.

(i) Find $\text{f}'(x)$, leaving your result in the form which involves one trigonometric function. [2]

(ii) Deduce that $\text{f}(x) < 1$ for $0 < x < \frac{1}{2}\pi$. [2]

The equation $\text{f}(x) = 0$ has a root, α, where $0 < \alpha < \frac{1}{2}\pi$.

(iii) A student wants to approximate the value of α using the x-intercept of the line passing through the points $(p, \text{f}(p))$ and $(q, \text{f}(q))$, where $0 < p < \alpha < q < \frac{1}{2}\pi$. Obtain an expression for the approximation of α, giving it in terms of p, q, $\text{f}(p)$ and $\text{f}(q)$. [2]

(iv) By finding $\text{f}''(x)$ and by considering the concavity of the graph $y = \text{f}(x)$, explain whether the estimate in part **(iii)** is an over-estimate or an under-estimate. [2]

7 A sequence $u_1,\ u_2,\ u_3,\ \ldots$ is such that $u_{n+2}+au_{n+1}+u_n=b$, where $a,\ b$ are constants and $n\geq 1$.

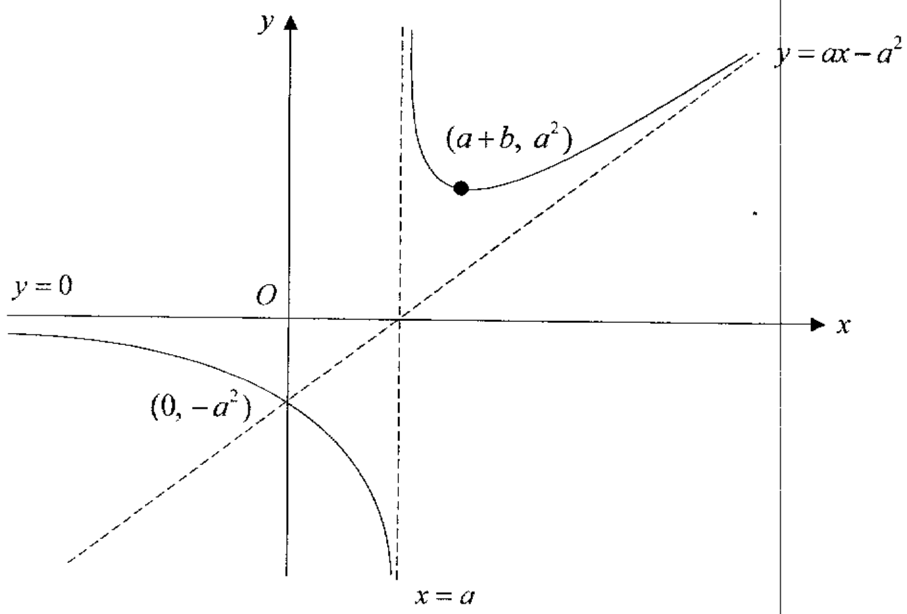
(i) Given that $u_1=c,\ u_2=0.5,\ u_3=5,\ u_4=14.5$ and $u_5=29$, where c is a constant, find $a,\ b$ and c. [3]

(ii) By using the substitution $v_n=u_{n+1}-u_n$, explain why the sequence $\{v_n\}$ is an arithmetic progression. [2]

(iii) By considering $\sum_{r=1}^{n-1} v_r=\sum_{r=1}^{n-1}(u_{r+1}-u_r)$, find u_n in terms of n. [4]

(iv) Describe the behaviour of the sequence $\{u_n\}$ for large values of n. [1]

8 The diagram below shows the graph of $y=\mathrm{f}(x)$, defined for $x\in\mathbb{R}\setminus\{a\}$, with asymptotes $x=a$, $y=0$ and $y=ax-a^2$, stationary point at $(a+b,\ a^2)$, and cuts the y-axis at $(0,\ -a^2)$, where a and b are positive constants such that $a>b$.



(i) Sketch the graph of $y=\dfrac{1}{\mathrm{f}(x)}$. [3]

(ii) Sketch the graph of $y=\mathrm{f}'(x)$. [3]

Another function g, is given by

$$\mathrm{g}(x)=\begin{cases} x & \text{for } 0\leq x<a, \\ a+b & \text{for } x\geq a. \end{cases}$$

(iii) Sketch the graph of g. [2]

(iv) Explain why fg exists. [3]

(v) Find the range of fg, leaving your answer in terms of a and b. [2]

9 Points A and B have position vectors $\begin{pmatrix}1\\q\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$ respectively, where q is a positive constant.

The equation of the line l is $\dfrac{4x-15}{-2}=y,\ z=\dfrac{5}{2}$.

(i) Given that the length of projection of vector $\overrightarrow{AB}$ onto line l is $\dfrac{8}{\sqrt{5}}$, show that $q=5$. [3]

(ii) Find the coordinates of the point of A', where A' is the image of point A reflected in the line l. [4]

(iii) The planes p_1 and p_2 have equations $\mathbf{r}\cdot\begin{pmatrix}2\\1\\k\end{pmatrix}=1$ and $\mathbf{r}=\mu\begin{pmatrix}1\\s\\1\end{pmatrix}+\gamma\begin{pmatrix}2\\-1\\1\end{pmatrix}$ respectively, for constants k, s and parameters μ, γ. Given that the two planes p_1 and p_2 intersect at l, determine the values of k and s. [3]

(iv) The plane p_3 has equation $2x+ty+z=u$. Given that the planes p_1, p_2 and p_3 have no point in common, what can be said about the values of t and u? [2]

10 Mr Safe takes up a bank loan of $\$M$ for an investment plan. The loan amount compounds at an annual interest rate of $100r\%$, where $0<r<1$.

(i) By considering the loan amount at the end of 6 years, find the range of values of r such that the amount of compounded interest will not exceed 20% of the initial loan. [3]

Take $M = 20\,000$ for the rest of the question.

Mr Safe invests \$10 000 and \$10 000 in Scheme A and Scheme B respectively for 6 years, with the two schemes starting at the same time.

Scheme A:
To receive payouts at the end of every month, starting from \$10 and increasing by \$0.30 every month.

Scheme B:
To receive payouts at the end of every month, starting from \$10 and increasing by 2% every month.

At the end of 6 years, the principal sum of \$20 000 is returned to Mr Safe.

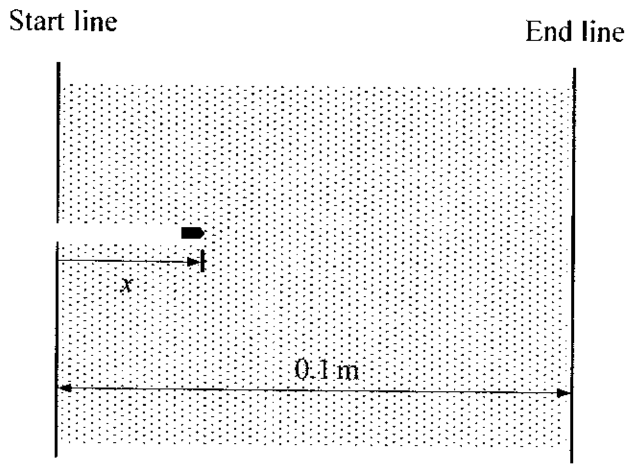
(ii) Find the total monthly payouts Mr Safe receives after n months, leaving your answer in exact form. [4]

(iii) Find the set of values of n such that the total monthly payouts from Scheme B exceeds that of Scheme A. [2]

(iv) Given that the annual interest rate of the bank is 2%, determine whether it is worthwhile for Mr Safe to take up the bank loan for the investment. [3]

11 A company conducts experiments to investigate the resistive power of armoured plates of different materials against a bullet fired from a rifle. The plates used for the experiment have a consistent thickness of 0.1 m. Once the plates are positioned, a bullet is fired at the start line of the plate with a fixed velocity of 100 m s^{-1}.

For a duration of t s after it is fired, the bullet is x m away from the start line and travels at a velocity $v\text{ m s}^{-1}$. The setup is shown below.



(i) When a plate of material P is used, v and x are modelled by the following differential equations:

$$\text{(A)} \quad \frac{dx}{dt} = v,$$

$$\text{(B)} \quad \frac{d^2x}{dt^2} + \beta\left(\frac{dx}{dt}\right)^2 = 0,$$

where β is a positive constant.

