1 A function is defined as $\mathrm{f}(x)=\dfrac{1-\mathrm{e}^{x}}{\mathrm{e}^{x-2}}$.

(i) Show that $\mathrm{f}(x)$ can be written in the form $\mathrm{e}^{\mathrm{g}(x)}+A$, where A is a constant and g(x) is a function in x to be determined. [2]

(ii) Hence describe a sequence of transformations that transforms the graph of $y=\mathrm{e}^{x}$ onto the graph of $y=\mathrm{f}(x)$. [3]

2 A curve C has equation $\dfrac{x^{3}-2y^{2}}{x^{2}+3xy}=1$.

(i) The points P and Q on C each have x-coordinate 1. Find the exact gradients of the tangents to C at the points P and Q. [5]

(ii) Find the acute angle between the tangents to C at the points P and Q. [2]

3 The function f is defined as $\mathrm{f}: x \mapsto \dfrac{a}{2-x},\ x\in\mathbb{R},\ x\neq 0, 2$.

(i) Find f^{2} and f^{-1} in terms of a, stating their domains clearly. [5]

(ii) Find the value of a such that $\mathrm{f}^{2}(x)=\mathrm{f}^{-1}(x)$ for all $x\in\mathbb{R},\ x\neq 0, 2, \dfrac{a}{2}$. [2]

(iii) Using the value of a found in **(ii)**, find $\mathrm{f}^{2021}(x)$. [2]

4 The curve C has equation $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=\dfrac{x(x+a)}{x-a}$ and a is a positive real constant.

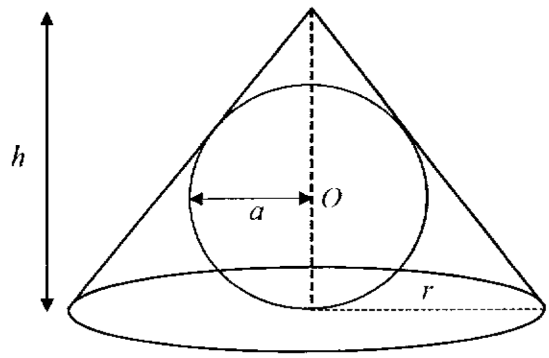
(i) Find algebraically, in terms of a, the set of values that y can take. [4]

(ii) Sketch C, indicating clearly the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [3]

(iii) By adding a suitable graph to your same diagram in **(ii)** and labelling it clearly, solve the inequality $\dfrac{x(x+a)}{x-a}<|3x|$. [3]

5 [It is given that a cone of radius r and height h has volume $\frac{1}{3}\pi r^2 h$.]

A sphere with fixed radius a and centre O is inscribed in a right cone with base radius r and height h. The sphere is in contact with the base and the curved surface of the cone, as shown in the diagram below.



(i) Show that $r^2 = \dfrac{a^2 h}{h-2a}$. [2]

(ii) Use differentiation to find, in terms of a, the minimum volume of the cone, proving that it is a minimum. [6]

6 It is given that $e^y = 2 + \sin x$.

(i) Show that $\dfrac{d^2 y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 2e^{-y} - 1$. [2]

(ii) By further differentiation of the result in **(i)**, find the Maclaurin series expansion for y in ascending powers of x, up to and including the term in x^3. [3]

(iii) By using the standard series from the List of Formulae (MF26), verify your answer in **(ii)**. [3]

7 Mr Chan is interested in investing in a savings account with an interest rate of 3.2% per year, so that on the last day of each year, the amount in the savings account on that day is increased by 3.2%. He decided to invest \$8 000 on the first day of each year, starting from 2020.

(i) Show that the amount in the account at the end of n years after the interest has been added is given by $\$258\,000(1.032^n - 1)$. [3]

After 10 years, Mr Chan stopped investing in this savings account. If he does not withdraw any money from this savings account, the savings account will still continue to generate interest of 1.5% per year on the amount in the account, so that on the last day of each year the amount in the account on that day is increased by 1.5%. Assuming that Mr Chan does not withdraw any money from the savings account,

(ii) by the end of which year will the total interest be first more than \$35 000 from the day that Mr Chan first started his saving account? [3]

At the age of 55, Mr Chan is able to receive a monthly pay-out over a period of 20 years from the savings account if he did not withdraw before. The monthly pay-out in the first year is \$850. The monthly pay-out for each subsequent year is an increment of $\$D$ from the monthly pay-out of the previous year. The total pay-out to Mr Chan at the end of 20 complete years is \$352 200.

(iii) Given that the monthly pay-out is \$1 500 for the mth year of the 20 year period, where m is a positive integer, find m. [3]

8 (i) By using the substitution $x = \cos\theta$, find $\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x$, expressing your answer in terms of x. [4]

(ii) Verify that the curves with equations $y = \frac{x^3}{\sqrt{1-x^2}}$ and $y = \frac{1}{\sqrt{49-4x^2}}$ intersect at the point with x-coordinate $\frac{1}{2}$. [1]

(iii) Hence find the exact area bounded by the curves $y = \frac{x^3}{\sqrt{1-x^2}}$ and $y = \frac{1}{\sqrt{49-4x^2}}$, and the y-axis. [4]

9 (a) (i) Show that the cubic polynomial $x^3 + px^2 + p^2x + q$ can be reduced to $y^3 + \left(\frac{2p^2}{3}\right)y + \alpha$ by the substitution $x = y - \frac{p}{3}$, where α is to be determined in terms of p and q. [3]

(ii) Given that -3i is a root of the equation $y^3 + 6y - 9\text{i} = 0$, find the other two roots exactly in the form $a + b\text{i}$. [3]

(iii) Hence find the exact roots of the equation $x^3 + 3x^2 + 9x + 7 - 9\text{i} = 0$. [2]

(b) Given that $z = \text{e}^{\text{i}\theta}$, show that $1 + z + z^2 + z^3 + \cdots + z^{n-1} = z^{\frac{n-1}{2}}\left(\frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\right)$. [3]

10 The **Gompertz differential equation** provides a good model for gauging lung cancer growth. The differential equation is given as

$$\frac{\text{d}V}{\text{d}t} = aV\ln\left(\frac{K}{V}\right),$$

where V mm^3 is the volume of the tumour at time t days after an early discovery, and a and K are positive real constants.

(i) Describe the behaviour of $\frac{\text{d}V}{\text{d}t}$ as $V \to K$. [1]

(ii) By using the substitution $u = \ln\left(\frac{K}{V}\right)$, solve the given differential equation and show that $V = K\text{e}^{-A\text{e}^{-at}}$, where A is an arbitrary constant. [5]

(iii) What happens to V as $t \to \infty$? State the significance of K in the context of this question. [2]

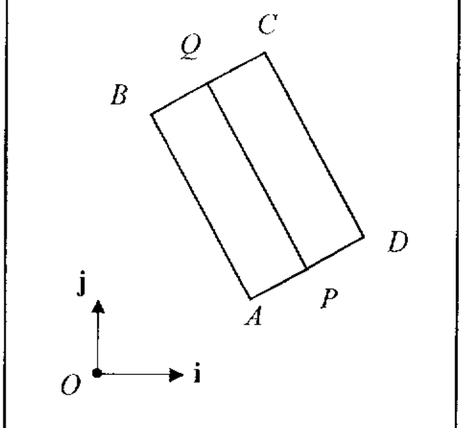
For the rest of this question, you may assume that $a = 0.01$ and $K = 8000$.

A patient was discovered early to have a lung tumour of volume 100 mm^3.

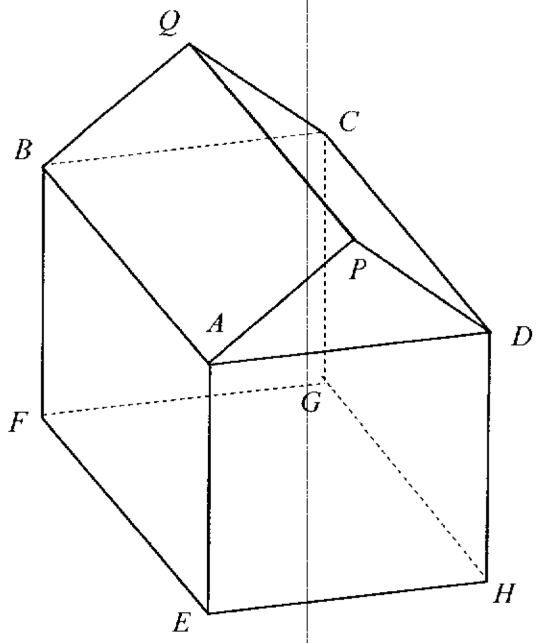
(iv) Find the size of the tumour after 100 days. Give your answer to the nearest mm^3. [2]

(v) Sketch the graph of V against t. [2]

11 Mr Neo wants to build a shed in his future garden as shown in the diagrams below.



Top view (Position of shed relative to an origin O)



The planes $ABCD$ and $EFGH$ are parallel to the horizontal plane, represented by the xy-plane. The triangles APD and BQC are congruent isosceles triangles and the lengths of the pillars AE, BF, CG and DH are of the same height.

The equation of the plane $ABQP$ is given by $-2x - y + 3z = 2$ and the point P has position vector $4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$.

(i) Explain why plane $CDPQ$ is perpendicular to $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. Hence show that the equation of the plane $CDPQ$ is $2x + y + 3z = 22$. [2]

(ii) Find the angle the roof $ABQP$ makes with the horizontal plane. [2]

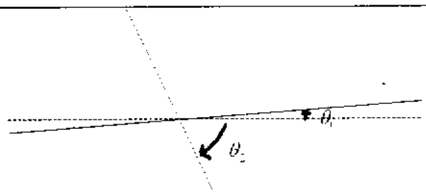
(iii) Find the vector equation of the line PQ and hence find the coordinates of the point Q given that PQ has length $3\sqrt{5}$ units. [5]

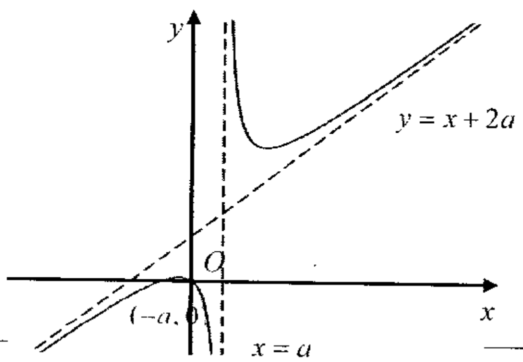
(iv) When the shed was completed, Mr Neo discovered a hole in the roof $ABQP$. When light shines perpendicularly onto the plane $ABQP$, the light passes through the hole and hits the ground at the point with coordinates (3, 6, 0).

