VICTORIA JUNIOR COLLEGE

JC 2 PRELIMINARY EXAMINATION 2018

H2 Mathematics **9758/01**

Paper 1 **3 hours**

Additional Materials: Answer Paper
Graph Paper (Upon Request)
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

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1 Abel, Mo and Paula train together for running. In a particular year, they spent time on three types of training programmes called "Circuit", "Intervals" and "Long Run".

For each type of training programme, the duration of each training session is the same for all three runners, except that Mo requires 10% more training time for each "Interval" session compared to Abel and Paula.

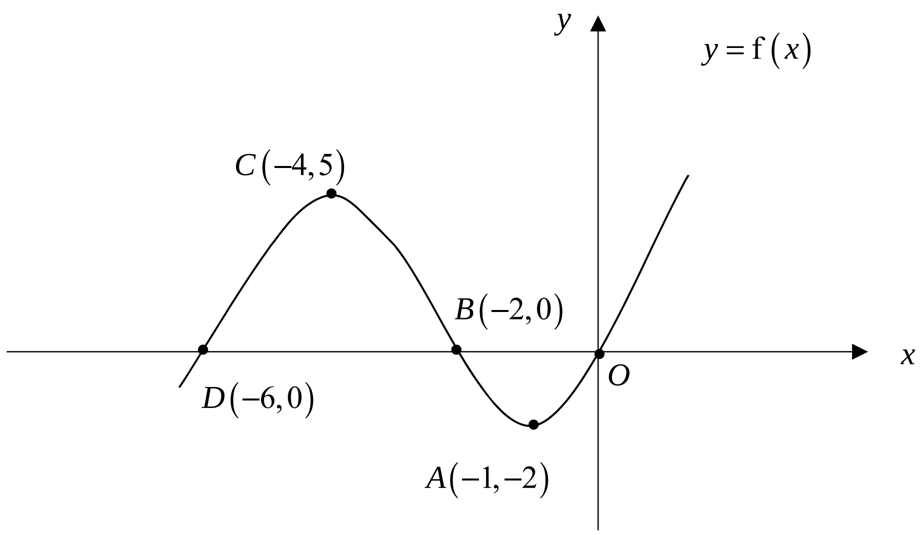
The table below shows the number of training sessions each runner attended, as well as their total training time for the year.

	Number of training sessions in the year			Total training time (in minutes)
	"Circuit"	"Intervals"	"Long Run"	
Abel	23	40	61	12600
Mo	34	67	75	17600.5
Paula	33	53	87	17725

Find the total amount of time (in minutes) each runner spent on "Intervals" in the year.

[4]

2

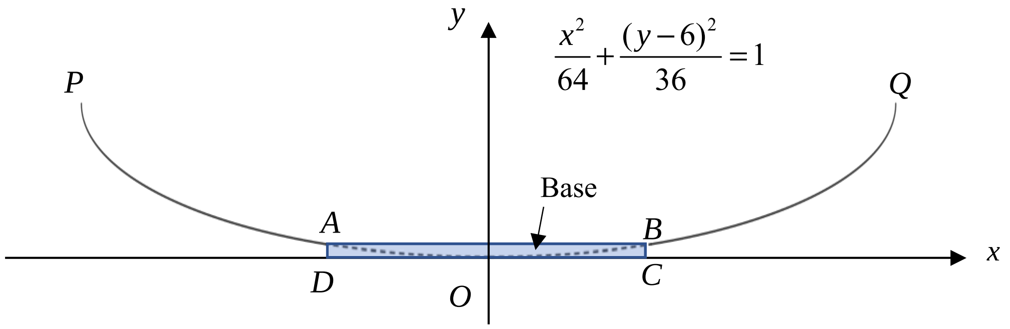


The diagram shows the graph of $y = \mathrm{f}(x)$. The curve passes through the points $A(-1,-2)$, $B(-2,0)$, $C(-4,5)$, $D(-6,0)$ and the origin O. Points A and C are the minimum point and maximum point respectively. On separate diagrams, sketch each of the following graphs indicating where possible, the coordinates of the stationary points, the points of intersections with the axes and the equations of any asymptotes.

(i) $y = \mathrm{f}\left(\frac{x}{2}+2\right)$, [2]

(ii) $y = \frac{1}{\mathrm{f}(x)}$. [3]

3



The diagram above shows the cross section of a bowl $PABQ$ and a base $ABCD$ of height h units. The curve $\frac{x^2}{64}+\frac{(y-6)^2}{36}=1$, $h \le y \le 6$, from point P to A and from point B to Q is rotated through π radians about the y-axis to form the curved surface of the bowl, while the rectangle $ABCD$ is rotated through π radians about the y-axis to form the solid cylindrical base.