(a) By substituting equation (A), show that equation (B) can be written as $\frac{dv}{dt} = -\beta v^2$. [1]

(b) Find v in terms of β and t and hence find x in terms of β and t. [5]

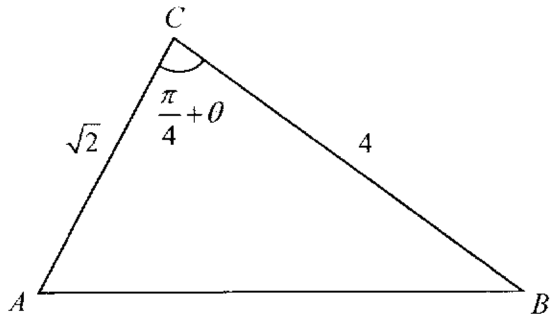
(c) Given that it is found that the bullet exits the end line of the plate with a velocity of 80 m s^{-1}, find the time taken for the bullet to pass through the plate of material P, leaving your answer to 5 decimal places. [3]

(ii) The company develops another plate made of a new material Q. When the experiment is conducted with a plate of material Q, it is found that the bullet is slowed down in such a way that $\frac{d^2x}{dt^2}$ is inversely proportional to e^{10000t}. It is given that $\frac{dx}{dt} = 100$ and $\frac{d^2x}{dt^2} = -10^6$ when $t = 0$. Write down a differential equation to model the bullet as it travels through the plate of material Q. Solve this differentiation equation to find x in terms of t for this model. [4]

(iii) The company claims that material Q provides better protection against bullets than material P. Explain, with justification, whether the company's claim is valid. [1]

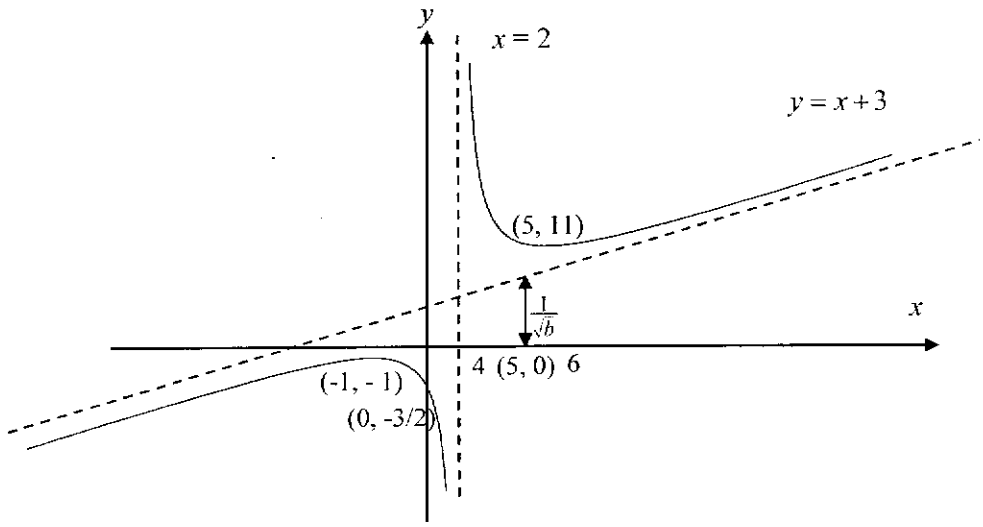
2021 Year 6 H2 Math Prelim Paper 1 Mark Scheme

	Suggested Solution
1	$3x^2 + xy = 5$ --------- (1) $\frac{d}{dt}$: $6x\frac{dx}{dt} + x\frac{dy}{dt} + y\frac{dx}{dt} = 0$ ------- (2) When $x = 1$, $\frac{dy}{dt} = 4$: (1): $3 + y = 5$ $\therefore y = 2$ (2): $6\frac{dx}{dt} + 4 + 2\frac{dx}{dt} = 0$ $\therefore \frac{dx}{dt} = -\frac{4}{8} = -\frac{1}{2}$ Rate of decrease of x is $\frac{1}{2}$. **Alternative** $3x^2 + xy = 5$ --------- (1) $\frac{d}{dx}$: $6x + x\frac{dy}{dx} + y = 0$ ------- (2) When $x = 1$, $\frac{dy}{dt} = 4$: (1): $3 + y = 5$ $\therefore y = 2$ (2): $6 + \frac{dy}{dx} + 2 = 0$ $\therefore \frac{dy}{dx} = -8$ Since $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ $\therefore \frac{dx}{dt} = -\frac{4}{8} = -\frac{1}{2}$ Rate of decrease of x is $\frac{1}{2}$.

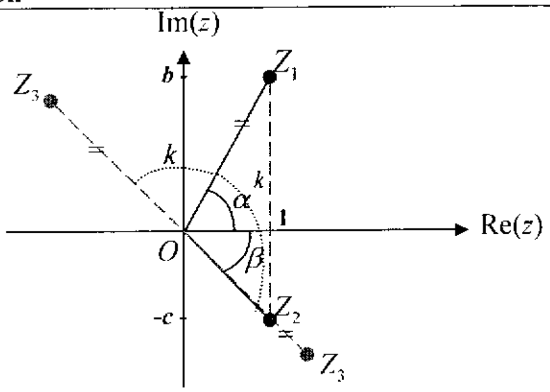
	Suggested Solution
2	 $AB^2=\left(\sqrt{2}\right)^2+4^2-2\sqrt{2}(4)\cos\left(\frac{\pi}{4}+\theta\right)$ $=2+16-8\sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right)$ $=18-8\sqrt{2}\left(\cos\frac{\pi}{4}\cos\theta-\sin\frac{\pi}{4}\sin\theta\right)$ $=18-8\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right)$ $=18-8(\cos\theta-\sin\theta)$ $\approx 18-8\left(1-\frac{\theta^2}{2}-\theta\right)$ (since θ is small) $=18-8+4\theta^2+8\theta$ $=10+8\theta+4\theta^2$

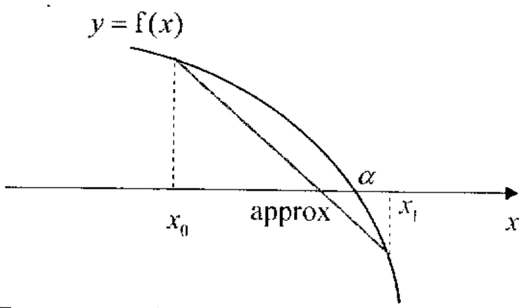
	$AB = \left(10+8\theta+4\theta^2\right)^{\frac{1}{2}}$ $= \sqrt{10}\left[1+\left(\frac{4}{5}\theta+\frac{2}{5}\theta^2\right)\right]^{\frac{1}{2}}$ $= \sqrt{10}\left[1+\frac{1}{2}\left(\frac{4}{5}\theta+\frac{2}{5}\theta^2\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{4}{5}\theta+\frac{2}{5}\theta^2\right)^2+\ldots\right]$ $= \sqrt{10}\left[1+\frac{2}{5}\theta+\frac{1}{5}\theta^2-\frac{1}{8}\left(\frac{16}{25}\theta^2+\ldots\right)+\ldots\right]$ $= \sqrt{10}\left(1+\frac{2}{5}\theta+\frac{1}{5}\theta^2-\frac{2}{25}\theta^2+\ldots\right)$ $\approx \sqrt{10}\left(1+\frac{2}{5}\theta+\frac{3}{25}\theta^2\right)$

Qn	Suggested Solution
3(i)	Since $x=2$ is a vertical asymptote, we have $q=-2$. <u>Method 1:</u> $y = x+3+\frac{k}{x-2}$ $= \frac{x^2+x-6+k}{x-2}$ Comparing coefficient of x in given equation, $p=1$ $\therefore y = \frac{x^2+x+3}{x-2} = x+3+\frac{9}{x-2}$ <u>Method 2:</u> $y = \frac{x^2+px+3}{x-2}$ $= (x+p+2)+\frac{7+2p}{x-2}$ (by long division) Comparing with the given oblique asymptote $p+2=3 \Rightarrow p=1$

	
(ii)	2 real roots for $(x-5)^2+b\left(\frac{x^2+px+3}{x+q}\right)^2=1$ implies that there are 2 intersection points between curve C and $(x-5)^2+by^2=1 \Rightarrow (x-5)^2+\frac{y^2}{\left(\frac{1}{\sqrt{b}}\right)^2}=1$ (ellipse) Sketch ellipse with centre (5,0) and horizontal axis 1 and vertical axis $\frac{1}{\sqrt{b}}$ $\frac{1}{\sqrt{b}}>11$ $0<\sqrt{b}<\frac{1}{11}$ $0<b<\frac{1}{121}$

Qn	Solution
4(i)	When $x = 2$, $2 = \frac{1}{\sqrt{1-y^2}}$ $\sqrt{1-y^2} = \frac{1}{2}$ $1-y^2 = \frac{1}{4}$ $y^2 = \frac{3}{4}$ $y = \frac{\sqrt{3}}{2}$ or $y = -\frac{\sqrt{3}}{2}$ Volume of solid $= \pi(2^2)(\sqrt{3}) - 2\pi\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1-y^2}\ \mathrm{d}y$ $= 4\sqrt{3}\pi - \frac{2\pi}{2}\left[\ln\left(\frac{1+y}{1-y}\right)\right]_0^{\frac{\sqrt{3}}{2}}$ $= 4\sqrt{3}\pi - \pi\left[\ln\left(\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right) - \ln 1\right]$ $= 4\sqrt{3}\pi - \pi\ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$
(ii)	Eqn of new curve: $x + 2 = \frac{1}{\sqrt{1-y^2}}$ $\therefore x = \frac{1}{\sqrt{1-y^2}} - 2$ Volume of solid $= 2\pi\int_0^{\frac{\sqrt{3}}{2}} \left(\frac{1}{\sqrt{1-y^2}} - 2\right)^2\ \mathrm{d}y$ $= 3.7213$ $= 3.72$ (3sf)