Find the coordinates of the hole in the roof $ABQP$. [3]

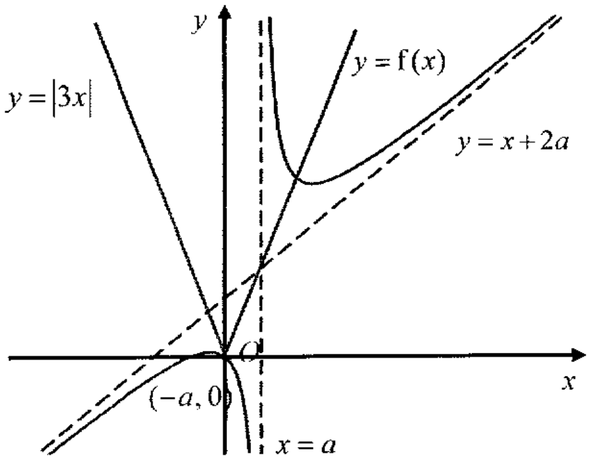
2021 ACJC H2 Math Prelim P1 Marking Scheme

Qn	Solutions		
1(i)	$\mathrm{f}(x)=\frac{1-\mathrm{e}^x}{\mathrm{e}^{x-2}}$ $=\frac{1}{\mathrm{e}^{x-2}}-\frac{\mathrm{e}^x}{\mathrm{e}^{x-2}}$ $=\mathrm{e}^{-x+2}-\mathrm{e}^2$ $\therefore A=-\mathrm{e}^2$ and $\mathrm{g}(x)=-x+2$		
1(ii)	1. A translation of 2 units in the negative x-axis direction. 2. A reflection in the y-axis. 3. A translation of e^2 units in the negative y-axis direction. **Alternative1:** 1. A reflection in the y-axis. 2. A translation of 2 units in the positive x-axis direction. 3. A translation of e^2 units in the negative y-axis direction. **Alternative2:** $\mathrm{f}(x)=\mathrm{e}^{-x+2}-\mathrm{e}^2=\mathrm{e}^2\bullet\mathrm{e}^{-x}-\mathrm{e}^2$ 1. A reflection in the y-axis. 2. A Scaling parallel to the y-axis by the scale factor e^2. 3. A translation of e^2 units in the negative y-axis direction. **Alternative3:** $\mathrm{f}(x)=\mathrm{e}^{-x+2}-\mathrm{e}^2=\mathrm{e}^2\left(\mathrm{e}^{-x}-1\right)$ 1. A reflection in the y-axis. 2. A translation of 1 unit in the negative y-axis direction. 3. A Scaling parallel to the y-axis by the scale factor e^2.		
2(i)	$\frac{x^3-2y^2}{x^2+3xy}=1$ $x^3-2y^2=x^2+3xy$ Differentiating implicitly wrt x: $3x^2-4y\frac{\mathrm{d}y}{\mathrm{d}x}=2x+\left(3x\frac{\mathrm{d}y}{\mathrm{d}x}+y(3)\right)$ $3x^2-2x-3y=(3x+4y)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x^2-2x-3y}{3x+4y}$ **Alternative (not advised)** Using quotient rule to differentiate:		

	$\dfrac{\left(x^2+3xy\right)\left(3x^2-4y\dfrac{dy}{dx}\right)-\left(x^3-2y^2\right)\left(2x+3x\dfrac{dy}{dx}+y(3)\right)}{\left(x^2+3xy\right)^2}=0$ $\Rightarrow\left(x^2+3xy\right)\left(3x^2-4y\dfrac{dy}{dx}\right)-\left(x^3-2y^2\right)\left(2x+3x\dfrac{dy}{dx}+y(3)\right)=0$ (because $x^2+3xy\neq 0$) $\Rightarrow 3x^2\left(x^2+3xy\right)-\left(x^3-2y^2\right)\left(2x+3y\right)=4y\left(x^2+3xy\right)\dfrac{dy}{dx}+3x\left(x^3-2y^2\right)\dfrac{dy}{dx}$ $\Rightarrow\dfrac{dy}{dx}=\dfrac{3x^2\left(x^2+3xy\right)-\left(x^3-2y^2\right)\left(2x+3y\right)}{4y\left(x^2+3xy\right)+3x\left(x^3-2y^2\right)}=\dfrac{x^4+6x^3y+4xy^2+6y^3}{3x^4+4x^2y+6xy^2}$ Sub $x=1$, $\dfrac{1^3-2y^2}{1^2+3(1)y}=1$ $\Rightarrow 1-2y^2=1+3y$ $\Rightarrow 2y^2+3y=0$ $\Rightarrow y(2y+3)=0 \quad \therefore y=0$ or $y=-\dfrac{3}{2}$ Sub $x=1$ and $y=0$ into $\dfrac{dy}{dx}$: $\dfrac{dy}{dx}=\dfrac{3(1)^2-2(1)-3(0)}{3(1)+4(0)}=\dfrac{1}{3}$ Sub $x=1$ and $y=-\dfrac{3}{2}$ into $\dfrac{dy}{dx}$: $\dfrac{dy}{dx}=\dfrac{3(1)^2-2(1)-3\left(\dfrac{-3}{2}\right)}{3(1)+4\left(\dfrac{-3}{2}\right)}=-\dfrac{11}{6}$	
2(ii)	$\theta_1=\tan^{-1}\left(\dfrac{1}{3}\right)$ $\theta_2=\tan^{-1}\left(\dfrac{11}{6}\right)$  acute angle between tangents : $\theta_1+\theta_2=\tan^{-1}\left(\dfrac{1}{3}\right)+\tan^{-1}\left(\dfrac{11}{6}\right)=79.8$ (to 1d.p.) (or 1.39 rad)	
3(i)	$f(x)=\dfrac{a}{2-x}, \quad x\neq 0, 2$ From graph, $R_f=\left\{x\in\mathbb{R}: x\neq 0, \dfrac{a}{2}\right\}$	

	$f^2(x)=f(f(x))=\dfrac{a}{2-\dfrac{a}{2-x}}=\dfrac{a(2-x)}{4-a-2x}$ $f^2: x \mapsto \dfrac{a(2-x)}{4-a-2x},\ x \neq 0, 2$ Let $y=\dfrac{a}{2-x},\ \ x \neq 0, 2$ $2-x=\dfrac{a}{y} \Rightarrow x=2-\dfrac{a}{y}$ $\therefore f^{-1}: x \mapsto 2-\dfrac{a}{x},\ x \neq 0, \dfrac{a}{2}$		
3(ii)	For $f^2(x)=f^{-1}(x)$, $\dfrac{a(2-x)}{4-a-2x}=2-\dfrac{a}{x} \Rightarrow \dfrac{ax-2}{2x-4+a}=\dfrac{2x-a}{x}$ By observation, $a=4$.		
3(iii)	$f^2(x)=f^{-1}(x) \Rightarrow f^3(x)=x$ Therefore $f^{2021}(x)=f^2 f^{2019}(x)=f^2(x)=\dfrac{2x-4}{x}=2-\dfrac{4}{x}$.		
4(i)	Consider $y=k$, k is a constant $\dfrac{x(x+a)}{x-a}=k$ $x^2+ax=xk-ak$ $x^2+(a-k)x+ak=0$ For the range of y can take, the line $y=k$ and the curve C should have point(s) of intersection. $b^2-4ac \geq 0$ $(a-k)^2-4ak \geq 0$ $a^2-2ak+k^2-4ak \geq 0$ $a^2-6ak+k^2 \geq 0$ Consider $k^2-6ak+a^2=0$ $k=\dfrac{6a \pm \sqrt{36a^2-4a^2}}{2}=\left(3 \pm 2\sqrt{2}\right)a$ $\therefore k \geq \left(3+2\sqrt{2}\right)a$ or $k \leq \left(3-2\sqrt{2}\right)a$ Hence, $y \geq \left(3+2\sqrt{2}\right)a$ or $y \leq \left(3-2\sqrt{2}\right)a$		
4(ii)			

4(iii)



Consider $\frac{x(x+a)}{x-a}=3x$

$x^2+ax=3x^2-3ax$

$2x^2-4ax=0$

$x(x-2a)=0$

$x=0 \quad or \quad x=2a$

Hence, $y=|3x|$ and $y=\mathrm{f}(x)$ intersect at $x=0$ and $x=2a$.

From the graph, $x<0$ or $0<x<a$ or $x>2a$.

Checking:

Consider $\frac{x(x+a)}{x-a}=-3x$

$x^2+ax=-3x^2+3ax$

$4x^2-2ax=0$

$x(2x-a)=0$

$x=0 \quad or \quad x=\frac{a}{2}$ (N.A. since $a>0$)

5(i)

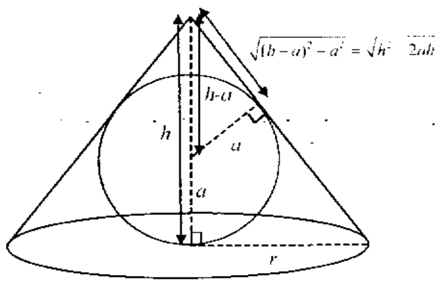
By similar triangles:

$\frac{\sqrt{h^2-2ah}}{h}=\frac{a}{r}$

$\Rightarrow \frac{h^2-2ah}{h^2}=\frac{a^2}{r^2}$ (shown)

$\Rightarrow \frac{h-2a}{h}=\frac{a^2}{r^2}$

$\Rightarrow r^2=\frac{a^2h}{h-2a}$



5(ii)

$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{a^2h}{h-2a}\right)h=\frac{1}{3}\pi a^2\left(\frac{h^2}{h-2a}\right)$

$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{3}\pi a^2\frac{(h-2a)(2h)-h^2(1)}{(h-2a)^2}=\frac{1}{3}\pi a^2\frac{h^2-4ah}{(h-2a)^2}=\frac{1}{3}\pi a^2\frac{h(h-4a)}{(h-2a)^2}$

When V is minimum, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$

$\Rightarrow h(h-4a)=0$

$\therefore h=0$ (rej $\because h>0$) or $h=4a$

$$\frac{dV}{dh}=\frac{1}{3}\pi a^2\frac{h(h-4a)}{(h-2a)^2}=\frac{\pi a^2 h}{3(h-2a)^2}(h-4a)$$

$$\left(\frac{\pi a^2 h}{3(h-2a)^2}>0 \text{ for } h>0\right)$$

h	$(4a)^-$	$4a$	$(4a)^+$
$\frac{dV}{dh}$	$\frac{dV}{dh}<0$ for $h=(4a)^-$ $\left(\begin{matrix}\because h<4a\\ \Rightarrow h-4a<0\end{matrix}\right)$	0	$\frac{dV}{dh}>0$ for $h=(4a)^+$ $\left(\begin{matrix}\because h>4a\\ \Rightarrow h-4a>0\end{matrix}\right)$
Shape	\	–	/

Or 2nd derivative test (by quotient rule):

$$\frac{d^2V}{dh^2}=\frac{d}{dh}\left(\frac{1}{3}\pi a^2\frac{h(h-4a)}{(h-2a)^2}\right)$$

$$=\frac{1}{3}\pi a^2\frac{(h-2a)^2(2h-4a)-h(h-4a)2(h-2a)}{(h-2a)^4}$$

$$=\frac{1}{3}\pi a^2\frac{2(h-2a)\left[(h-2a)(h-2a)-h(h-4a)\right]}{(h-2a)^4}$$

$$=\frac{2}{3}\pi a^2\frac{\left[\cancel{h^2}+4a^2\cancel{-4ah}\cancel{-h^2}+\cancel{4ah}\right]}{(h-2a)^3}$$

$$=\frac{8\pi a^4}{3(h-2a)^3}$$

$$\Rightarrow \left.\frac{d^2V}{dh^2}\right|_{h=4a}=\frac{8\pi a^4}{3(4a-2a)^3}=\frac{\pi a}{3}>0 \quad (\because a>0)$$

Or 2nd derivative test (by implicit differentiation):

$$\frac{dV}{dh}=\frac{\pi a^2 h}{3(h-2a)^2}(h-4a)$$

$$3(h-2a)^2\frac{dV}{dh}=\pi a^2\left(h^2-4ah\right)$$

$$6(h-2a)\frac{dV}{dh}+3(h-2a)^2\frac{d^2V}{dh^2}=\pi a^2(2h-4a)$$

When $h=4a$ $\left(\text{and } \frac{dV}{dh}=0\right)$,

$$0+3(4a-2a)^2\frac{d^2V}{dh^2}=\pi a^2\left(2(4a)-4a\right)$$

$$\Rightarrow \frac{d^2V}{dh^2}=\frac{4\pi a^3}{12a^2}=\frac{\pi a}{3}>0 \quad (\because a>0)$$

Therefore volume of cone is minimum when $h=4a$.