The bowl is assumed to have negligible thickness. Determine, correct to 3 decimal places, the greatest height of the base such that volume of the bowl is at least 790 units^3. [5]

4 A curve C has parametric equations

$$x = a\cos(2t), \ y = 2a\tan t,$$

where $0 \le t < \frac{\pi}{2}$ and a is a positive constant.

(i) Sketch C, showing clearly the coordinates of any point(s) of intersection with the axes and the equations of any asymptotes. [3]

(ii) It is given that $a = 3$. Find the equation of the tangent to C at the point where $t = \frac{\pi}{3}$. [3]

5 Using an algebraic method, solve the inequality $\frac{7x+6}{x+2} \le 3x-1$. [4]

Hence, find the set of values of x that satisfy $\frac{7e^{-x}+6}{e^{-x}+2} \le 3e^{-x}-1$. [2]

6 **Do not use a calculator in answering this question.**

(a) Express $\left(\frac{\sqrt{2}+\mathrm{i}\sqrt{2}}{1-\mathrm{i}\sqrt{3}}\right)^8$ in the cartesian form a + ib, where a and b are in a non-trigonometrical form. [3]

(b) Solve the simultaneous equations

$$2\mathrm{i}z^* - w = 1 + 7\mathrm{i} \text{ and } z + 2w = 16 - 2\mathrm{i},$$

giving z and w in the cartesian form $p+\mathrm{i}q$.. [4]

7 Given that $y = \tan^{-1}(ax+1)$, where a is a constant, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = a\cos^2 y$. Use this result to find the Maclaurin series for y in terms of a, up to and including the term in x^3. [6]

Given that the first two terms in the series expansion of $\sqrt{9-x}$ are equal to the first two terms in the series expansion of $b\tan^{-1}(ax+1)$, find the exact values of a and b. [3]

8 **(a)** The straight line l has equation $\mathbf{r} = \mathbf{i} + 6\mathbf{j} - 3\mathbf{k} + s(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, $s \in$ ¡ . The plane p has equation $\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}) = 6$. The line l intersects the plane p at a point C.

(i) Show that the position vector of C is $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$. [2]

(ii) Find a vector equation of the line which lies in p, passes through C and is perpendicular to l. [3]

(b) Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.

(i) Show that $|\mathbf{a}\times\mathbf{b}|^2 = |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2$. [3]

It is further given that **a** is a unit vector, **b** has magnitude 3 and **a.b** = $\frac{1}{3}$.

(ii) Find the exact area of triangle OAB. [2]

(iii) Give the geometrical meaning of $|\mathbf{a}\cdot\mathbf{b}|$. [1]

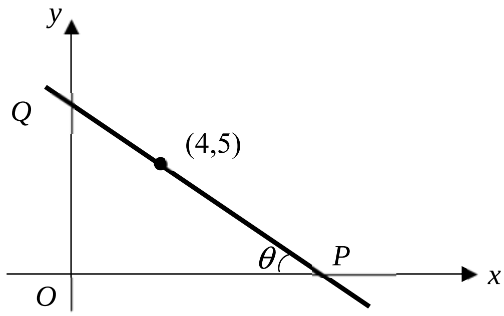
9 The curve C has equation $y=\dfrac{2x+5}{4-x^2}$.

(i) Sketch C, giving the coordinates of the axial intercepts, turning points and equations of any asymptotes. [4]

(ii) Find the numerical value of the area bounded by C and the line $y = 2$. [3]

(iii) Find the set of values of k such that $\left|\dfrac{2x+5}{x^2-4}\right|=k$ has 3 distinct negative real solutions. [3]

10 **(a)**

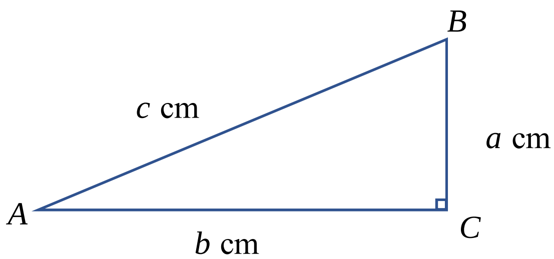


A line passes through the point $(4,5)$ and cuts the x-axis and y-axis at points P and Q respectively. It is given that angle $OPQ=\theta$, where $0<\theta<\dfrac{\pi}{2}$ and O is the origin. You may assume that the same scale is used on both axes.