Qn	Solution
5 (i) (ii)	 Note: (i) $\alpha > 0$, $\beta < 0$ (ii) $z_3 = e^{ki} z_1 \Rightarrow \|z_3\| = \|z_1\|$ and $\arg(z_3) = k + \alpha$
(iii)	From the Argand diagram **Case 1:** z_3 on same side as z_2 (in 4th quadrant) $\arg(z_3) = k + \alpha$ ie, $k + \alpha = \beta \Rightarrow k = \beta - \alpha$ (note $k < 0$) **Case 2:** z_3 on opposite side as z_2 (in 2nd quadrant) $-\beta + \alpha + k = \pi$ $\therefore k = \pi + \beta - \alpha$ (Note $k > 0$)
(iv)	For $k = \frac{\pi}{2}$, $OZ_1 \perp OZ_3$, hence base = $\|z_3\| + \|z_2\| = \|z_1\| + \|z_2\|$ ht = $\|z_1\|$ triangle area = $\frac{1}{2}(\|z_1\| + \|z_2\|)\|z_1\|$

Qn	Solution	
6(i)	$\mathrm{f}(x)=2x+1-2\ln(\sec x+\tan x)$ $\mathrm{f}'(x)=2-2\dfrac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}$ $\quad=2-2\sec x$	
(ii)	For $0<x<\frac{1}{2}\pi$, $0<\cos x<1$ $\sec x>1$ $\therefore \mathrm{f}'(x)=2-2\sec x<0$ Together with the fact that $\mathrm{f}(0)=20+1-2\ln(\sec 0+\tan 0)=1$ $\mathrm{f}(x)<1$ for $0<x<\frac{1}{2}\pi$	
(iii)	Line passing through $(p,\ \mathrm{f}(p))$ and $(q,\ \mathrm{f}(q))$: $y-\mathrm{f}(p)=\dfrac{\mathrm{f}(q)-\mathrm{f}(p)}{q-p}(x-p)$ $\alpha\approx p-\mathrm{f}(p)\dfrac{q-p}{\mathrm{f}(q)-\mathrm{f}(p)}$ **Alternative:** $y-\mathrm{f}(q)=\dfrac{\mathrm{f}(q)-\mathrm{f}(p)}{q-p}(x-q)$ $\alpha\approx q-\mathrm{f}(q)\dfrac{q-p}{\mathrm{f}(q)-\mathrm{f}(p)}$	
(iv)	$\mathrm{f}''(x)=-2\sec x\tan x<0$ for $0<x<\frac{1}{2}\pi$ I.e. curve is concave downwards  From curve, the approximation in part **(iii)** is an under-estimate.	

Qn	Suggested Solution
7(i)	$u_3 + au_2 + u_1 = b \rightarrow 5 + a(0.5) + c = b \rightarrow 0.5a - b + c = -5$ $u_4 + au_3 + u_2 = b \rightarrow 14.5 + a(5) + 0.5 = b \rightarrow 5a - b = -15$ $u_5 + au_4 + u_3 = b \rightarrow 29 + a(14.5) + 5 = b \rightarrow 14.5a - b = -34$ Solving, $a = -2, b = 5, c = 1$
(ii)	$v_{n+1} - v_n$ $= (u_{n+2} - u_{n+1}) - (u_{n+1} - u_n)$ $= u_{n+2} - 2u_{n+1} + u_n$ From (i), $u_{n+2} + au_{n+1} + u_n = b$ where $a = -2,\ b = 5$ Hence, $u_{n+2} - 2u_{n+1} + u_n = 5$ Thus, $v_{n+1} - v_n = 5$ (constant) Therefore, $\{v_n\}$ is an arithmetic progression. **ALTERNATIVE** $v_n = u_{n+1} - u_n$ $= (5 + 2u_n - u_{n-1}) - u_n$ $= 5 + u_n - u_{n-1}$ $= 5 + v_{n-1}$ $v_n - v_{n-1} = 5$ Since the difference between consecutive terms is a constant, $\{v_n\}$ is an arithmetic progression with common difference 5.

(iii)

$$\sum_{r=1}^{n-1} v_r = \sum_{r=1}^{n-1}\left(u_{r+1}-u_r\right)$$

$$\text{LHS}=\sum_{r=1}^{n-1} v_r$$
$$=\frac{n-1}{2}\left(2(-0.5)+(n-2)5\right)$$
$$=\frac{n-1}{2}(5n-11)$$

Alternative:

$$\text{LHS}=\sum_{r=1}^{n-1} v_r$$
$$=\sum_{r=1}^{n-1}(5r-5.5)$$
$$=\frac{n-1}{2}(-0.5+5n-10.5)$$
$$=\frac{n-1}{2}(5n-11)$$

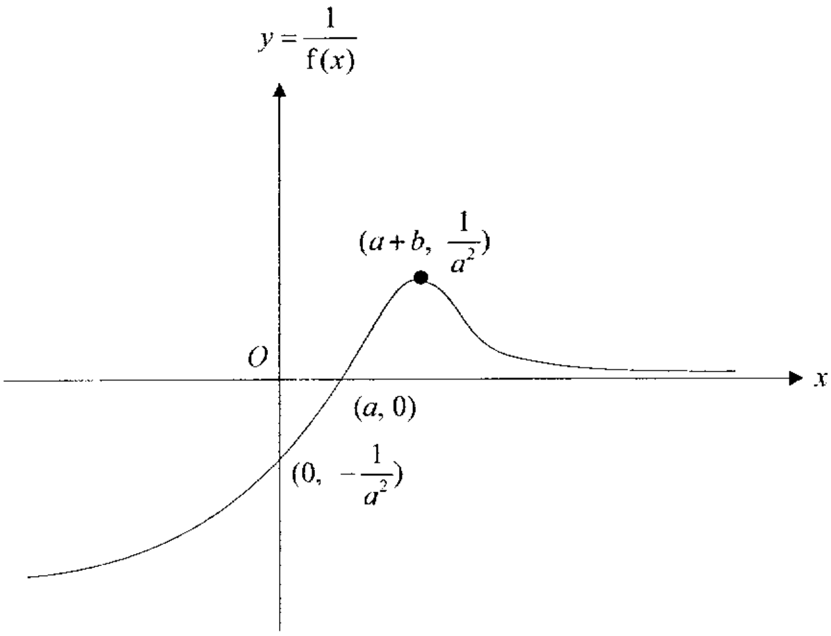
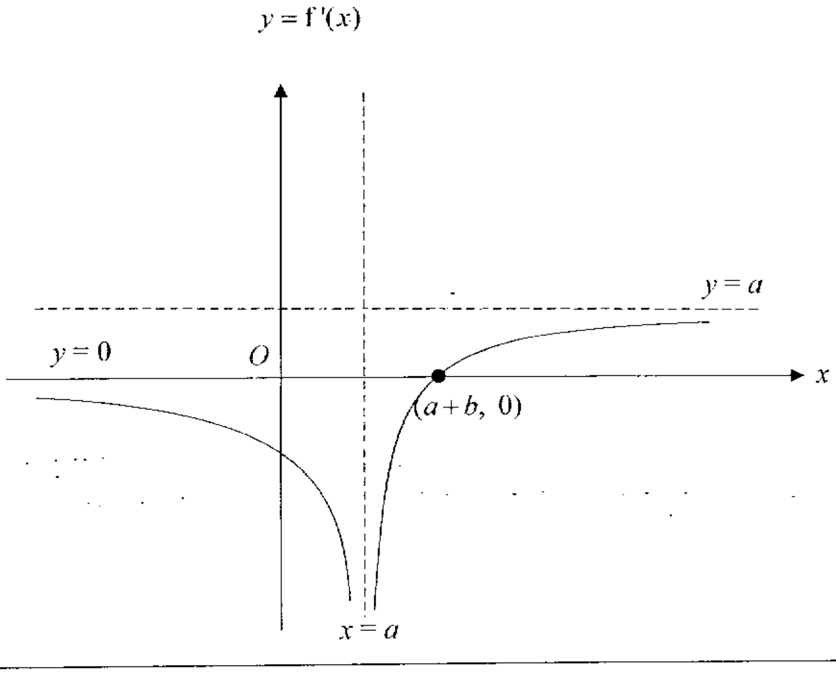
$$\text{RHS}=\sum_{r=1}^{n-1}\left(u_{r+1}-u_r\right)$$
$$=\begin{cases} u_2-u_1 \\ u_3-u_2 \\ u_4-u_3 \\ \quad\ldots \\ u_{n-1}-u_{n-2} \\ u_n-u_{n-1} \end{cases}$$
$$=u_n-u_1$$

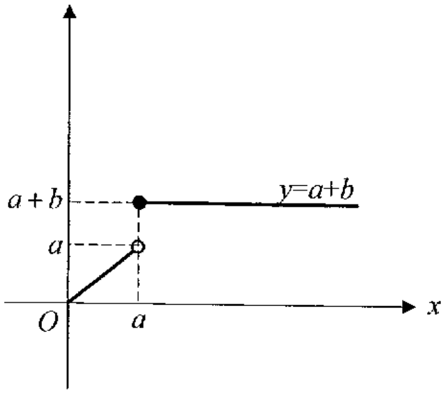
Thus, $u_n-u_1=\frac{n-1}{2}(5n-11)$

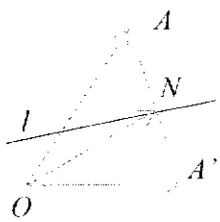
$$u_n=\frac{n-1}{2}(5n-11)+1$$
$$=\frac{5n^2-16n+13}{2}$$

(iv)

Possible responses:
-Sequence **increases** and **diverges** for large values of n.
-Sequence **increases to infinity** for large values of n.