$\therefore$ Minimum volume of the cone $=\frac{1}{3}\pi a^2\left(\frac{(4a)^2}{4a-2a}\right)=\frac{8}{3}\pi a^3$

6(i)

$e^y=2+\sin x$

Differentiating w.r.t. x,

$$e^y\frac{dy}{dx}=\cos x$$

Differentiating w.r.t. x again,

$$e^y\frac{d^2y}{dx^2}+e^y\frac{dy}{dx}\left(\frac{dy}{dx}\right)=-\sin x$$

$$\Rightarrow e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2=2-e^y \quad \left(\because e^y=2+\sin x \Rightarrow -\sin x=2-e^y\right)$$

$$\Rightarrow \frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=2e^{-y}-1 \quad \text{(shown)}$$

Alternative Method 1:

$e^y=2+\sin x \Rightarrow y=\ln(2+\sin x)$

$$\frac{dy}{dx}=\frac{\cos x}{2+\sin x}\Rightarrow (2+\sin x)\frac{dy}{dx}=\cos x......(1)$$

$$(2+\sin x)\frac{d^2y}{dx^2}+\frac{dy}{dx}(\cos x)=-\sin x$$

$$\left(\because e^y=2+\sin x \text{ and fr (1) } \cos x=(2+\sin x)\frac{dy}{dx}=e^y\frac{dy}{dx}\right)$$

$$e^y\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(e^y\frac{dy}{dx}\right)=2-e^y$$

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=2e^{-y}-1 \quad \text{(shown)}$$

Alternative Method 2:

$e^y=2+\sin x \Rightarrow y=\ln(2+\sin x)$

$$\frac{dy}{dx}=\frac{\cos x}{2+\sin x}$$

$$\frac{d^2y}{dx^2}=\frac{(2+\sin x)(-\sin x)-\cos x(\cos x)}{(2+\sin x)^2}$$

$$=\frac{-2\sin x-\sin^2 x-\cos^2 x}{(2+\sin x)^2}$$

$$=\frac{-2\sin x-1}{(2+\sin x)^2}$$

$$\text{LHS}=\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=\frac{-2\sin x-1}{(2+\sin x)^2}+\left(\frac{\cos x}{2+\sin x}\right)^2$$

$$=\frac{-2\sin x-1+\cos^2 x}{(2+\sin x)^2}$$

$$=\frac{-2\sin x\cancel{-1}\cancel{+1}-\sin^2 x}{(2+\sin x)^2}$$

$$=\frac{-\sin x(2+\sin x)}{(2+\sin x)^2}$$

$$=\frac{-\sin x}{2+\sin x}=-1+\frac{2}{2+\sin x}$$

$$=-1+\frac{2}{e^y}=2e^{-y}-1=\text{RHS}\quad\text{(shown)}$$

6(ii)

Differentiate $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=2e^{-y}-1$ implicitly w.r.t. x:

$$\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=-2e^{-y}\frac{dy}{dx}$$

When $x=0$, $e^y=2+\sin 0\Rightarrow y=\ln 2$

$$e^y\frac{dy}{dx}=\cos x\Rightarrow(2)\frac{dy}{dx}=\cos 0\Rightarrow\frac{dy}{dx}=\frac{1}{2}$$

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=2e^{-y}-1\Rightarrow\frac{d^2y}{dx^2}+\left(\frac{1}{2}\right)^2=2\left(\frac{1}{2}\right)-1\Rightarrow\frac{d^2y}{dx^2}=-\frac{1}{4}$$

$$\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=-2e^{-y}\frac{dy}{dx}$$

$$\Rightarrow\frac{d^3y}{dx^3}+2\left(\frac{1}{2}\right)\left(-\frac{1}{4}\right)=-2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\Rightarrow\frac{d^3y}{dx^3}=-\frac{1}{4}$$

$$\therefore y=\ln 2+\left(\frac{1}{2}\right)x+\frac{\left(-\frac{1}{4}\right)}{2!}x^2+\frac{\left(-\frac{1}{4}\right)}{3!}x^3+\ldots$$

$$=\ln 2+\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{24}x^3+\ldots$$

6(iii)

$$e^y=2+\sin x$$

$$y=\ln(2+\sin x)$$

$$\approx\ln\left(2+x-\frac{x^3}{6}\right)=\ln\left(2\left(1+\frac{x}{2}-\frac{x^3}{12}\right)\right)$$

$$=\ln 2+\ln\left(1+\frac{x}{2}-\frac{x^3}{12}\right)$$

$$=\ln 2+\left(\frac{x}{2}-\frac{x^3}{12}\right)-\frac{\left(\frac{x}{2}-\frac{x^3}{12}\right)^2}{2}+\frac{\left(\frac{x}{2}-\frac{x^3}{12}\right)^3}{3}+\ldots$$

$$\approx\ln 2+\frac{x}{2}-\frac{x^3}{12}-\frac{x^2}{8}+\frac{x^3}{24}=\ln 2+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{24}\quad\text{(verified)}$$

7(i)	See table below

*n*th year	Beginning	End
1	8000	8000(1.032)
2	8000(1.032)+8000	$8000(1.032)^2+8000(1.032)$
3	$8000(1.032)^2+8000(1.032)$ +8000	$8000(1.032)^3+8000(1.032)^2$ +8000(1.032)
…		…
n		$8000(1.032)^n+8000(1.032)^{n-1}$ $+\ldots 8000(1.032)^2+8000(1.032)$

$$\text{Total amount} = 8000(1.032)+8000(1.032)^2+\ldots+8000(1.032)^n$$
$$= \frac{8000(1.032)\left(1.032^n-1\right)}{1.032-1}$$
$$= 258000\left(1.032^n-1\right)$$

7(ii)

After 10 years, total amount in the account $= 258000\left(1.032^{10}-1\right)$
$= 95522.18995$

Let k be the additional number of years required.

$$95522.18995(1.015)^k \geq 8000\times 10+35000$$
$$(1.015)^k \geq 1.2039$$
$$k \geq \frac{\ln 1.2039}{\ln 1.015} \quad \Rightarrow \quad k \geq 12.5$$
$$2020+10+13-1=2042$$

Hence, he will have a total interest be first more than \$35 000 at the end of 2042.

7(iii)

Year	Pay-out per month	Pay-out per year
1	850	12(850)
2	850+D	12(850+D)
3	850+2D	12(850+2D)
⋮	⋮	

$$\text{Total pay-out} = 12\times\frac{20}{2}\left(2\times 850+(20-1)D\right)$$
$$120\left(1700+(20-1)D\right)=352200$$
$$19D=1235$$
$$D=65$$

To find the year m with a pay-out of \$1500:

$$850+(m-1)(65)=1500$$
$$(m-1)(65)=650$$
$$m=11$$

∴ The pay-out is \$1500 in the 11^{th} year.

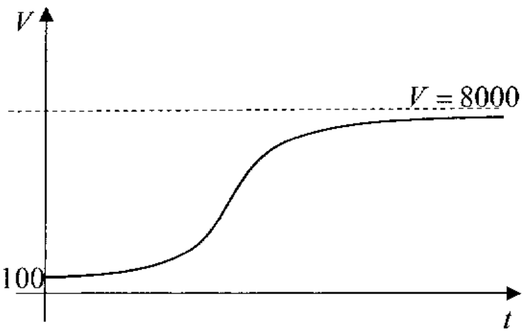
8(i)

$$x=\cos\theta \Rightarrow \frac{dx}{d\theta}=-\sin\theta$$

	$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\cos^3\theta}{\sqrt{1-\cos^2\theta}}(-\sin\theta)\,d\theta$ $= -\int \frac{\cos^3\theta}{\sin\theta}(\sin\theta)\,d\theta$ $= -\int \cos^3\theta\,d\theta = -\int \cos^2\theta\cos\theta\,d\theta$ $= -\int (1-\sin^2\theta)\cos\theta\,d\theta = -\int \cos\theta - \cos\theta\sin^2\theta\,d\theta$ $= -\left(\sin\theta - \frac{\sin^3\theta}{3}\right) + c = \frac{\sin^3\theta}{3} - \sin\theta + c$ $\cos\theta = x \Rightarrow \cos^2\theta = x^2 \Rightarrow 1-\sin^2\theta = x^2$ $\therefore \sin\theta = \sqrt{1-x^2}$ $\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{\sin^3\theta}{3} - \sin\theta + c$ $= \frac{1}{3}(1-x^2)^{\frac{3}{2}} - (1-x^2)^{\frac{1}{2}} + c$		
8(ii)	$y = \frac{\left(\frac{1}{2}\right)^3}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = \frac{\left(\frac{1}{2}\right)^3}{\frac{\sqrt{3}}{2}} = \frac{1}{4\sqrt{3}}$ $\quad y = \frac{1}{\sqrt{49-4\left(\frac{1}{2}\right)^2}} = \frac{1}{\sqrt{49-1}} = \frac{1}{4\sqrt{3}}$ Hence, the 2 curves intersect at the point $\left(\frac{1}{2}, \frac{1}{4\sqrt{3}}\right)$.		
8(iii)	[Using GC to identify which graph is 'on top'] $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{49-4x^2}} - \frac{x^3}{\sqrt{1-x^2}}\,dx$ $= \int_0^{\frac{1}{2}} \frac{1}{2\sqrt{\left(\frac{7}{2}\right)^2 - x^2}} - \frac{x^3}{\sqrt{1-x^2}}\,dx$ $= \left[\frac{1}{2}\sin^{-1}\frac{2x}{7} - \left(-\frac{1}{3}\left(\sqrt{1-x^2}\right)(x^2+2)\right)\right]_0^{\frac{1}{2}}$ $= \left[\frac{1}{2}\sin^{-1}\frac{2x}{7} + \frac{1}{3}\left(\sqrt{1-x^2}\right)(x^2+2)\right]_0^{\frac{1}{2}}$ $= \frac{1}{2}\sin^{-1}\frac{1}{7} + \frac{1}{3}\sqrt{\frac{3}{4}}\left(\frac{9}{4}\right) - 0 - \frac{2}{3}$ $= \left(\frac{1}{2}\sin^{-1}\frac{1}{7} + \frac{9\sqrt{3}-16}{24}\right)$ units^2 $\quad \left(\text{or } \frac{1}{2}\sin^{-1}\frac{1}{7} + \frac{9}{8\sqrt{3}} - \frac{2}{3}\right)$		
9(a)(i)	Sub $x = y - \frac{p}{3}$ into $x^3 + px^2 + p^2x + q$		