(i) Show that the equation of line PQ is given by $y=(4-x)\tan\theta+5$. [3]

(ii) Hence or otherwise, show that $OP+OQ=9+4\tan\theta+5\cot\theta$. By differentiation, find the stationary value of $OP+OQ$ as θ varies. Determine the nature of this stationary value. [5]

(b)

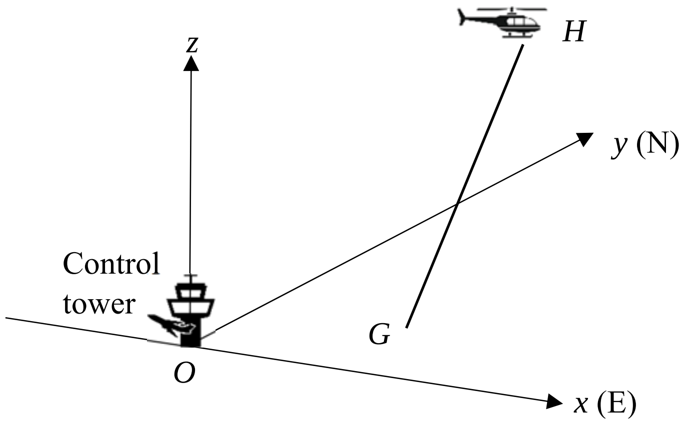


The diagram above shows a right angled triangle ABC. The lengths AB, BC and AC are denoted by c cm, a cm and b cm respectively. The triangle has a fixed area of 100 cm^2.

Express b in terms of a. [1]

Given that BC increases at a rate of 3 cm s^{-1}, find the rate of change of c when $a=20$. [3]

11



The diagram illustrates the initial flight path of a helicopter H taking off from an airport. The origin O is taken to be at the base of the control tower. The x-axis is due east, the y-axis due north, and the z-axis is vertical. The units of distances are measured in kilometres.

The helicopter takes off from the point G on the ground. The position vector $\mathbf{r}$ of the helicopter t minutes after take-off is given by

$$\mathbf{r} = (1 + t)\mathbf{i} + (0.5 + 2t)\mathbf{j} + 2t\mathbf{k}.$$

(i) Write down the coordinates of G and describe the initial flight path. [3]

(ii) Find the acute angle that the helicopter's flight path makes with the horizontal. [3]

(iii) A mountain top is situated at the point M (5, 4.5, 3). Determine how long after take off the helicopter will be nearest to M. [2]

(iv) An eagle sets off from the mountain top to hunt for food. The position of the eagle satisfies the equation

$$\frac{x-2}{3} = \frac{z-2}{1},\ y = 4.5\,.$$

Determine if the flight path of the helicopter will intersect the path traced out by the eagle, showing your reasoning clearly. [3]

(v) The helicopter enters a cloud at a height of 2 km. Given that the visibility on that day is 3.75 km, determine if the air traffic controller who is situated at 70 m above ground level, in the control tower, will be able to sight the helicopter as it enters the cloud. [2]

12 The logistic equation, sometimes called the *Verhulst model*, is a model of population growth. Letting N be the population size at any time, t (in years), this model is formalized by the differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN\left(1 - \frac{N}{K}\right),$$

where r and K are real positive constants.

(i) By considering the value of $\frac{\mathrm{d}N}{\mathrm{d}t}$ when N approaches K, explain, in the context of the question, the significance of K. [2]

(ii) Solve the differential equation, and show that $N = \frac{K}{1 + Be^{-rt}}$, where B is a constant. [5]

(iii) It is now given that the initial population in a small town is 10,000. The rate of population growth at that time is 100 per year. When the population is 15,000, the population growth rate is 75 per year. Use the *Verhulst model* to find N in terms of t.

Hence, sketch the graph of N against t. [5]

[End of Paper]

VICTORIA JUNIOR COLLEGE

JC 2 PRELIMINARY EXAMINATION 2018

H2 Mathematics 9758/02

Paper 2 3 hours

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Section A: Pure Mathematics [40 marks]

1 The function f is defined by

$$\mathrm{f} : x \mapsto \lambda + \frac{1}{1-x}, \quad x \in \mathbb{R},\ x \neq 1,$$

where λ is a negative constant.