Qn	Solution
8(i)	
(ii)	

(iii)	$y = g(x)$ 	
(iv)	Range of g $= [0, a) \cup \{a+b\}$ Since range of g is subset of domain of f, $\mathbb{R} \setminus \{a\}$, Thus fg exists. (explained)	
(v)	To find range of fg: $D_{fg} = D_g \xrightarrow{g} R_g = [0, a) \cup \{a+b\} \xrightarrow{f} (-\infty, -a^2] \cup \{a^2\} = R_{fg}$	

Qn	Suggested Solutions
9(i)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-2\\-q\\-1\end{pmatrix}$ $l: r=\begin{pmatrix}\frac{15}{4}\\0\\\frac{5}{2}\end{pmatrix}+\lambda\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}, \lambda\in\mathbb{R}$ $\frac{\left\lvert\overrightarrow{AB}\cdot\mathbf{d}_l\right\rvert}{\left\lvert\mathbf{d}_l\right\rvert}=\frac{8}{\sqrt{5}}$ $\frac{\left\lvert\begin{pmatrix}-2\\-q\\-1\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}\right\rvert}=\frac{8}{\sqrt{5}}$ $\lvert 1-q\rvert=\frac{8}{\sqrt{5}}\left(\frac{\sqrt{5}}{2}\right)$ $1-q=4$ or $1-q=-4$ $q=-3$ (reject $\because q>0$) or $q=5$
(ii)	 $\overrightarrow{ON}=\begin{pmatrix}\frac{15}{4}-\frac{1}{2}\lambda\\\lambda\\\frac{5}{2}\end{pmatrix}$, for some $\lambda\in\mathbb{R}$

	$\overrightarrow{AN}=\overrightarrow{ON}-\overrightarrow{OA}=\begin{pmatrix}\frac{15}{4}-\frac{1}{2}\lambda\\ \lambda\\ \frac{5}{2}\end{pmatrix}-\begin{pmatrix}1\\5\\2\end{pmatrix}=\begin{pmatrix}\frac{11}{4}-\frac{1}{2}\lambda\\ \lambda-5\\ \frac{1}{2}\end{pmatrix}$ $\overrightarrow{AN}\bullet\mathbf{d}_l=0$ $\begin{pmatrix}\frac{11}{4}-\frac{1}{2}\lambda\\ \lambda-5\\ \frac{1}{2}\end{pmatrix}\bullet\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}=0$ $\lambda=\frac{51}{10}$ $\therefore\overrightarrow{ON}=\begin{pmatrix}\frac{15}{4}-\frac{1}{2}\left(\frac{51}{10}\right)\\ \frac{51}{10}\\ \frac{5}{2}\end{pmatrix}=\begin{pmatrix}\frac{12}{10}\\ \frac{51}{10}\\ \frac{5}{2}\end{pmatrix}=\frac{1}{10}\begin{pmatrix}12\\51\\25\end{pmatrix}$ Using ratio theorem, $\overrightarrow{ON}=\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ $\overrightarrow{OA'}=2\overrightarrow{ON}-\overrightarrow{OA}=\frac{2}{10}\begin{pmatrix}12\\51\\25\end{pmatrix}-\begin{pmatrix}1\\5\\2\end{pmatrix}=\frac{1}{10}\begin{pmatrix}14\\52\\30\end{pmatrix}=\frac{1}{5}\begin{pmatrix}7\\26\\15\end{pmatrix}$ Coordinates of A' $=\left(\frac{7}{5},\frac{26}{5},3\right)$	
(iii)	Since the point $\left(\frac{15}{4},0,\frac{5}{2}\right)$ lies on the p_1, $2\left(\frac{15}{4}\right)+\frac{5}{2}k=1$ $k=-\frac{13}{5}$ Normal of $p_2=\begin{pmatrix}1\\s\\1\end{pmatrix}\times\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}s+1\\1\\-1-2s\end{pmatrix}$	

	Since line l lies on p_2, $\mathbf{d}_l \bullet \mathbf{n}_{p_2} = 0$ $\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} s+1 \\ 1 \\ -1-2s \end{pmatrix} = 0$ $-\frac{1}{2}(s+1)+1=0$ $s=1$ **ALT 1:** Alternatively, solve SLE $\begin{pmatrix} \frac{15}{4} \\ 0 \\ \frac{5}{2} \end{pmatrix} = \mu \begin{pmatrix} 1 \\ s \\ 1 \end{pmatrix} + \gamma \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ to obtain $\mu = \gamma = \mu s = \frac{5}{4}$ so $s=1$ **ALT 2:** Can also use $\begin{pmatrix} \frac{15}{4} \\ 1 \\ \frac{5}{2} \end{pmatrix} \bullet \begin{pmatrix} s+1 \\ 1 \\ -1-2s \end{pmatrix} = 0$ since origin must lie on p2.
(iv)	For no points of intersection between the three planes, p_3 is parallel to line l, and line l does not lie on p_3. $\mathbf{d}_l \bullet \mathbf{n}_{p_3} = 0$ $\begin{pmatrix} \frac{15}{4} \\ 0 \\ \frac{5}{2} \end{pmatrix} \bullet \mathbf{n}_{p_3} \neq u$ $\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ t \\ 1 \end{pmatrix} = 0$ and $u \neq \begin{pmatrix} \frac{15}{4} \\ 0 \\ \frac{5}{2} \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{15}{2} + \frac{5}{2} = 10$ $-1+t=0$ $u \neq 10$ $t=1$

Qn	Solution
10(i)	Loan amount = $M Loan amount after 6 yrs = $M(1+r)^6$ For amount of compounded interest to not exceed 20% of the initial loan, $M(1+r)^6 - M \le 0.2M$ $(1+r)^6 \le 1.2$ $r \le 1.2^{\frac{1}{6}} - 1$ $r \le 0.030853$ $\therefore r \le 0.0308$
(ii)	Scheme A amount $S_A = 10 + (10+0.3) + ...$ (for n terms) $= \frac{n}{2}(2(10) + (n-1)(0.3))$ $= 9.85n + 0.15n^2$ Scheme B amount $S_B = 10 + 10(1.02) + 10(1.02)^2 + ... + 10(1.02)^{n-1}$ $= \frac{10(1.02^n - 1)}{1.02 - 1}$ $= 500(1.02^n - 1)$ Total amt = $9.85n + 0.15n^2 + 500(1.02^n - 1)$
(iii)	$9.85n + 0.15n^2 < 500(1.02^n - 1)$ From GC, $n \ge 58$ $\therefore \{n \in \mathbb{Z} : 58 \le n \le 72\}$
(iv)	At end of 6 years, total payout $= 9.85(72) + 0.15(72)^2 + 500(1.02^{72} - 1)$ $= 3067.3701$ $= 3067.37$ bank interest $= 20000(1.02^6) - 20000 = 2523.25 <$ total payout Thus it is worthwhile for Mr Safe to take up the bank loan for the investment.

	Suggested Solution
11(i) (a)	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\beta\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=0$ Let $v=\frac{\mathrm{d}x}{\mathrm{d}t}$, $\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$, we have $\frac{\mathrm{d}v}{\mathrm{d}t}+\beta v^2=0$ $\frac{\mathrm{d}v}{\mathrm{d}t}=-\beta v^2$
(i)(b)	$v^{-2}\frac{\mathrm{d}v}{\mathrm{d}t}=-\beta$ $\int v^{-2}\mathrm{d}v=-\beta\int 1\ \mathrm{d}t$ $-\frac{1}{v}=-\beta t+C$ When $t=0$, $v=100$, $C=-\frac{1}{100}$. $-\frac{1}{v}=-\beta t-\frac{1}{100}$ $\frac{1}{v}=\beta t+\frac{1}{100}$ $v=\frac{1}{\beta t+0.01}$ --- (1) $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{\beta t+0.01}$ $x=\int\frac{1}{\beta t+0.01}\ \mathrm{d}t$ $x=\frac{1}{\beta}\ln(\beta t+0.01)+D$ (since $\beta>0$) When $t=0$, $x=0$, $0=\frac{1}{\beta}\ln(0.01)+D$ $D=-\frac{1}{\beta}\ln(0.01)$

	$x=\frac{1}{\beta}\ln(\beta t+0.01)-\frac{1}{\beta}\ln(0.01)$ $x=\frac{1}{\beta}\left[\ln\left(\frac{\beta t+0.01}{0.01}\right)\right]$ $x=\frac{1}{\beta}\ln(100\beta t+1)$ --- (2)
(i)(c)	Let $t = T$ when the bullet leaves plate of material X. When $t = T$, $v = 80$, from (1), $80=\frac{1}{\beta T+0.01}$ $\beta T+0.01=\frac{1}{80}$ $\beta T=\frac{1}{400}=0.0025$ $\beta=\frac{1}{400T}$ When $t = T$, $x = 0.1$ and $\beta=\frac{1}{400T}$, from (2), $0.1=\frac{1}{\left(\frac{1}{400T}\right)}\ln\left(100\left(\frac{1}{400T}\right)T+1\right)$ $0.1=400T\ln\left(\frac{5}{4}\right)$ $T=\frac{0.1}{400\ln(1.25)}$ $=0.00112035502$ $=0.00112$ seconds
(ii)	$\frac{d^2x}{dt^2}=\frac{k}{e^{10000t}}=ke^{-10^4t}$ When $t=0$, $\frac{d^2x}{dt^2}=-10^6$. $-10^6=ke^{-10^4(0)}$ $k=-10^6$ Hence, $\frac{d^2x}{dt^2}=-10^6e^{-10^4t}$