	$\left(y-\frac{p}{3}\right)^3+p\left(y-\frac{p}{3}\right)^2+p^2\left(y-\frac{p}{3}\right)+q$ $=\left(y^3-3\left(\frac{p}{3}\right)y^2+3\left(\frac{p}{3}\right)^2y-\left(\frac{p}{3}\right)^3\right)$ $+p\left(y^2-2\left(\frac{p}{3}\right)y+\left(\frac{p}{3}\right)^2\right)$ $+p^2\left(y-\frac{p}{3}\right)+q$ $=y^3+y^2\left(-p+p\right)+y\left(\frac{p^2}{3}-\frac{2p^2}{3}+p^2\right)+\left(-\frac{p^3}{27}+\frac{p^3}{9}-\frac{p^3}{3}+q\right)$ $=y^3+\frac{2p^2}{3}y+\left(q-\frac{7p^3}{27}\right)$ $\therefore \alpha=q-\frac{7p^3}{27}$	
9(a)(ii)	-3i is a root of the equation $y^3+6y-9\text{i}=0$ $y^3+6y-9\text{i}=(y-(-3\text{i}))\left(y^2+ay+b\right)=(y+3\text{i})\left(y^2+ay+b\right)$ Compare coefficients $y^0:-9i=3\text{i}b\Rightarrow b=-3$ $y^2:0=a+3\text{i}\Rightarrow a=-3\text{i}$ $y^3+6y-9\text{i}=(y+3\text{i})\left(y^2-3\text{i}y-3\right)$ For roots of $y^2-3\text{i}y-3=0$ $y^2-3\text{i}y-3$ $y=\frac{-(-3\text{i})\pm\sqrt{(-3\text{i})^2-4(1)(-3)}}{2}=\frac{3\text{i}\pm\sqrt{3}}{2}$ Therefore, the other two roots are $\frac{3\text{i}+\sqrt{3}}{2}$ and $\frac{3\text{i}-\sqrt{3}}{2}$.	
9(a)(iii)	For $x^3+3x^2+9x+7-9\text{i}=0$, let $p=3, q=7-9\text{i}$, then $\frac{2p^2}{3}=6,\quad \alpha=\left(7-9\text{i}\right)-\frac{7(3)^3}{27}=-9\text{i}$ gives the equation in (ii). Hence roots of $x^3+3x^2+9x+7-9\text{i}=0$ are $x=y-1=-3\text{i}-1,\ \frac{3\text{i}+\sqrt{3}-2}{2},\ \frac{3\text{i}-\sqrt{3}-2}{2}$	

9(b)	$1+z+z^2+z^3+\cdots+z^{n-1}$ $=\dfrac{z^n-1}{z-1}=\dfrac{e^{in\theta}-1}{e^{i\theta}-1}$ $=\dfrac{e^{i\frac{n\theta}{2}}\left(e^{i\frac{n\theta}{2}}-e^{-i\frac{n\theta}{2}}\right)}{e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}\right)}$ $=\dfrac{e^{i\frac{(n-1)\theta}{2}}\left(2i\sin\dfrac{n\theta}{2}\right)}{2i\sin\dfrac{\theta}{2}}$ $\left(\begin{array}{c}\because e^{i\alpha}-e^{-i\alpha}=\cos\alpha+i\sin\alpha\\ -(\cos\alpha-i\sin\alpha)\\ =2i\sin\alpha\end{array}\right)$ $=z^{\frac{n-1}{2}}\left(\dfrac{\sin\dfrac{n\theta}{2}}{\sin\dfrac{\theta}{2}}\right)$		
10(i)	As $V\to K$, $\ln\left(\dfrac{K}{V}\right)\to 0$. $\therefore\dfrac{dV}{dt}\to 0$		
10(ii)	$u=\ln\left(\dfrac{K}{V}\right)\Rightarrow\dfrac{du}{dt}=\left(\dfrac{V}{K}\right)\left(-\dfrac{K}{V^2}\right)\dfrac{dV}{dt}$ $\Rightarrow\dfrac{dV}{dt}=-V\dfrac{du}{dt}$ $\dfrac{dV}{dt}=aV\ln\left(\dfrac{K}{V}\right)$ $-V\dfrac{du}{dt}=aVu$ $\dfrac{du}{dt}=-au$ $\int\dfrac{1}{u}\,du=\int -a\,dt$ $\int\dfrac{1}{u}\,du=\int -a\,dt$ $\ln\|u\|=-at+c\Rightarrow\|u\|=e^{-at+c}$ $u=Ae^{-at}$, where $A=\pm e^c$ $\ln\left(\dfrac{K}{V}\right)=Ae^{-at}$ $\dfrac{K}{V}=e^{Ae^{-at}}\Rightarrow V=Ke^{-Ae^{-at}}$		
10(iii)	As $t\to\infty$, $e^{-at}\to 0$ and hence $e^{-Ae^{-at}}\to e^0=1$ $\therefore V\to K$ The size of the lung tumour approaches K (mm^3). K is the maximum possible size of the lung tumour that can be achieved.		
10(iv)	$V=8000e^{-Ae^{-0.01t}}$		

	When $t=0$, $V=100$. $100=8000\mathrm{e}^{-A\mathrm{e}^0} \Rightarrow \frac{1}{80}=\mathrm{e}^{-A} \Rightarrow A=\ln 80$. When $t=100$, $V=8000\mathrm{e}^{-\ln 80\mathrm{e}^{-1}}=1595.81\approx 1596\text{ mm}^3$	
10(v)	 $V=8000\mathrm{e}^{-\ln 80\mathrm{e}^{-0.01t}}$	
11(i)	*ABQP* and *CDPQ* are symmetric about the vertical plane passing through *PQ*. Hence direction of the normal is $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$. $\mathbf{r}.\begin{pmatrix}2\\1\\3\end{pmatrix}=\begin{pmatrix}4\\2\\4\end{pmatrix}.\begin{pmatrix}2\\1\\3\end{pmatrix}=22$. $\quad 2x+y+3z=22$.	
11(ii)	Normal of horizontal plane is **k**. Therefore $\cos\theta=\frac{\begin{pmatrix}-2\\-1\\3\end{pmatrix}.\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{2^2+1^2+3^2}\sqrt{1}}=\frac{3}{\sqrt{14}}\Rightarrow\theta=36.7°$	
11(iii)	*PQ* is the line of intersection of the planes *ABQP* and *CDPQ*. $-2x-y+3z=2$ $2x+y+3z=22$ Solving using GC gives $\mathbf{r}=\begin{pmatrix}5\\0\\4\end{pmatrix}+\lambda'\begin{pmatrix}-\frac{1}{2}\\1\\0\end{pmatrix}=\begin{pmatrix}5\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}$ Given $\lvert\overrightarrow{PQ}\rvert=3\sqrt{5}$.	

	$\overrightarrow{OQ}=\begin{pmatrix}5\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}\Rightarrow\overrightarrow{PQ}=\begin{pmatrix}5\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}-\begin{pmatrix}4\\2\\4\end{pmatrix}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}$ $\therefore\left\|\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\|=3\sqrt{5}\Rightarrow\left\|(\lambda-1)\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\|=3\sqrt{5}\Rightarrow\|\lambda-1\|\left\|\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\|=3\sqrt{5}$ $\Rightarrow\|(\lambda-1)\|=3\Rightarrow\lambda=4$ or -2 $\Rightarrow\overrightarrow{OQ}=\begin{pmatrix}1\\8\\4\end{pmatrix}$ or $\begin{pmatrix}7\\-4\\4\end{pmatrix}$ (reject, from top view y-coordinate >2) Alternatively, $\overrightarrow{OQ}=\overrightarrow{OP}\pm3\sqrt{5}\left(\widehat{PQ}\right)$ $=\begin{pmatrix}4\\2\\4\end{pmatrix}\pm3\sqrt{5}\left(\frac{1}{\sqrt{1^2+2^2}}\begin{pmatrix}-1\\2\\0\end{pmatrix}\right)$ $=\begin{pmatrix}4\\2\\4\end{pmatrix}\pm3\begin{pmatrix}-1\\2\\0\end{pmatrix}=\begin{pmatrix}1\\8\\4\end{pmatrix}$ or $\begin{pmatrix}7\\-4\\4\end{pmatrix}$ (reject, from top view y-coordinate >2)	
11(iv)	Let l be the line passing through $(3, 6, 0)$ and perpendicular to the plane $ABQP$. $l:\mathbf{r}=\begin{pmatrix}3\\6\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\3\end{pmatrix}$ For foot of perpendicular on the $ABQP$, $\left(\begin{pmatrix}3\\6\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\-1\\3\end{pmatrix}\right).\begin{pmatrix}-2\\-1\\3\end{pmatrix}=2$ $\Rightarrow(-6-6)+\mu(4+1+9)=2$ $\Rightarrow\mu=1$ $\Rightarrow\begin{pmatrix}3\\6\\0\end{pmatrix}+\begin{pmatrix}-2\\-1\\3\end{pmatrix}=\begin{pmatrix}1\\5\\3\end{pmatrix}$ Hence hole on roof has coordinates $(1, 5, 3)$. Alternatively, Let the point $(3, 6, 0)$ be R, then $\overrightarrow{RP}=\begin{pmatrix}4\\2\\4\end{pmatrix}-\begin{pmatrix}3\\6\\0\end{pmatrix}=\begin{pmatrix}1\\-4\\4\end{pmatrix}$	

	$\overrightarrow{RN} = \overrightarrow{RP}.\frac{\begin{pmatrix}-2\\-1\\3\end{pmatrix}}{\sqrt{2^2+1^2+3^2}}\frac{\begin{pmatrix}-2\\-1\\3\end{pmatrix}}{\sqrt{2^2+1^2+3^2}} = \frac{\begin{pmatrix}1\\-4\\4\end{pmatrix}.\begin{pmatrix}-2\\-1\\3\end{pmatrix}}{14}\begin{pmatrix}-2\\-1\\3\end{pmatrix} = \begin{pmatrix}-2\\-1\\3\end{pmatrix}$ Therefore, $\overrightarrow{ON} = \overrightarrow{OR} + \overrightarrow{RN} = \begin{pmatrix}3\\6\\0\end{pmatrix} + \begin{pmatrix}-2\\-1\\3\end{pmatrix} = \begin{pmatrix}1\\5\\3\end{pmatrix}$ Hence hole on roof has coordinates $(1, 5, 3)$.	

Section A: Pure Mathematics [40 marks]

1 Three digits are chosen from 0, 1, 2, …, 9 and arranged to form a 3-digit number, with no digit being repeated. The sum of the digits in the 3-digit number is 8. When the digits in the 3-digit number are reversed, the new number is 297 less than the original 3-digit number. Find all possibilities of the 3-digit number. [3]

2 Relative to the origin O, the points A, B, C have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, such that $OACB$ is a quadrilateral. Let P, Q, R and S be the midpoints of the line segments OA, AC, CB and OB respectively.