(i) Show that f^{-1} exists. [2]

(ii) Find the set of values of λ such that the equation $\mathrm{f}(x) = 2x$ has real solutions. [3]

The functions g and h are defined by

$$\mathrm{g} : x \mapsto \mathrm{f}(x),\ x < 0,$$

$$\mathrm{h} : x \mapsto \left(x - \left(\lambda + \frac{2}{3}\right)\right)^2, x \in \mathbb{R}.$$

(iii) Find the range of hg. [3]

2 **(i)** Show that $2\sin\left(\frac{1}{2}\right)\sin(n) = \cos\left(\frac{2n-1}{2}\right) - \cos\left(\frac{2n+1}{2}\right)$. [2]

(ii) By expressing $\sin(n)$ as $\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left(2\sin\left(\frac{1}{2}\right)\sin(n)\right)$, find

$\sin(1) + \sin(2) + \sin(3) + \ldots + \sin(N)$ in terms of N, where $N \in \mathbb{Z}^+$. [3]

(iii) Explain why the above series will not be equal to $\frac{1}{2}\cot\left(\frac{1}{2}\right)$ for all $N \in \mathbb{Z}^+$. [1]

(iv) Use your answer in part **(ii)** to find the numerical value of $\sum_{n=10}^{25} \sin(n)$. [2]

3 It is given that

$$\mathrm{f}(x) = \begin{cases} 2\sqrt{1+x^2} & \text{for } -1 \le x \le 1, \\ 1 & \text{for } 1 < x < 3, \end{cases}$$

and that $\mathrm{f}(x+4) = \mathrm{f}(x)$ for all real values of x.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-1 \le x < 7$. [4]

(ii) Using the substitution $x = \tan\theta$, show that $\int_3^5 \mathrm{f}(x)\,\mathrm{d}x = 2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^3\theta\,\mathrm{d}\theta$.

By writing $\sec^3\theta$ as $\sec\theta\sec^2\theta$, find the value of $\int_3^5 \mathrm{f}(x)\,\mathrm{d}x$, leaving your answer in the form $a + \ln b - \ln c$, where a, b and c are constants to be determined exactly. [8]

4 Refinement is the process that mined material undergoes so as to recover the desired metals and separate all the undesirable minerals. Recovery rate measures the amount of metal extracted, as a percentage of the actual metal content. There are various refinement methods where the mined material undergoes numerous repetitions of separation reactions to reach the desired recovery rate. We will take the desired recovery rate to be 60% for this question.

(i) The following claim is made by a company using Method A:
Method A recovers 2% of the metal content after the first reaction. For each subsequent reaction, the amount of metal recovered is an additional t % compared to the amount recovered in the previous reaction. Hence, the amount of metal recovered during the first reaction is 2%, the amount of metal recovered during the second reaction is $(2 + t)$%, the amount of metal recovered during the third reaction is $(2 + 2t)$%, and so on.

Find the value of t such that the total percentage of metals recovered after 20 reactions reaches the desired recovery rate. [2]

Give a reason why the claim made by the company may not be realistic in the long run. [1]

(ii) Method B recovers 10% of the initial amount of the desired metals after the first reaction, but each subsequent reaction will recover r times as much as the previous reaction. If 10 reactions are required to reach the same desired recovery rate, show that r satisfies the equation $r^{10} - 6r + 5 = 0$. Explain why r cannot be 1, even though $r = 1$ is a root of this equation. Find the value of r. [4]

Each reaction for method B will require chemical L. At the start of the refinement process, there is 100 kg of chemical L and after each reaction, 30% of L will be used. 15 kg of chemical L will be added at the start of the next reaction.

(iii) Calculate the amount of chemical L left after 3 reactions. [2]

(iv) Express the amount of chemical L left after n reactions in the form $k\left(0.7\right)^{n-1} + m$, where k and m are constants to be determined. Hence, find the amount of chemical L left at the end of a reaction in the long run. [3]

Section B: Statistics [60 marks]

5 Twelve cards, numbered from 1 to 12 are arranged in a straight line. Find the number of ways this can be done if

(i) there are no restrictions, [1]

(ii) the cards numbered 2 and 3 are together, and all the six even numbered cards are adjacent. [3]

The twelve cards are arranged in a circle. Find the number of ways this can be done if all the cards numbered as multiples of 3 are separated. [2]

Six of the cards are selected at random, without replacement. Find the probability that at least two of the chosen cards are even numbered. [3]

6 A wildlife biologist wishes to test frogs for a genetic trait. He caught 1200 frogs from the wild and randomly packed them into boxes of 6. The number, x, of frogs found with the genetic trait in each box is recorded. The results obtained from 200 boxes are shown in the table below.