	$\frac{dx}{dt}=\frac{-10^6}{-10^3}e^{-10^4 t}+E$ $=100e^{-10^4 t}+E$ When $t=0$, $\frac{dx}{dt}=100$, $100=100e^{-10^4(0)}+E$ $E=0$ $\frac{dx}{dt}=100e^{-10^4 t}$ --- (3) $x=-\frac{100}{10^4}e^{-10^4 t}+F$ $x=-0.01e^{-10^4 t}+F$ When $t=0, x=0$, $0=-0.01e^{-10^4(0)}+F$ $F=0.01$ $x=0.01-0.01e^{-10^4 t}$ $x=0.01\left(1-e^{-10^4 t}\right)$ --- (4)
(iii)	**Method 1** When $t=0.0011204$, from (4), $x=0.01\left(1-e^{-10^4(0.0011204)}\right)$ $\approx 0.001<0.1$ In the same amount of time it takes for the bullet to penetrate the entire plate of material P of 0.1m, it only travels 0.01m through plate of material Q. Hence, the company's claim is valid. **Method 2** From (4), as $t\to\infty$, $x\to 0.01$. This means the bullet will not be able to penetrate the plate of material Q which is 0.1m thick, unlike what it has done to material P. Hence, the company's claim is valid. **Method 3** When $t=0.0011204$, from (3), $\frac{dx}{dt}=100e^{-10^4(0.0011204)}=0.00136\approx 0$.

	In the same amount of time it takes for the bullet to penetrate the entire plate of material P to exit with a velocity of 80 ms^{-1}, the bullet that travels through the plate of material Q has its velocity reduced to almost 0 ms^{-1}. Hence, the company's claim is valid.

Name:		Centre/Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination 2021
Year 6

MATHEMATICS

9758/02

Paper 2

22 September 2021

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Centre number, index number, name and class on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	5	7	8	10	10	10	7	8	11	12	12	100

Section A: Pure Mathematics [40 marks]

1 **Do not use a graphing calculator in answering this question. Show all workings clearly.**

The complex numbers w, z_1 and z_2 are given by $w = 3 - 4\text{i}$, $z_1 = \text{Im}\left(\frac{w}{w^*}\right)$ and $z_2 = w - w^*$.

(i) Find z_1 and z_2. [2]

(ii) Given that z_1 and z_2 are roots of the equation $bz^3 + cz^2 + dz + 1536 = 0$, find the values of the real numbers b, c and d. [3]

2 **(i)** By expressing $\frac{1}{2^{k+1}-3} - \frac{1}{2^k - 3}$ as a single fraction, find in terms of N, an expression for $\sum_{k=1}^{N} \frac{2^{k+1}}{(2^{k+1}-3)(2^k-3)}$. [4]

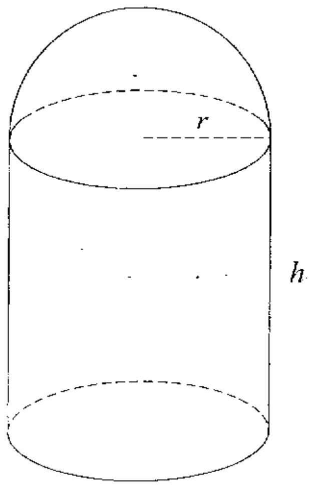
(ii) Use your result in **(i)**, or otherwise, to evaluate

$$\frac{16}{(61)(29)} + \frac{32}{(125)(61)} + \frac{64}{(253)(125)} + \ldots + \frac{2^{k-1}}{(2^{k+1}-3)(2^k-3)} + \ldots,$$

leaving your answer as a fraction in its lowest terms. [3]

3 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

The solid below is made up of a hemisphere, with radius r cm, placed on top of a cylinder with radius r cm and height h cm. The total surface area of the solid is kept at a constant k cm^2.



(i) Find r when the volume of the solid is at its maximum, leaving your answer in terms of k and π. [5]

(ii) Find this maximum volume in terms of k and π, simplifying your answer. Find also the ratio of $r : h$ when this occurs. [3]

4 **(a)** Find $\int \sin^{-1} 4x \, dx$. [3]

(b) A curve C is defined by the following parametric equations, $x = a\left(t - \frac{1}{3}t^3\right)$, $y = at^2$, $t \geq 0$, for a positive constant a.

(i) Find the equation of the tangent to the curve at the point $P\ \left(a\left(p - \frac{1}{3}p^3\right),\ ap^2\right)$. [3]

The arc length between two points on C, where $t = t_1$ and $t = t_2$ is given by the formula

$$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.$$

(ii) The tangent at P meets the x-axis at Q. Prove that the arc length OP is twice the length of OQ where O is the origin that C passes through. [4]

5 The points A, B and R have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{r}$ respectively.

(a) The point C has position vector $\frac{2}{7}\mathbf{a} - \frac{3}{7}\mathbf{b}$ and the point D is such that the origin O is the midpoint of the line segment CD. The point R lies on BD extended such that the ratio of BD to BR is $4:7$. Show that the points A, O and R are collinear and state the ratio of OA to OR. [4]

(b) It is given that the point R has position vector $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and that $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$.

(i) Determine the exact area of the triangle AOB. [2]

(ii) Give the geometrical interpretation of the point R, given that $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. [2]

(iii) Find the shortest distance between the point $(-8, -2, 9)$ and the collection of all points R satisfying $\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. [2]

Section B: Probability and Statistics [60 marks]

6 There are ten cards of which 4 are black, 3 are blue, 2 are green and 1 is yellow. Cards of the same colour are indistinguishable.

(a) For parts **(a)(i)** to **(a)(iii)**, four cards are to be selected from these ten cards and the order of selection is not relevant.

(i) Find the number of possible selections that can be made if exactly 3 cards are of the same colour. [2]

(ii) Find the number of possible selections that can be made if at least 2 of the cards are of the same colour. [3]

(iii) Find the number of possible selections that can be made if either at least one green card is selected or yellow card is selected or both. [2]

(b) Find the number of arrangements of all ten cards in a row if the blue cards are not next to one another. [3]

7 Mike plays a game at a fun-fair. Each game costs \$3 to play. The probability distribution of the amount of money \$$X$, that Mike wins from a game is given by

$$\mathrm{P}(X=x)=\begin{cases} kx & \text{for } x=1,\ 2,\ 3, \\ \frac{1}{3}k(x-1) & \text{for } x=4,\ 5,\ 6, \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

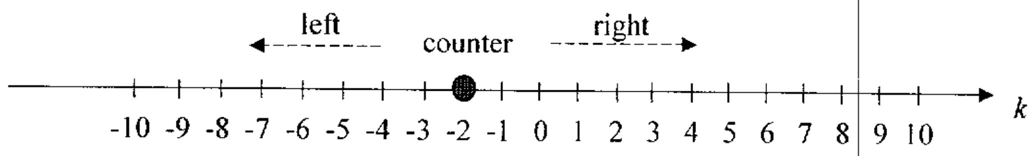
(i) Show that $k=\frac{1}{10}$. [1]

(ii) Find the expectation and variance of his net gain for one game. [3]

(iii) Assuming that each game is independent of one another, find the probability that his total net gain in 30 games is more than \$20. [3]

8 In a computer game, a counter moves along a straight line and is originally placed at $k = 0$. At each stage, it takes one step to the right with probability p or one step to the left with probability q, where $q = 1 - p$. Each step is of length 1 unit and the step taken at each stage is independent of one another.

For illustration purpose, the counter is seen to be at $k = -2$ in the diagram below.



The counter takes 10 consecutive steps.

(i) Explain why the counter cannot possibly end at $k = 7$. [1]

(ii) Show that the probability that the counter ends at $k = 6$ is $45p^8q^2$. [2]

(iii) Given that the most probable end-point of the counter is $k = 6$, find exactly the range of values of p in the form $p_1 < p < p_2$ where p_1 and p_2 are constants to be determined. [4]

(iv) Write down the most probable end-point(s) of the counter if $p = 1 - p_1$. [1]

9 A manufacturer produces sweets in 2 flavours, lime and mint. On average, a proportion p of the sweets are mint. The sweets are packed randomly in bags of 12 and the random variable X denotes the number of mint sweets in each bag.

(i) State, in context, two assumptions needed for the number of mint sweets in a bag to be well-modelled by a binomial distribution. [2]

Assume now that the number of mint sweets in a bag has a binomial distribution.

(ii) Explain why the variance of X does not exceed 3. [1]

(iii) Find the range of values of p such that in a bag, the probability that there are 5 mint sweets is less than 0.1. Give your answer to 2 decimal places. [2]

Take $p = 0.65$ for the rest of the question.

(iv) Find the probability that in a randomly selected bag, at least half of the sweets are mint. [2]

The bags are now randomly packed into boxes. Each box contains 5 bags.

(v) Find the probability that in each of the bags in a randomly selected box, at least half of the sweets are mint. [1]

(vi) Find the probability that there are at least 30 mint sweets in a randomly selected box. [2]

(vii) Explain why the answer to part **(vi)** is greater than that in **(v)**. [1]

10 The queue time for customers, in minutes, at a particular hair salon has a mean of 15 minutes and a standard deviation of 10 minutes.