(i) Show that $PQRS$ is a parallelogram. [2]

(ii) For any vectors $\mathbf{p}$ and $\mathbf{q}$, state the condition for $|\mathbf{p}+\mathbf{q}|=|\mathbf{p}|+|\mathbf{q}|$. [1]

(iii) Hence, by considering vector products, show that the area of $OACB$ is twice the area of $PQRS$. [3]

3 **(i)** It is given that $\mathrm{f}(r)=\dfrac{1}{(r-1)r!}$ for $r \geq 2, r \in \mathbb{Z}$.

Show that $\mathrm{f}(r)-\mathrm{f}(r+1)=\dfrac{r^2+1}{r(r-1)(r+1)!}$. [1]

The sum $\displaystyle\sum_{r=2}^{n} \frac{r^2+1}{r(r-1)(r+1)!}+3^{-r}$ is denoted by S_n.

(ii) Find an expression for S_n in terms of n. [4]

(iii) Explain why S_∞ is a convergent series and state its value. [2]

4 A curve C has parametric equations

$$x=\ln(3+\theta), \qquad y=\cos^{-1}\left(\frac{\theta}{3}\right) \qquad \text{for } -3<\theta\leq 3.$$

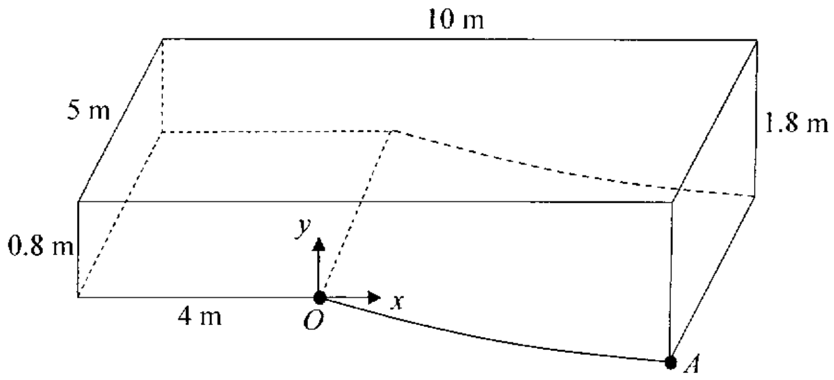
(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{\dfrac{3+\theta}{3-\theta}}$. What can be said about the tangent to C as $\theta \to 3$? [3]

(ii) Sketch the curve C, stating the exact equations of any asymptotes and coordinates of any axial intercepts, showing clearly the feature of the curve at the point where $\theta=3$. [3]

(iii) Show that the gradient of C at the point with y-coordinate p is $-\cot\dfrac{p}{2}$. [3]

(iv) The normal to C at the point with y-coordinate p passes through the point $(0,1)$. Find the value of p. [3]

5 Tom wishes to build a pool in his backyard. The following diagram shows the dimensions of the pool he desires.



The surface of the pool is rectangular, while the floor consists of a rectangular piece and a sloped piece with a side from point O to point A as shown (with the depth of the pool gradually deepening from 0.8 m to 1.8 m). Taking O as the origin, the curve OA can be modelled by the equation $y = -\sqrt{hx + k}$, where h and k are real constants.

(i) Find the values of h and k. [2]

(ii) Hence find the volume of water required to fill the pool to the brim. [2]

Tom's wife, Jane, prefers a more artistic pool design, involving the curve C given by the equation $y = e^{0.04x^2} - 3$. Her preferred pool is formed by rotating the region bounded by the curve C and the x-axis π radians about the y-axis.

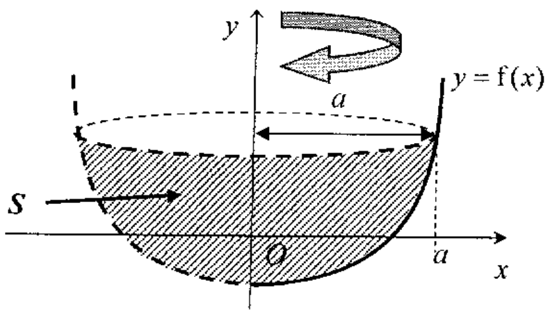
(iii) What is the exact volume of water required to fill Jane's preferred pool? Give your answer in the form $(a \ln b - c)\pi$ m^3, where a, b and c are real constants to be determined. [5]

5 [Continued]

For a curve $y = \mathrm{f}(x)$, where $0 \le x \le \alpha$, to be rotated 2π radians about the y-axis, the formula for the **surface area of the revolution** is given by

$$S = 2\pi\int_0^{\alpha} x\sqrt{1+\left(\mathrm{f}'(x)\right)^2}\,\mathrm{d}x,$$

where S is the surface area of the revolution and $\mathrm{f}'(x)$ is the derivative of $\mathrm{f}(x)$ with respect to x (see diagram below for illustration).



(iv) To ensure that there is no seepage in her preferred pool, Jane wants to laminate the entire curved side/floor of the pool. By using the formula given, find the area of laminating material required. Give your answer to the nearest m^2. [3]

Section B: Probability and Statistics [60 marks]

6 (a) The events A and B are such that $\mathrm{P}(A \cap B) = 0.03$ and $\mathrm{P}(A \cup B) = 0.37$.

(i) Find $\mathrm{P}\left[(A \cap B)' \mid (A \cup B)\right]$. [2]

It is given that events A and B are independent.

(ii) If $\mathrm{P}(A) > \mathrm{P}(B)$, find the possible values of $\mathrm{P}(A)$ and $\mathrm{P}(B)$. [3]

(b) The events X, Y and Z are such that events X and Z are mutually exclusive, $\mathrm{P}(X) = 0.2$, $\mathrm{P}(Y \cap Z) = 0.05$ and $\mathrm{P}(X' \cap Y' \cap Z') = 0.28$. Find the maximum and minimum possible values of $\mathrm{P}(Z)$. [3]

7 A group of 12 students consists of 3 students from class A, 4 students from class B and 5 students from class C.

(i) Find the number of ways in which a committee of 8 students can be chosen from the 12 students if it includes at least 1 student from each class. [2]

(ii) The 12 students from the 3 classes sit at random at a round table. Albert is a student from class A and Bob is a student from class B. Find the probability that Albert and Bob are seated together and no two students from class A are next to each other. [2]

(iii) Each of the 12 students attends one of 3 leadership programmes X, Y and Z. The table below shows the number of students from each class attending the various leadership programmes.

	Leadership Programmes		
	X	Y	Z
Class A	3	0	0
Class B	0	3	1
Class C	0	0	5

4 students are selected from the 12 students to participate in a group interview about the leadership programmes. They are arranged to sit in a row of 8 labelled seats such that there is exactly one empty seat between every 2 students as part of safe management measures.

Find the number of possible arrangements if each arrangement must include students from all 3 classes with representation from all 3 leadership programmes. [4]

8 3 discs are taken, at random and without replacement, from a bag containing 5 red discs and n white discs, where $n \geq 3$. The random variable R is the number of red discs taken and the random variable W is the number of white discs taken.

(i) Determine the probability distribution of W. [3]

(ii) Show that $\mathrm{E}(W) = \dfrac{3n}{n+5}$ and $\mathrm{Var}(W) = \dfrac{\mathrm{g}(n)}{(n+5)^2(n+4)}$, where $\mathrm{g}(n)$ is a quadratic polynomial to be determined. [5]

(iii) Hence write down an expression for $\mathrm{Var}(R)$. [1]

9 The Particle Filtration Efficiency (PFE) of a mask is a measure of how well a mask filters airborne particles such as pollen or dust. A mask with higher PFE is deemed to be of better efficiency as it filters more particles. A mask with a PFE of 95% would have met the requirement for surgical masks.

A factory manufactures Brand BEY surgical masks that is known to have expected PFE of 95.8%. During a routine check of the manufacturing process, the quality control manager suspects that the efficiency of the Brand BEY surgical masks produced is compromised such that the mean PFE is reduced. The PFE, x%, of a random sample of 50 masks is taken and the summarised results are as follows.

$$\sum(x-90)=289 \qquad \sum(x-90)^2=1670.56$$

(i) State what it means for a sample to be random in this context. [1]

The manager carries out a hypothesis test at 1% level of significance.

(ii) Explain whether there is a need for the manager to make any assumption about the population distribution of the PFE of the masks. [1]

(iii) State the appropriate hypotheses, defining any symbols that you use. Test whether the manager's suspicion is justified at 1% level of significance. [5]

The manager discovered an error in the data collection process of the sample and discarded 10 out of the 50 readings. From the remaining random sample of 40 masks, the manager found that the mean PFE is 95.5% and the standard deviation is k %. Given that the manager concludes that there is insufficient evidence to justify his suspicion at 1% level of significance, find the range of possible values of k used in calculating the test statistic. [3]

10 On average, 91% of the people who have taken a vaccine will develop immunity to a particular virus. A random sample of n people who have taken the vaccine is chosen. The number of people in the sample who develop immunity after taking the vaccine is denoted by A.

(i) State, in context, what must be assumed for A to be well modelled by a binomial distribution. [2]

Assume now that A has a binomial distribution.

(ii) Given that the most likely number of people who develop immunity after taking the vaccine is 19, find the value of n without using a graphing calculator. [4]

(iii) Find the probability that in a randomly chosen sample of 25 people, more than 3 people did not develop immunity after taking the vaccine. [1]

A sample of 25 people with at most 21 people who developed immunity after taking the vaccine is deemed to be undesirable.

(iv) Find the probability that out of 30 randomly chosen samples of 25 people, there are at most 2 undesirable samples and the eighth chosen sample is the first undesirable sample. [3]

When a randomly chosen patient develops immunity to the virus after vaccination, there is a 30% chance that he exhibits an allergic reaction. When a randomly chosen patient does not develop immunity to the virus after vaccination, there is a 10% chance that he exhibits an allergic reaction.

(v) Find the probability that a randomly chosen patient who exhibited an allergic reaction after vaccination has developed immunity to the virus. [2]

11 In this question you should state the parameters of any distribution that you use. You should also assume that X, Y and S follow independent normal distributions.

Billy leaves his workplace X minutes past 6 pm daily from Monday to Friday to pick his son up from the childcare centre before its closing time at 7 pm. X follows the distribution $N(12,3^2)$. The time taken for the journey from his workplace to the childcare centre, Y minutes, follows the distribution $N(m,10^2)$, where m is a positive constant. The childcare centre imposes a penalty fine on parents who arrive at the childcare centre after 7 pm.

Given that Billy pays a penalty fine 22.176% of the time, show that $m = 40.0$. [3]

For the rest of this question, assume that $m = 40.0$.

(i) The time in minutes after 6 pm at which Billy arrives at the childcare centre each day is denoted by W.

Sketch the distribution of W for the period from 6.20 pm to 7.20 pm. [2]

(ii) Find the latest time Billy has to leave his workplace in order for him to have at least a 98% chance that he will not have to pay a penalty fine. [2]

(iii) In the month of July, the childcare centre is in operation on 22 workings days. It is given that on n randomly chosen working days in July, the probability that Billy's mean journey time from his workplace to the childcare centre exceeds 42 minutes is at most 0.2. Find the possible values of n. [3]

Billy's workplace is relocated to a new address and the time taken for the journey from his new workplace to the childcare centre, S minutes, now follows an independent distribution $N(65,6^2)$. The childcare centre imposes a penalty fine of \$1.50 per late minute on parents who arrive at the childcare centre after 7 pm. Each day, Billy leaves his workplace X minutes past 6 pm to pick his son up from the childcare centre.