(i) Explain why a normal model would not be suitable for the queue times. [1]

The time duration of a haircut for customers, in minutes, at 2 different hair salons, Qcut, an express service, and SPcut, a specialist service, are normally distributed with mean and standard deviation as given in the table.

	Mean	Standard Deviation
Qcut	9.2	σ
SPcut	20.7	3.1

(ii) If the probability that a Qcut customer having a haircut duration of more than 10 minutes does not exceed 0.35, find the range of values σ can take up. [2]

Use $\sigma = 1.5$ for the rest of the question.

(iii) Given that the difference between a haircut duration of a randomly chosen Qcut customer and its mean has a 95% chance of being larger than k, find the value of k. [2]

(iv) Find the probability that the mean haircut duration of two Qcut customers and three SPcut customers is less than 15 minutes. [3]

(v) State an assumption needed for your calculations in part **(iv)** to be valid. [1]

The salons fee-charging system comprises 2 components. Qcut charges a fixed component of \$3 and a variable component of \$0.50 per minute. Similarly, SPcut charges a fixed component of \$10 and a variable component of \$2 per minute.

(vi) Find the probability that the fees paid by a randomly chosen SPcut customer is at least 6 times the fees paid by a randomly chosen Qcut customer. [3]

11 **In this question you should state clearly the variables and the values of the parameters of any distribution you use.**

A pharmaceutical company has 300 employees from the research department and 330 employees from the manufacturing department.

(i) Billy wishes to find out about the stress level of employees in the research department, so he sends a survey to the 300 employees in the research department. Explain whether these 300 employees form a sample or a population. [1]

(ii) Charlie wishes to conduct a survey regarding the employees' experiences with a new facility. 100 of them are randomly selected from the research department and the other 100 are randomly selected from the manufacturing department. Explain whether the method will form a random sample. [2]

The pharmaceutical company produces pills against a particular bacteria. The manager claims that the average time taken to manufacture a batch of pills is less than 12 hours. The time taken in hours, x was measured on 80 occasions and the results are summarised as follows:

$$\sum(x-12) = -12.8, \quad \sum(x-12)^2 = 16.19.$$

(iii) Find the unbiased estimates of the population mean and variance of the time taken to manufacture a batch of pills. [2]

(iv) Carry out a hypothesis test on the manager's claim. Find the p-value for this test and explain what it indicates about the manager's claim. [4]

It was found that the testing process had not been rigorous enough and the same hypothesis test was conducted in another random sample of 80 observations. The sample mean was found to be k hours and the sample standard deviation was found to be 1.5 hours. It was found that the manager's claim was not supported at 1% level of significance.

(v) Find the range of values of k, correct to 3 decimal places. [3]

Y6 Prelim Paper 2 Mark Scheme

Qn	Suggested Solutions
1(i)	Given : $w = 3-4\mathrm{i}$ $z_1 = \mathrm{Im}\left(\frac{w}{w^*}\right) = \mathrm{Im}\left(\frac{w^2}{\lvert w\rvert^2}\right)$ $= \left(\frac{1}{\lvert w\rvert^2}\right)\mathrm{Im}(-7-24\mathrm{i})$ $= -\frac{24}{25}$ Or $\frac{w}{w^*} = \frac{3-4\mathrm{i}}{3+4\mathrm{i}}.\frac{3-4\mathrm{i}}{3-4\mathrm{i}}$ $= \frac{9-24\mathrm{i}-16}{9+16}$ $= \frac{-7-24\mathrm{i}}{25}$ $z_1 = \mathrm{Im}\left(\frac{w}{w^*}\right) = \mathrm{Im}\left(\frac{-7-24\mathrm{i}}{25}\right) = -\frac{24}{25}$ $z_2 = w - w^* = -8\mathrm{i}$
(ii)	Since coefficients of polynomial eqn are all real, by complex conjugate root theorem, $z_3 = z_2^{\ *} = 8\mathrm{i}$ Hence $[z-8\mathrm{i}][z+8\mathrm{i}]\left[z-\left(-\frac{24}{25}\right)\right] = 0$ $\left[z^2+64\right]\left[z+\frac{24}{25}\right] = 0$ Simplifying: $25z^3+24z^2+1600z+1536=0$ $\therefore \quad b=25,\ c=24,\ d=1600$

	Suggested Solution
2 (i)	$\frac{1}{2^{k+1}-3}-\frac{1}{2^k-3}=\frac{(2^k-3)-(2^{k+1}-3)}{(2^{k+1}-3)(2^k-3)}$ $=\frac{2^k-2^{k+1}}{(2^{k+1}-3)(2^k-3)}$ $=\frac{2^k-2^k(2)}{(2^{k+1}-3)(2^k-3)}$ $=\frac{2^k(1-2)}{(2^{k+1}-3)(2^k-3)}$ $=\frac{-2^k}{(2^{k+1}-3)(2^k-3)}$ $\sum_{k=1}^{N}\frac{2^{k+1}}{(2^{k+1}-3)(2^k-3)}=-2\sum_{k=1}^{N}\frac{-2^k}{(2^{k+1}-3)(2^k-3)}$ $=-2\sum_{k=1}^{N}\left(\frac{1}{2^{k+1}-3}-\frac{1}{2^k-3}\right)$ $=-2\left[\frac{1}{2^2-3}-\frac{1}{2^1-3}\right.$ $+\frac{1}{2^3-3}-\frac{1}{2^2-3}$ $+\frac{1}{2^4-3}-\frac{1}{2^3-3}$ $+\ \vdots$ $+\frac{1}{2^{N-1}-3}-\frac{1}{2^{N-2}-3}$ $+\frac{1}{2^N-3}-\frac{1}{2^{N-1}-3}$ $\left.+\frac{1}{2^{N+1}-3}-\frac{1}{2^N-3}\right]$ $=-2\left[\frac{1}{2^{N+1}-3}-\frac{1}{2^1-3}\right]$ $=-2\left(\frac{1}{2^{N+1}-3}+1\right)$

(ii)	$\frac{16}{(61)(29)}+\frac{32}{(125)(61)}+\frac{64}{(253)(125)}+\ldots+\frac{2^{k-1}}{(2^{k+1}-3)(2^k-3)}+\ldots$ $=\frac{1}{4}\sum_{k=5}^{\infty}\frac{2^{k+1}}{(2^{k+1}-3)(2^k-3)}$ $=\frac{1}{4}\left[-2(0+1)-\sum_{k=1}^{4}\frac{2^{k+1}}{(2^{k+1}-3)(2^k-3)}\right]$ $=\frac{1}{4}\left[-2(0+1)-\left(-\frac{60}{29}\right)\right]$ $=\frac{1}{58}$

Qn	Solution
3(i)	Surface area $=2\pi r^2+\pi r^2+2\pi rh$ $=3\pi r^2+2\pi rh$ Since $3\pi r^2+2\pi rh=k$, $h=\frac{k-3\pi r^2}{2\pi r}$ Let the volume be V. $V=\frac{2}{3}\pi r^3+\pi r^2 h$ $=\frac{2}{3}\pi r^3+\pi r^2\left(\frac{k-3\pi r^2}{2\pi r}\right)$ $=\frac{2}{3}\pi r^3+\frac{k}{2}r-\frac{3}{2}\pi r^3$ $=-\frac{5}{6}\pi r^3+\frac{k}{2}r$ $\frac{dV}{dr}=\frac{d}{dr}\left(-\frac{5}{6}\pi r^3+\frac{k}{2}r\right)$ $=-\frac{15}{6}\pi r^2+\frac{k}{2}$

When V max, $\dfrac{dV}{dr} = 0$:

$$-\frac{15}{6}\pi r^2 + \frac{k}{2} = 0$$

$$r^2 = \frac{\frac{k}{2}}{\frac{15}{6}\pi} = \frac{6k}{30\pi} = \frac{k}{5\pi} \text{ ---------------(*)}$$

$$r = \sqrt{\frac{k}{5\pi}}$$

$$\frac{dV}{dr} = -\frac{15}{6}\pi r^2 + \frac{k}{2} \Rightarrow \frac{d^2V}{dr^2} = -5\pi r < 0 \ (\because r > 0)$$

Hence, V max when $r = \sqrt{\dfrac{k}{5\pi}}$

Alternative

$$\frac{dV}{dr} = \frac{5\pi}{2}\left(\frac{k}{5\pi} - r^2\right) = \frac{5\pi}{2}\left(\sqrt{\frac{k}{5\pi}} + r\right)\left(\sqrt{\frac{k}{5\pi}} - r\right)$$

r	$r < \sqrt{\frac{k}{5\pi}}$	$r = \sqrt{\frac{k}{5\pi}}$	$r < \sqrt{\frac{k}{5\pi}}$
$\frac{dV}{dr}$	$(+)(+) = +$	0	$(+)(-) = -$

(ii)	$V=-\frac{5}{6}\pi r^3+\frac{k}{2}r$ $=-\frac{5}{6}\pi\left(\frac{k}{\sqrt{5\pi}}\right)^3+\frac{k}{2}\left(\sqrt{\frac{k}{5\pi}}\right)$ $=-\frac{k^{\frac{3}{2}}}{6\sqrt{5\pi}}+\frac{k^{\frac{3}{2}}}{2\sqrt{5\pi}}$ $=\frac{k^{\frac{3}{2}}}{3\sqrt{5\pi}}$ From (*), we have $k=5\pi r^2$ Hence, $h=\frac{k-3\pi r^2}{2\pi r}=\frac{5\pi r^2-3\pi r^2}{2\pi r}=\frac{2\pi r^2}{2\pi r}=r$ Ratio of $r:h$ is 1:1.