(iv) Find the distribution of T, where T is the time in minutes after 7 pm at which Billy arrives at the childcare centre on a randomly chosen day. Hence find the probability that the total penalty fine paid by Billy for 10 randomly chosen days is more than \$180. [3]

2021 ACJC H2 Math Prelim P2 Marking Scheme

Qn	Solutions		
1			

Let x, y and z be the digits on the hundreds, tens and ones position respectively.

$$x+y+z=8 \quad -----(1)$$

$$(100x+10y+z)-(100z+10y+x)=297----(2)$$

$$\Rightarrow \begin{aligned} x+y+z&=8 \\ x-z&=3 \end{aligned}$$

Using GC: $x=3+z,\ \ y=5-2z,\ \ z=z$

Since x, y and z are non-negative integer values,

$3+z\geq 0 \Rightarrow z\geq -3$

$5-2z\geq 0 \Rightarrow z\leq 2.5$

$z\geq 0$

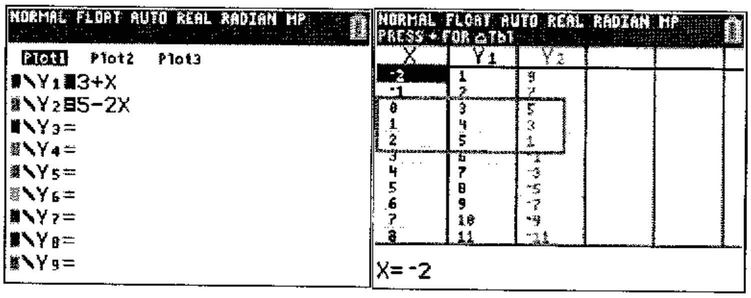
$\therefore 0\leq z\leq 2.5$

When $z=0, x=3, y=5$

When $z=1, x=4, y=3$

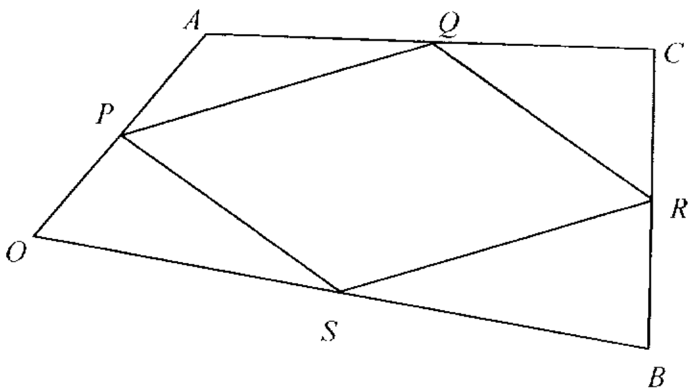
When $z=2, x=5, y=1$

Alternative: using GC



Hence, the possible original numbers are 350, 431 or 512.

Qn	Solutions		
2(i)			



	$\mathbf{p}=\frac{1}{2}\mathbf{a}$ $\quad \mathbf{q}=\frac{1}{2}(\mathbf{a}+\mathbf{c})$ $\mathbf{r}=\frac{1}{2}(\mathbf{b}+\mathbf{c})$ $\quad \mathbf{s}=\frac{1}{2}\mathbf{b}$ $\overrightarrow{PQ}=\mathbf{q}-\mathbf{p}=\frac{1}{2}(\mathbf{a}+\mathbf{c})-\frac{1}{2}\mathbf{a}=\frac{1}{2}\mathbf{c}$ $\overrightarrow{SR}=\mathbf{r}-\mathbf{s}=\frac{1}{2}(\mathbf{b}+\mathbf{c})-\frac{1}{2}\mathbf{b}=\frac{1}{2}\mathbf{c}=\overrightarrow{PQ}$ Therefore $PQRS$ is a parallelogram.	
2(ii)	$\lvert\mathbf{p}+\mathbf{q}\rvert=\lvert\mathbf{p}\rvert+\lvert\mathbf{q}\rvert$ if and only if **p** and **q** are parallel and in the same direction.	
2(iii)	Area of $PQRS = \lvert\overrightarrow{PQ}\times\overrightarrow{PS}\rvert=\left\lvert\frac{1}{2}\mathbf{c}\times\frac{1}{2}(\mathbf{b}-\mathbf{a})\right\rvert=\frac{1}{4}\lvert\mathbf{c}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}\rvert=\frac{1}{4}\lvert\mathbf{c}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}\rvert$ Since O, A, B and C are coplanar, then $\mathbf{c}\times\mathbf{b}$ and $\mathbf{a}\times\mathbf{c}$ are normal to the plane containing $OACB$ and in the same direction. Hence Area of $PQRS = \frac{1}{4}\left(\lvert\mathbf{c}\times\mathbf{b}\rvert+\lvert\mathbf{a}\times\mathbf{c}\rvert\right)$ Area of $OACB$ = Area of OAC + Area of OCB $=\frac{1}{2}\lvert\mathbf{a}\times\mathbf{c}\rvert+\frac{1}{2}\lvert\mathbf{b}\times\mathbf{c}\rvert=2(\text{Area of PQRS})$.	
3(i)	$\mathrm{f}(r)-\mathrm{f}(r+1)=\frac{1}{(r-1)r!}-\frac{1}{r(r+1)!}$ $=\frac{1}{(r-1)r!}-\frac{1}{r(r+1)!}$ $=\frac{r(r+1)-(r-1)}{r(r-1)(r+1)!}$ $=\frac{r^2+1}{r(r-1)(r+1)!}$	
3(ii)	$\sum_{r=2}^{n}\frac{r^2+1}{r(r-1)(r+1)!}+3^{-r}=\sum_{r=2}^{n}\mathrm{f}(r)-\mathrm{f}(r+1)+\sum_{r=2}^{n}\left(\frac{1}{3}\right)^r$ $=\begin{bmatrix}\mathrm{f}(2)-\mathrm{f}(3)\\+\mathrm{f}(3)-\mathrm{f}(4)\\+\mathrm{f}(5)-\mathrm{f}(6)\\ \vdots\\+\mathrm{f}(n-1)-\mathrm{f}(n)\\+\mathrm{f}(n)-\mathrm{f}(n+1)\end{bmatrix}+\frac{\frac{1}{3}^2\left(1-\left(\frac{1}{3}\right)^{n-1}\right)}{1-\frac{1}{3}}$ $=\mathrm{f}(2)-\mathrm{f}(n+1)+\left(\frac{3}{2}\right)\left(\frac{1}{9}\right)\left(1-\frac{1}{3^{n-1}}\right)$ $=\frac{1}{(2-1)2!}-\frac{1}{(n+1-1)(n+1)!}+\frac{1}{6}\left(1-\frac{3}{3^n}\right)$ $=\frac{2}{3}-\frac{1}{n(n+1)!}-\frac{1}{2}\left(\frac{1}{3}\right)^n$	
3(iii)	As $n\to\infty$, $\frac{1}{n(n+1)!}\to 0$ and $\left(\frac{1}{3}\right)^n\to 0$.	

Hence $S_n \to \frac{2}{3}$ which is a constant, hence S_n is convergent.

$\therefore S_\infty = \frac{2}{3}$.

4(i)

$x = \ln(3+\theta)$, $\qquad y = \cos^{-1}\left(\frac{\theta}{3}\right)$

$$\frac{dx}{d\theta} = \frac{1}{3+\theta} \qquad \frac{dy}{d\theta} = -\frac{1}{\sqrt{1-\left(\frac{\theta}{3}\right)^2}} \times \frac{1}{3} = -\frac{1}{\sqrt{9-\theta^2}}$$

$$\frac{dy}{dx} = \frac{-\frac{1}{\sqrt{9-\theta^2}}}{\frac{1}{3+\theta}} = -\frac{3+\theta}{\sqrt{9-\theta^2}}$$

$$= -\frac{3+\theta}{\sqrt{(3+\theta)(3-\theta)}}$$

$$= -\frac{\cancel{\sqrt{(3+\theta)}}\sqrt{(3+\theta)}}{\cancel{\sqrt{(3+\theta)}}\sqrt{(3-\theta)}}$$

$$= -\sqrt{\frac{3+\theta}{3-\theta}}$$

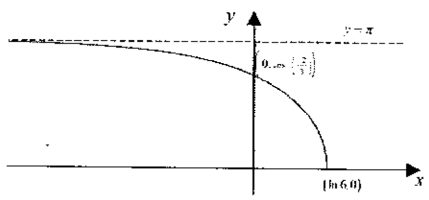
As $\theta \to 3$, $\frac{dy}{dx} \to \infty \Rightarrow$ The tangent is parallel to the y-axis.

4(ii)

As $\theta \to -3$, $x = \ln(3+\theta) \to -\infty$

and $y = \cos^{-1}\left(\frac{\theta}{3}\right) \to \cos^{-1}\left(\frac{-3}{3}\right) = \pi$

$\therefore y = \pi$ is a horizontal asymptote of the curve C.



when $x = 0 \Rightarrow \ln(3+\theta) = 0 \Rightarrow \theta = -2 \Rightarrow y = \cos^{-1}\left(\frac{-2}{3}\right)$

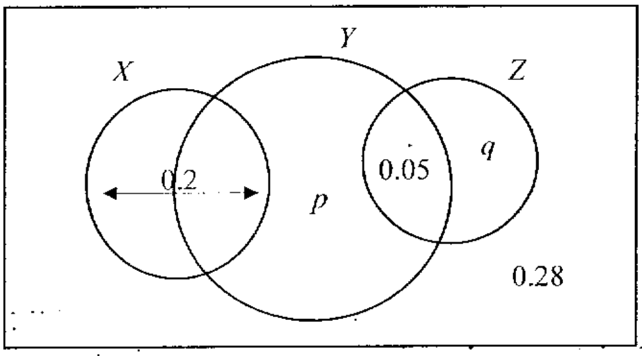
$\Rightarrow y$-intercept at $\left(0, \cos^{-1}\left(\frac{-2}{3}\right)\right)$

when $y = 0 \Rightarrow \cos^{-1}\left(\frac{\theta}{3}\right) = 0 \Rightarrow \theta = 3 \Rightarrow x = \ln(3+3) = \ln 6$