Qn	Solution
4(a)	$\int\sin^{-1}4x\ \mathrm{d}x=x\sin^{-1}4x-\int\frac{4x}{\sqrt{1-16x^2}}\ \mathrm{d}x$ $=x\sin^{-1}4x+\frac{1}{8}\int-32x(1-16x^2)^{-\frac{1}{2}}\ \mathrm{d}x$ $=x\sin^{-1}4x+\frac{1}{8}\frac{\sqrt{1-16x^2}}{0.5}+C$ $=x\sin^{-1}4x+\frac{1}{4}\sqrt{1-16x^2}+C$
(b) **(i)**	$\frac{\mathrm{d}x}{\mathrm{d}t}=a(1-t^2),\ \frac{\mathrm{d}y}{\mathrm{d}t}=2at$ Tangent at P: $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2at}{a(1-t^2)}=\frac{2t}{1-t^2}$ $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{t=p}=\frac{2p}{1-p^2}$ $y-ap^2=\frac{2p}{1-p^2}\left(x-a\left(p-\frac{p^3}{3}\right)\right)$

	Alternative $y = mx + c$ $ap^2 = \frac{2pa}{1-p^2}\left(p - \frac{p^3}{3}\right) + c$ $c = ap^2 - \frac{2pa}{1-p^2}\left(p - \frac{p^3}{3}\right)$ $\therefore y = \frac{2p}{1-p^2}x + ap^2 - \frac{2pa}{1-p^2}\left(p - \frac{p^3}{3}\right)$ $y = \frac{2p}{1-p^2}x + ap^2 - \frac{2p^2a}{1-p^2} + \frac{2p^4a}{3(1-p^2)}$
(ii)	At Q, $y = 0$: $-ap^2 = \frac{2p}{1-p^2}\left(x - a\left(p - \frac{p^3}{3}\right)\right)$ $\frac{-ap(1-p^2)}{2} = x - ap + \frac{ap^3}{3}$ $x = ap - \frac{ap^3}{3} - \frac{ap}{2} + \frac{ap^3}{2} = \frac{ap}{2} + \frac{ap^3}{6}$ $\therefore OQ = \frac{ap}{2} + \frac{ap^3}{6}$ Arc length OP $= \int_0^p \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$ $= \int_0^p \sqrt{a^2\left(1-t^2\right)^2 + \left(2at\right)^2}\ dt$ $= a\int_0^p \sqrt{1 - 2t^2 + t^4 + 4t^2}\ dt$ $= a\int_0^p \sqrt{(t^2+1)^2}\ dt$ $= a\int_0^p (t^2+1)\ dt$ $= a\left[\frac{t^3}{3} + t\right]_0^p$ $= a\left(\frac{p^3}{3} + p\right)$ $OP = a\left(\frac{p^3}{3} + p\right) = 2\left(\frac{ap}{2} + \frac{ap^3}{6}\right) = 2OQ$

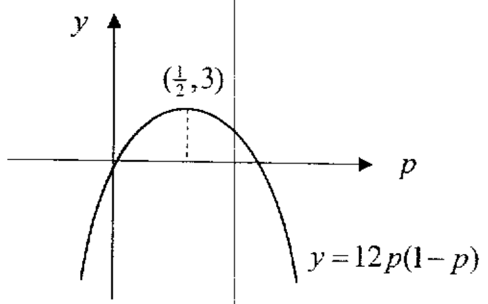
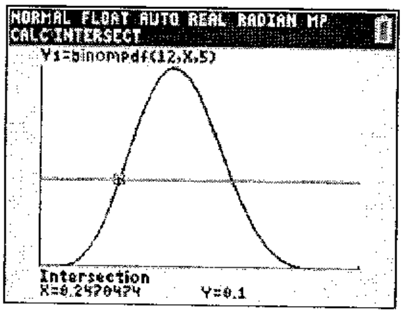
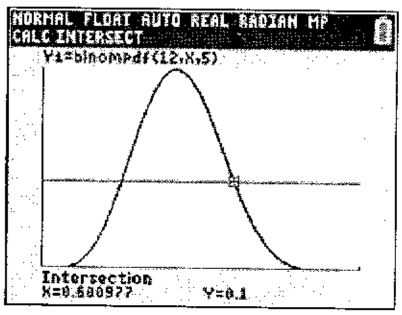
Qn	Suggested Solutions
5(a)	$\overrightarrow{OD} = -\overrightarrow{OC}$ $\mathbf{d} = -\left(\frac{2}{7}\mathbf{a} - \frac{3}{7}\mathbf{b}\right)$ $= \frac{1}{7}(3\mathbf{b} - 2\mathbf{a})$ By ratio theorem, $\overrightarrow{OD} = \frac{4\overrightarrow{OR} + 3\overrightarrow{OB}}{7}$ $4\overrightarrow{OR} = 7\overrightarrow{OD} - 3\overrightarrow{OB}$ $\overrightarrow{OR} = \frac{1}{4}\left(7\left(\frac{3}{7}\mathbf{b} - \frac{2}{7}\mathbf{a}\right) - 3\mathbf{b}\right)$ $= -\frac{1}{2}\mathbf{a} = -\frac{1}{2}\overrightarrow{OA}$ Since $\overrightarrow{OR}$ can be expressed as a scalar multiple of $\overrightarrow{OA}$ and O is a common point, the points A, O and R are collinear. The ratio of AO to OR is $1 : \frac{1}{2} = 2 : 1$
(b)(i)	Area of triangle AOB $= \frac{1}{2}\left\|\overrightarrow{OA} \times \overrightarrow{OB}\right\|$ $= \frac{1}{2}\left\|\begin{pmatrix}1\\3\\2\end{pmatrix} \times \begin{pmatrix}-1\\5\\3\end{pmatrix}\right\|$ $= \frac{1}{2}\left\|\begin{pmatrix}-1\\-5\\8\end{pmatrix}\right\|$ $= \frac{1}{2}\sqrt{90}$ $= \frac{3}{2}\sqrt{10}$
(ii)	$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ refers to the collection of all points on the plane that is perpendicular to $\mathbf{a} \times \mathbf{b}$ and containing the origin.
(iii)	$\mathbf{r} \cdot \begin{pmatrix}-1\\-5\\8\end{pmatrix} = 0$ Distance required $= \frac{\left\|0 - \begin{pmatrix}-8\\-2\\9\end{pmatrix} \cdot \begin{pmatrix}-1\\5\\8\end{pmatrix}\right\|}{\left\|\begin{pmatrix}-1\\5\\8\end{pmatrix}\right\|} = \frac{90}{\sqrt{90}} = \sqrt{90}$

	Suggested Solution
6(a)(i)	# of selections $= {}^2C_1 \cdot {}^3C_1 = 6$. **Alt:** 3C1 +3C1 = 6
(ii)	Case 1: 2 of the same colour $\underbrace{\left({}^3C_1 \cdot {}^3C_2\right)}_{\text{Exactly 1 pair same colour}} + \underbrace{{}^3C_2}_{\text{2 pairs same colour}}$ Case 2: 3 of the same colour (answer from (a)) ${}^2C_1 \cdot {}^3C_1$ Case 3: 4 (all) the same colour ${}^1C_1$ Total # of selections $= \left[\left({}^3C_1 \cdot {}^3C_2\right) + {}^3C_2\right] + \left({}^2C_1 \cdot {}^3C_1\right) + {}^1C_1$ $= (9+3)+6+1$ $= 19$
(iii)	Total # of selections w/o restrictions $= \underbrace{19}_{\text{Ans from (b)}} + \underbrace{1}_{\text{All different}}$ $= 20$ # of selections w/o green and yellow $= \underbrace{4}_{\text{2 blue, 2 black; 3 bllue, 1 black; 3 black, 1 blue; 4 black}}$ # of selections required $= 20 - 4 = 16$
(b)	Total # of arrangements $= \frac{7!}{4!2!}\left({}^8C_3\right) = 5880$

	Solution	
7i)	$\sum P(X=x)=1$ $k+2k+3k+\frac{3k}{3}+\frac{4k}{3}+\frac{5k}{3}=1$ $k=\frac{1}{10}$ (shown)	
ii)	$E(X)$ $=\frac{1}{10}+2\left(\frac{2}{10}\right)+3\left(\frac{3}{10}\right)+4\left(\frac{1}{10}\right)+5\left(\frac{2}{15}\right)+6\left(\frac{1}{6}\right)$ $=\frac{52}{15}\approx 3.46666.....$ Expected net gain = $E(X-3)=E(X)-3$ $=\frac{52}{15}-3=0.46666...\approx 0.47$ (2dp) or 0.467 (3sf) Variance of net gain $=\text{Var}(X-3)$ $=\text{Var}(X)$ $=1.5860^2=2.5153\approx 2.52$ (3sf) (or $\frac{566}{225}$)	
iii)	Let $Y=X-3$ which denotes the net gain of one game. Since $n=30$ is large, by Central Limit Theorem, $Y_1+Y_2+\cdots+Y_{30}\sim N\left(30(0.46666),30(2.5153)\right)$ approximately $P\left(Y_1+Y_2+\cdots+Y_{30}>20\right)=0.24486=0.245$(3sf)	