$\Rightarrow x$-intercept at $(\ln 6, 0)$

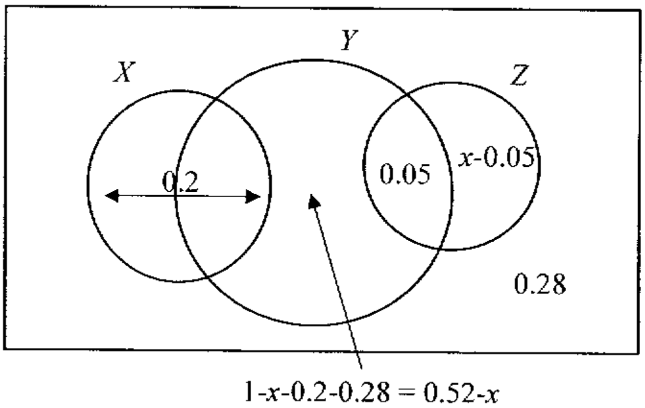
4(iii)	$y=p \Rightarrow p=\cos^{-1}\left(\frac{\theta}{3}\right) \Rightarrow \theta = 3\cos p$ $\Rightarrow \left.\frac{dy}{dx}\right\|_{\theta=3\cos p} = -\sqrt{\frac{3+3\cos p}{3-3\cos p}}$ $= -\sqrt{\frac{3+3\left(2\cos^2\frac{p}{2}-1\right)}{3-3\left(1-2\sin^2\frac{p}{2}\right)}}$ $= -\sqrt{\frac{6\cos^2\frac{p}{2}}{6\sin^2\frac{p}{2}}}$ $= -\sqrt{\cot^2\frac{p}{2}} = -\cot\frac{p}{2}$ (shown) $(\because 0 \le y < \pi \Rightarrow \cot\frac{p}{2} > 0)$ Or $x = \ln(3+\theta), y=\cos^{-1}\left(\frac{\theta}{3}\right) \Rightarrow \theta = 3\cos y$ $\Rightarrow x = \ln(3+3\cos y)$ $\Rightarrow \frac{dx}{dy} = \frac{-3\sin y}{3+3\cos y} = -\frac{\sin y}{1+\cos y} \Rightarrow \frac{dy}{dx} = -\frac{1+\cos y}{\sin y}$ $\left.\frac{dy}{dx}\right\|_{y=p} = -\frac{1+\cos p}{\sin p}$ $= -\frac{1+\left(2\cos^2\frac{p}{2}-1\right)}{2\sin\frac{p}{2}\cos\frac{p}{2}}$ $= -\frac{2\cos^2\frac{p}{2}}{2\sin\frac{p}{2}\cos\frac{p}{2}}$ $= -\frac{\cos\frac{p}{2}}{\sin\frac{p}{2}} = -\cot\frac{p}{2}$ (shown)	
4(iv)	Gradient of normal $= -\frac{1}{\left(-\cot\frac{p}{2}\right)} = \tan\frac{p}{2}$ when $y=p \Rightarrow p=\cos^{-1}\left(\frac{\theta}{3}\right) \Rightarrow \theta = 3\cos p$ (may have been found in (iii)) $\Rightarrow x = \ln(3+3\cos p)$ $\therefore$ equation of normal: $y-p=\tan\frac{p}{2}\left(x-\ln(3+3\cos p)\right)$	

	$(0,1)$ lies on the normal and thus satisfies normal equation $\Rightarrow 1-p=\tan\frac{p}{2}\left(0-\ln\left(3+3\cos p\right)\right)$ $\Rightarrow \tan\frac{p}{2}\left(\ln\left(3+3\cos p\right)\right)+1-p=0$ Using GC, since $0\le p<\pi \Rightarrow p=1.94$ (to 3sf)		
5(i)	$y=-\sqrt{hx+k}$ $(0,0):0=-\sqrt{k} \Rightarrow k=0$ Point A has coordinates $(6,-1)$. $(6,-1):-1=-\sqrt{6h+0} \Rightarrow h=\frac{1}{6}$		
5(ii)	Volume of water required: $(10\times 5\times 0.8)+5\times\int_0^6\left\|-\sqrt{\frac{x}{6}}\right\|\mathrm{d}x=60\ \mathrm{m}^3$		
5(iii)	$y=\mathrm{e}^{0.04x^2}-3$ When $x=0,\ y=1-3 \Rightarrow y=-2$. $y+3=\mathrm{e}^{0.04x^2}$ $\ln(y+3)=0.04x^2$ $x^2=25\ln(y+3)$ Volume of water required to fill Jane's preferred pool: $\pi\int_{-2}^{0}x^2\ \mathrm{d}y=25\pi\int_{-2}^{0}\ln(y+3)\,\mathrm{d}y$ $=25\pi\left(\left[y\ln(y+3)\right]_{-2}^{0}-\int_{-2}^{0}\frac{y}{y+3}\,\mathrm{d}y\right)$ $=-25\pi\left(\int_{-2}^{0}1-\frac{3}{y+3}\,\mathrm{d}y\right)$ $=-25\pi\left(\left[y-3\ln(y+3)\right]_{-2}^{0}\right)$ $=-25\pi\left(-3\ln 3-(-2)\right)$ $=(75\ln 3-50)\pi\ \mathrm{m}^3$ $a=75,\ b=3,\ c=50$.		
5(iv)	$y=\mathrm{e}^{0.04x^2}-3$ When $y=0,\ \mathrm{e}^{0.04x^2}=3 \Rightarrow 0.04x^2=\ln 3 \Rightarrow x=\pm 5\sqrt{\ln 3}$. (or $x=\pm 5.24$) $\frac{\mathrm{d}y}{\mathrm{d}x}=0.08x\mathrm{e}^{0.04x^2}$ $\therefore S=2\pi\int_0^{5\sqrt{\ln 3}}x\sqrt{1+\left(0.08x\mathrm{e}^{0.04x^2}\right)^2}\ \mathrm{d}x$ $=102.077\ \mathrm{m}^2\approx 102\ \mathrm{m}^2$ (if $x=5.24$ used, $S\approx 102.038$)		

6(a) (i)	$$\mathrm{P}\left((A\cap B)'\mid A\cup B\right)=\frac{\mathrm{P}\left[(A\cap B)'\cap(A\cup B)\right]}{\mathrm{P}(A\cup B)}$$ $$=\frac{\mathrm{P}(A\cup B)-\mathrm{P}(A\cap B)}{\mathrm{P}(A\cup B)}$$ $$=\frac{0.37-0.03}{0.37}$$ $$=\frac{34}{37}\text{ or }0.919\text{ (3sf)}$$	
6(a) (ii)	A and B are independent events, $\mathrm{P}(A\cap B)=\mathrm{P}(A)\cdot\mathrm{P}(B)=0.03$ $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)=0.37$ $\Rightarrow \mathrm{P}(A)+\mathrm{P}(B)-0.03=0.37$ $\Rightarrow \mathrm{P}(A)+\mathrm{P}(B)=0.4$ Let $\mathrm{P}(A)=a$, $\mathrm{P}(B)=b$ So $ab=0.03$ --- (1) & $a+b=0.4$ Sub. $b=0.4-a$ into (1), $\therefore a(0.4-a)=0.03$ $a^2-0.4a+0.03=0$ Solving $a=0.3$ or 0.1 $b=0.1$ or 0.3 Since $\mathrm{P}(A)>\mathrm{P}(B)$, $P(A)=0.3$, $\mathrm{P}(B)=0.1$	
6(b)	 <u>Method 1</u> From the Venn Diagram, $0.2+p+0.05+q+0.28=1\Rightarrow p=0.47-q$ $p=0.47-q$ $p\geq 0\Rightarrow q\leq 0.47$ Thus $0\leq q\leq 0.47$ Since $\mathrm{P}(Z)=0.05+q$ $\therefore$ min $\mathrm{P}(Z)=0.05$, max $\mathrm{P}(Z)=0.47+0.05=0.52$	

Alternatively

Let $P(Z) = x$



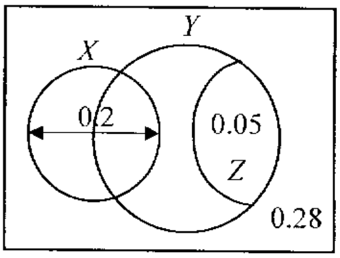
$x - 0.05 \geq 0 \Rightarrow x \geq 0.05$

$0.52 - x \geq 0 \Rightarrow x \leq 0.52$

Thus $0.05 \leq x \leq 0.52$

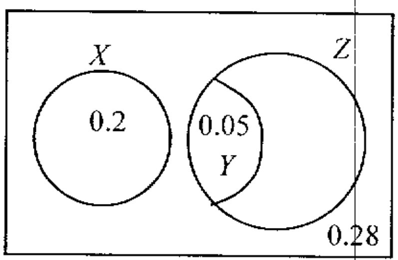
<u>Method 2</u>

For min P(Z),
Z is a subset of Y.



So min P(Z) = 0.05

For max P(Z),
Y is a subset of Z.



max P(Z) = $1 - 0.28 - 0.2 = 0.52$

7(i)

No. of ways $= {}^{12}C_8 - {}^{9}C_8 - {}^{8}C_8$

$= 495 - 9 - 1$

$= 485$

7(ii)

Probability $= \dfrac{(9-1)! \times 2 \times {}^{8}C_2 \times 2!}{(12-1)!} = \dfrac{56}{495}$

7(iii)

<u>Case 1: 2A, 1B, 1C</u>

No of choices of 4 students = ${}^{3}C_2 \times {}^{3}C_1 \times {}^{5}C_1$ (AX, AX, BY, CZ) = 45

<u>Case 2: 1A, 2B, 1C</u>

No of choices of 4 students = ${}^{3}C_1 \times {}^{3}C_2 \times {}^{5}C_1$ (AX, BY, BY, CZ)

$+ {}^{3}C_1 \times {}^{3}C_1 \times {}^{1}C_1 \times {}^{5}C_1$ (AX, BY, BZ, CZ)

$= 45 + 45 = 90$

<u>Case 3: 1A 1B 2C</u>

No of choices of 4 students = ${}^{3}C_1 \times {}^{3}C_1 \times {}^{5}C_2$ (AX, BY, CZ, CZ) = 90

Total no of arrangements = $(45 + 90 + 90) \times 4! \times 2 = 10800$

8(i)	$P(W=0)=\frac{5}{(n+5)}\cdot\frac{4}{(n+4)}\cdot\frac{3}{(n+3)}=\frac{60}{(n+5)(n+4)(n+3)}$ $P(W=1)=\frac{n}{(n+5)}\cdot\frac{5}{(n+4)}\cdot\frac{4}{(n+3)}\times 3=\frac{60n}{(n+5)(n+4)(n+3)}$ $P(W=2)=\frac{n}{(n+5)}\cdot\frac{n-1}{(n+4)}\cdot\frac{5}{(n+3)}\times 3=\frac{15n(n-1)}{(n+5)(n+4)(n+3)}$ $P(W=3)=\frac{n}{(n+5)}\cdot\frac{n-1}{(n+4)}\cdot\frac{n-2}{(n+3)}=\frac{n(n-1)(n-2)}{(n+5)(n+4)(n+3)}$	

w	0	1	2	3
$P(W=w)$	$\frac{60}{(n+5)(n+4)(n+3)}$	$\frac{60n}{(n+5)(n+4)(n+3)}$	$\frac{15n(n-1)}{(n+5)(n+4)(n+3)}$	$\frac{n(n-1)(n-2)}{(n+5)(n+4)(n+3)}$

8(ii)

$$E(W)=\frac{0\times 60+1\times 60n+2\times 15n(n-1)+3\times n(n-1)(n-2)}{(n+5)(n+4)(n+3)}$$

$$=\frac{60n+30n(n-1)+3n(n-1)(n-2)}{(n+5)(n+4)(n+3)}$$

$$=\frac{3n\left[20+10(n-1)+(n-1)(n-2)\right]}{(n+5)(n+4)(n+3)}$$

$$=\frac{3n\left(n^2+7n+12\right)}{(n+5)(n+4)(n+3)}=\frac{3n}{(n+5)}$$

$$E\left(W^2\right)=\frac{0\times 60+1\times 60n+4\times 15n(n-1)+9\times n(n-1)(n-2)}{(n+5)(n+4)(n+3)}$$

$$=\frac{60n+60n(n-1)+9n(n-1)(n-2)}{(n+5)(n+4)(n+3)}$$

$$=\frac{3n\left[20+20(n-1)+3(n-1)(n-2)\right]}{(n+5)(n+4)(n+3)}$$

$$=\frac{3n\left(3n^2+11n+6\right)}{(n+5)(n+4)(n+3)}$$

$$=\frac{3n(3n+2)(n+3)}{(n+5)(n+4)(n+3)}=\frac{3n(3n+2)}{(n+5)(n+4)}$$

$$\mathrm{Var}(W)=E\left(W^2\right)-\left[E(W)\right]^2$$

$$=\frac{3n(3n+2)}{(n+5)(n+4)}-\left[\frac{3n}{n+5}\right]^2$$

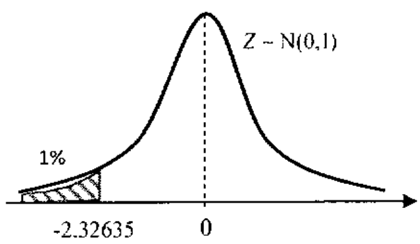
$$=\frac{3n}{n+5}\cdot\left[\frac{3n+2}{n+4}-\frac{3n}{n+5}\right]$$

$$=\frac{3n}{n+5}\cdot\left[\frac{(3n+2)(n+5)-3n(n+4)}{(n+5)(n+4)}\right]$$

$$=\frac{3n}{n+5}\cdot\left[\frac{5n+10}{(n+5)(n+4)}\right]=\frac{15n(n+2)}{(n+5)^2(n+4)}$$

$g(n)=15n(n+2)$ OR $15n^2+30n$

8(iii)	$R+W=3 \Rightarrow R=3-W$ $\text{Var}(R)=\text{Var}(3-W)=\text{Var}(W)=\dfrac{15n(n+2)}{(n+5)^2(n+4)}$	
9(i)	A sample is random if <u>every mask has an equal chance of being selected</u> and that the selections are independent of each other. Or A sample is random if <u>every subset of n masks has an equal chance of being selected to be part of the sample</u>.	
9(ii)	There is no need for any assumptions to be made about the population distribution of PFE of masks since $n=50$ is large, by central limit theorem, the sample mean PFE will follow a normal distribution approximately.	
9(iii)	Let X be random variable for the PFE of Brand BEY masks. Unbiased estimate of population mean, $\bar{x}=\dfrac{289}{50}+90=95.78$ Unbiased estimate of population variance, $s^2=\dfrac{1}{49}\left[1670.56-\dfrac{(289)^2}{50}\right]=0.0028571$ Let μ be the population mean PFE of Brand BEY masks. To test $H_0: \mu=95.8$ against $H_1: \mu<95.8$ at 1% level of significance. Under H_0, since $n=50$ is large, $\bar{X} \sim N\left(95.8, \dfrac{0.0028571}{50}\right)$ approximately by central limit theorem $Z=\dfrac{\bar{X}-95.8}{\sqrt{0.0028571/50}} \sim N(0,1)$ Value of test statistic $z=-2.646$ p-value $=0.00408$ Since $z=-2.646<-2.326$ or p-value $=0.00408<0.01$, reject H_0 at 1% level of significance There is sufficient evidence at 1% level of significance that the manager's suspicion is justified, that is, mean PFE of the Brand BEY surgical masks is compromised.	
9(iv)	$H_0: \mu=95.8$ $H_1: \mu<95.8$ at 1% level of significance. Now $\bar{x}=95.5$, $s^2=\dfrac{n}{n-1}(\text{sample variance})=\dfrac{40}{39}(k^2)$	

	Under H_0, $\overline{X} \sim N\left(95.8, \frac{40k^2/39}{40}\right)$ by CLT since n is large i.e. $\overline{X} \sim N\left(95.8, \frac{k^2}{39}\right)$ Value of Test Statistic, $z = \frac{\bar{x} - 95.8}{\sqrt{k^2/39}}$  H_0 is not rejected if $z > -2.32635$ $\frac{95.5-95.8}{\sqrt{\frac{k^2}{39}}} > -2.32635$ $-0.3 > -2.32635\sqrt{\frac{k^2}{39}}$ $\frac{0.3}{2.32635} < \frac{k}{\sqrt{39}}$ $k > 0.80534$ $k > 0.805$ (to 3 s.f.)	
10(i)	The probability that a randomly selected person who will develop immunity after taking the vaccine is constant at 0.91. Whether a randomly selected person will develop immunity after taking the vaccine is independent of whether other selected people will develop immunity.	
10(ii)	A is the random variable for the number of people, out of n, who develop immunity after taking the vaccine. $A \sim B(n, 0.91)$ Mode of $A = 19$ $P(A=19) > P(A=18)$ & $P(A=19) > P(A=20)$	

$$P(A=19) > P(A=18)$$

$${}^{n}C_{19}(0.91)^{19}(0.09)^{n-19} > {}^{n}C_{18}(0.91)^{18}(0.09)^{n-18}$$

$$\frac{n!}{(n-19)!19!}\frac{(0.09)^{n-19}}{(0.09)^{n-18}} > \frac{n!}{(n-18)!18!}\frac{(0.91)^{18}}{(0.91)^{19}}$$

$$(n-18)(0.91) > 19(0.09)$$

$$n-18 > \frac{19(0.09)}{0.91}$$

$$n > 19.879$$

$$P(A=19) > P(A=20)$$

$${}^{n}C_{19}(0.91)^{19}(0.09)^{n-19} > {}^{n}C_{20}(0.91)^{20}(0.09)^{n-20}$$

$$\frac{n!}{(n-19)!19!}\frac{(0.09)^{n-19}}{(0.09)^{n-20}} > \frac{n!}{(n-20)!20!}\frac{(0.91)^{20}}{(0.91)^{19}}$$

$$20(0.09) > (n-19)(0.91)$$

$$n-19 < \frac{20(0.09)}{0.91}$$

$$n < 20.978$$

Hence $19.879 < n < 20.978 \quad \therefore n = 20$.

10(iii)

$A' \sim B(25, 0.09)$

$P(A' > 3) = 1 - P(A' \le 3) = 0.18315 = 0.183$

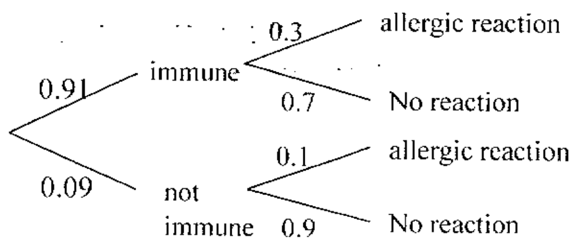
10(iv)

Let X_k be the random variable for the number of undesirable samples, out of k.

$X_k \sim B(k, 0.18315)$

$$\text{Probability} = P(X_7 = 0) \times 0.18315 \times P(X_{22} \le 1)$$
$$= 0.24267 \times 0.18315 \times 0.069250$$
$$= 0.00308$$

10(v)



$$P(\text{immune} \mid \text{allergic reaction}) = \frac{P(\text{immune} \cap \text{allergic reaction})}{P(\text{allergic reaction})}$$
$$= \frac{0.91 \times 0.3}{0.91 \times 0.3 + 0.09 \times 0.1}$$
$$= 0.968$$

11(1st part)

X denotes the random variable for the time in minutes past 6pm at which Billy leaves his workplace

Y denotes the random variable for the time taken for the journey from Billy's workplace to the childcare centre $X \sim \mathrm{N}(12, 3^2), Y \sim \mathrm{N}(m, 10^2)$

$\therefore X + Y \sim \mathrm{N}(12 + m,\ 3^2 + 10^2)$

Given: $\mathrm{P}(X + Y > 60) = 0.22176$

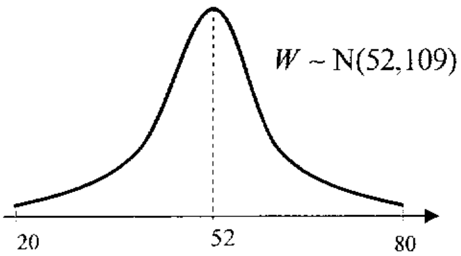
$\Rightarrow \mathrm{P}\left(Z > \dfrac{60 - 12 - m}{\sqrt{109}}\right) = 0.16$

$\Rightarrow \dfrac{60 - 12 - m}{\sqrt{109}} = 0.76626$

$\therefore m = 40.0$ (to 3 s.f.)

11(i)

$W = X + Y \sim \mathrm{N}(52,\ 109)$



Note: $\mathrm{P}(20 < W < 80) = 0.995$

11(ii)

$W = X + Y \sim \mathrm{N}(52,\ 109)$

$\mathrm{P}(W < k) \geq 0.98$

From GC, $k \geq 73.44177$

Smallest $k = 73.4$ mins = 1 h 13.4min

Latest time to leave workplace is 5.46pm.

11(iii)

$\overline{Y} = \dfrac{Y_1 + Y_2 + \ldots + Y_n}{n} \sim \mathrm{N}(40.0, \dfrac{10^2}{n})$

Given: $\mathrm{P}(\overline{Y} > 42) \leq 0.2$

By GC,

n	$\mathrm{P}(\overline{Y} > 42)$
17	$0.20479 > 0.2$
18	$0.019807 < 0.2$
19	$0.19166 < 0.2$

Since $n \leq 22$, $\therefore n = 18, 19, 20, 21, 22$

Alternatively

Given: $\mathrm{P}(\overline{Y} > 42) \leq 0.2$

$\Rightarrow \mathrm{P}\left(Z > \dfrac{42 - 40}{\sqrt{\dfrac{10^2}{n}}}\right) \leq 0.2$

$\Rightarrow \dfrac{42 - 40}{\sqrt{\dfrac{10^2}{n}}} \geq 0.84162$

	$\Rightarrow \dfrac{2\sqrt{n}}{10} \geq 0.84162$ $\Rightarrow n \geq 17.7$ Since $n \leq 22$, $\therefore n = 18, 19, 20, 21, 22$		
11(iv)	Let T be random variable for the time in minutes after 7pm at which Billy arrives at the childcare centre. Given: $S \sim \mathrm{N}(65,\ 6^2)$ $T = X + S - 60$ $\mathrm{E}(T) = 12 + 65 - 60$, $\mathrm{Var}(T) = 6^2 + 3^2 = 45$ $T \sim \mathrm{N}(17,\ 45)$ $T_1 + T_2 + ... + T_{10} \sim \mathrm{N}(170,\ 450)$ $\mathrm{P}(\text{Total fine} > 180) = \mathrm{P}(T_1 + T_2 + ... + T_{10} > \dfrac{180}{1.5})$ $= \mathrm{P}(T_1 + T_2 + ... + T_{10} > 120)$ $= 0.991$ Alternatively, F be random variable for total penalty fines for 10 randomly chosen days. $F = 1.5(T_1 + T_2 + ... + T_{10})$ $\mathrm{E}(F) = 1.5(10)(17) = 255$, $\mathrm{Var}(F) = 1.5^2(10)(45) = 1012.5$ $F \sim \mathrm{N}(255, 1012.5)$ $\mathrm{P}(F > 180) = 0.991$		