Qn	Solution
8	Let R and L be the number of right and left steps taken respectively.
(i)	Note that since counter starts at 0, $R-L$ gives the number of the ending position. For the counter to end at $k=7$, $R-L=7 \Rightarrow R=L+7$ Also, since game is played for 10 stages, $R+L=10$. Hence, $(L+7)+L=10 \Rightarrow L=\frac{3}{2}$ (Or show that $R=\frac{17}{2}$) But L must be integer, hence it's not possible for counter to end at $k=7$. **Alternative 1** Case 1: R is odd, L is odd (since must add up 10) $\Rightarrow$ $R-L$ is even. Case 2: R is even, L is even $\Rightarrow$ $R-L$ is even Hence, one can never end at an odd numbered position with 10 steps starting at 0. **Alternative 2** Any combination of 9 right steps and 1 left step leads to $k=8$. Any combination of 8 right steps and 2 left steps leads to $k=6$. Hence $k \neq 7$. **Alternative 3** Any combination of 9 right steps and 2 left step with a total of 11 steps leads to $k=7$. Any combination of 8 right steps and 1 left step with a total of 9 steps leads to $k=7$. Hence it is not possible to get to $k=7$ in exactly 10 steps.
(ii)	For the counter to end at $k=6$, $R-L=6$ and $R+L=10$ Hence $R=8$ and $L=2$. Any combination of 8 right steps and 2 left steps occurs with probability p^8q^2. Number of such combinations is $\binom{10}{8}=45$. Hence the probability that the counter ends at $k=6$ is $45p^8q^2$.
(iii)	$R \sim \mathrm{B}(10,p)$ For the most probable end-point to be $k=6$, the mode of R is 8. As R is binomial, it suffices to ensure the two inequalities below are satisfied: $\binom{10}{7}p^7q^3 < \binom{10}{8}p^8q^2 \quad \ldots(1)$ $\binom{10}{9}p^9q^1 < \binom{10}{8}p^8q^2 \quad \ldots(2)$ From (1), $8(1-p)<3p \Rightarrow p>\frac{8}{11}$ From (2) $2p<9(1-p) \Rightarrow p<\frac{9}{11}$ Hence we have $\frac{8}{11}<p<\frac{9}{11}$ i.e. $p_1=\frac{8}{11}$, $p_2=\frac{9}{11}$.

(iv)	Note that when $p = p_1$, the modes of R are 7 and 8 i.e. most probable end-points for the counter are $k = 4$ and $k = 6$. When $p = 1 - p_1$ (complement probability), most probable end-points for the counter will be $k = -4$ and $k = -6$. (by symmetry since interchange right with left) **Alternative** $p = 1 - \frac{8}{11} = \frac{3}{11}$ $R \sim \mathrm{B}\left(10, \frac{3}{11}\right)$ Modes of $R = 2$ and 3 The two most probable end-points for the counter will be $k = -4$ and $k = -6$.

Qn	Suggested Solutions
9(i)	1. The probability of a randomly chosen sweet being mint is a constant p for each sweet in a bag. 2. The event that a sweet in the bag is mint is independent of another sweet being a mint sweet.
(ii)	Let X = number of mint sweets in a bag of 12 $X \sim \mathrm{B}(12, p)$ $\mathrm{Var}(X) = 12p(1-p)$  From the graph, we can see that maximum $\mathrm{Var}(X)$ is 3 (when $p = \frac{1}{2}$) Hence, variance of X does not exceed 3. **Alternative** $\mathrm{Var}(X) = 12p(1-p) = -12(p - \frac{1}{2})^2 + 3$ Since $(p - \frac{1}{2})^2 \geq 0$, $-12(p - \frac{1}{2})^2 \leq 0$ $-12(p - \frac{1}{2})^2 + 3 \leq 3$ $\therefore \mathrm{Var}(X) \leq 3$
(iii)	$\mathrm{P}(X = 5) < 0.1$  

	From GC, $p < 0.24704$ or $p > 0.60097$ $\therefore p < 0.24$ or $p > 0.60$ (2 d.p.)
(iv)	$X \sim B(12, 0.65)$ Required probability = $P(X \geq 6) = 1 - P(X \leq 5)$ $= 0.91536$ $= 0.915$ (3sf)
(v)	Required probability = $(0.91536)^5 = 0.643$ (3sf)
(vi)	Let Y be the number of mint sweets in a box. $Y \sim B(60, 0.65)$ Required probability = $P(Y \geq 30) = 1 - P(Y \leq 29)$ $= 0.994$ (3sf)
(vii)	The event in part **(v)** a subset of the event in part **(vi)**.

Qn	Suggested Solutions
10 (i)	Let X denote queue time. If $X \sim N(15, 10^2)$, then $P(X < 0) = 0.0668$, not close to zero. Since queue time cannot be negative, a normal model would not be suitable.
(ii)	Let Q denote haircut duration of a randomly chosen customer at Qcut (in min). $Q \sim N(9.2, \sigma^2)$ $P(Q > 10) \leq 0.35$ $P\left(Z > \frac{10 - 9.2}{\sigma}\right) \leq 0.35$ $\frac{0.8}{\sigma} \geq 0.38532$ $\sigma \leq 2.0761$ $\sigma \leq 2.08$ (3sf)
(iii)	Let Q denote haircut duration of a randomly chosen customer at Qcut (in min). $Q \sim N(9.2, 1.5^2)$ Given $P(\lvert Q - 9.2 \rvert > k) = 0.95$, $9.2 + k = 9.29406$ $k = 0.0941$

	Alternative By complement, $P(\lvert Q-9.2\rvert < k) = 0.05$ $P\left(\lvert Z\rvert < \frac{k}{1.5}\right) = 0.05$ $\frac{k}{1.5} = 0.062707$ $k = 0.0941$
(iv)	Let S denote haircut duration of a randomly chosen customer at SPcut (in min). $S \sim N(20.7, 3.1^2)$ Let $M = \frac{Q_1+Q_2+S_1+S_2+S_3}{5}$ $\sim N\left(\frac{2\times 9.2+3\times 20.7}{5}, \frac{2\times 1.5^2+3\times 3.1^2}{25}\right)$ i.e. $M \sim N(16.1, 1.3332)$ $P(M<15) = 0.17037 = 0.170$ (3 s.f.)
(v)	The haircut duration of each customer must be independent of each other / All the haircut durations are independent.
(vi)	$10+2S \geq 6(3+0.5Q)$ $2S-3Q \geq 8$ $2S-3Q \sim N(13.8, 58.69)$ $P(2S-3Q \geq 8) = 0.77550 = 0.776$ (3 sf)

Qn	Suggested Solution
11(i)	Billy is interested in the stress levels of employees in the research department, and surveyed **all 300 employees** in the research department, thus the 300 employees constitute a population.
(ii)	Probability of selecting an employee from research department $= \frac{100}{300} = \frac{1}{3}$ Probability of selecting an employee from manufacturing department $= \frac{100}{330} = \frac{10}{33}$ Since not every employee has the same chance of being selected, the method does not give us a random sample.

(iii)

$$\bar{x} = \frac{\sum(x-12)}{80} + 12 = \frac{-12.8}{80} + 12 = 11.84$$

$$s^2 = \frac{1}{80-1}\left[\sum(x-12)^2 - \frac{\left(\sum(x-12)\right)^2}{80}\right]$$

$$= \frac{1}{79}\left[16.19 - \frac{(-12.8)^2}{80}\right]$$

$$= 0.17901 = 0.179 \text{ (3 s.f.)}$$

(iv)

$H_0 : \mu = 12$
$H_1 : \mu < 12$

where μ is the population mean time taken

Under H_0, $\overline{X} \sim N\left(12, \frac{0.17901}{80}\right)$ approximately by Central Limit Theorem since sample size of 80 is large.
From GC, p-value = 0.000359 (3 s.f.)
Since the ***p*-value is very small,** there is very strong evidence to reject H_0 and conclude that the **manager's claim is valid.**

(v)

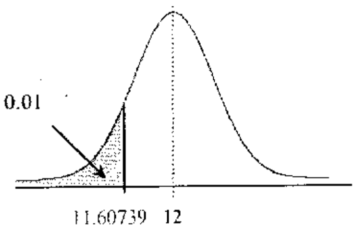
Given,
$\bar{x} = k, \ n = 80$

$$s^2 = \frac{n}{n-1}(\text{sample variance}) = \frac{80}{79}\left(1.5^2\right)$$

Under H_0, $\overline{X} \sim N\left(12, \frac{1.5^2}{79}\right)$ approximately by Central Limit Theorem since $n = 80$ is large.

Manager's claim not supported $\Rightarrow$ we do not reject H_0

__Method 1 (critical value)__



Critical value is 11.60739 at 1% significance level.
$k > 11.608$ (3 d.p) or $k \geq 11.608$ (3 d.p)

__Method 2 (p-value)__

p-value > 0.01

$P(\overline{X} \leq k) > 0.01$

$k > 11.608$ (3 d.p) or $k \geq 11.608$ (3 d.p)