Number of frogs in box with genetic trait (x)	0	1	2	3	4	5	6
Number of occurrences	17	53	65	45	18	2	0

(i) Calculate, from the above data, the mean value of x. [1]

(ii) State, in context, two assumptions needed for the number of frogs carrying the genetic traits in a box to be well modelled by a binomial distribution. [2]

(iii) Assuming that X can be modelled by a binomial distribution having the same mean as the one calculated in part **(i)**, state the values for the binomial parameters n and p. [1]

Another 60 frogs are caught from the same place and packed randomly into ten boxes of 6. Find the probability that at least three of the boxes contain exactly 2 frogs with the genetic trait. [3]

7 A machine in a factory produces steel rods. The machine is reset periodically. After resetting, it should produce rods with mean length 27.0 cm. In order to test whether the mean length of rods produced differs significantly from 27.0 cm, a sample of fifty rods is taken and their lengths, x cm, are summarised as follows.

$$\sum(x-27)=12.0, \quad \sum(x-27)^2=36.4.$$

(i) Calculate unbiased estimates of the population mean and variance of the lengths of steel rods. [2]

(ii) Carry out an appropriate test at the 2.5% level of significance, explaining whether there is a need for the population distribution of the lengths of the steel rods to be known. [5]

(iii) Explain, in the context of the question, the meaning of 'at the 2.5% level of significance'. [1]

Another machine in the factory produces steel rods with mean length l cm and standard deviation 3 cm. In order to carry out a test to determine whether the mean length of rods produced is more than l cm at 5% significance level, a sample of fifty rods is taken and the sample mean is found to be 23.8 cm.

Given that the test concluded the population mean is more than l cm , find the set of possible values of l. [4]

8 The probability of obtaining a '6' on a biased cubical dice is thrice the probability of rolling every other number on the dice.

Show that probability of obtaining a '6' on the biased dice is $\frac{3}{8}$. [1]

This biased dice is put into a bag together with 3 fair cubical dice.

(i) One of the dice is chosen randomly from the bag and rolled once. Find the probability of obtaining a '6'. [2]

(ii) One of the dice is chosen randomly and is rolled n times.

(a) Find the probability that '6' is obtained on all the n rolls. [1]

(b) Given that '6' is obtained for all the n rolls, the probability that the biased dice is chosen is more than 0.95. By forming an inequality in terms of n, solve for the least value of n. [3]

(iii) One of the dice is chosen randomly from the bag and rolled once. The dice is then put back into the bag. Another dice is chosen randomly from the bag and rolled once.

For every '6' obtained, the player gains \$2, otherwise the player loses \$$k$. If the game is fair, that is, expected winnings of the player is \$0, show that the value of k is 0.56. [3]

Find the variance of the player's winnings. [2]

9 AppleC is a fruit farm that produces Cameo apples. It is known that 8.1% of the apples have a mass more than 125 g and 14.7% of the apples have a mass less than 90 g. It is also known that the masses of this batch of Cameo apples follows a normal distribution.

(i) Show that the mean and standard deviation of the masses for this batch of Cameo apples, correct to 3 significant figures, are 105 g and 14.3 g respectively. [4]

The Cameo apples are packed into bags of 8 apples.

(ii) Find the probability that the mass of a randomly chosen bag of Cameo apples is less than 845 g. State the distribution used and its parameters. [3]

A local distributor imports Cameo apples from AppleC. The distributor also imports Jazz apples from another supplier. It is known that the masses of bags of Jazz apples follow a normal distribution with mean 700 g with standard deviation of 35 g. The retail price of Cameo apples and Jazz apples are \$8.50 per kg and \$7.30 per kg respectively.

(iv) Find the probability that the cost of three bags of Jazz apples is more than twice the cost of a bag of Cameo apples by at least \$1. [4]

10 In an agricultural experiment, a certain fertilizer is applied at different rates to ten identical plots of land. Seeds of a type of grass are then sown and several weeks later, the mean height of the grass on each plot is measured. The results are shown in the table.

Rate of application of fertilizer, x g/m^2	10	20	30	40	50	60	70	80	90	100
Mean height of grass, y cm	6.2	11.4	13.2	14.8	15.8	17.0	19.4	19.4	20.6	20.8

(i) Draw the scatter diagram for these values, labelling the axes clearly. [1]

It is thought that the mean height of grass, y cm, can be modelled by one of the formulae

$$y = ax + b \quad \text{or} \quad y = c\ln x + d$$

where a, b, c and d are constants.

(ii) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) x and y, **(b)** $\ln x$ and y. [2]

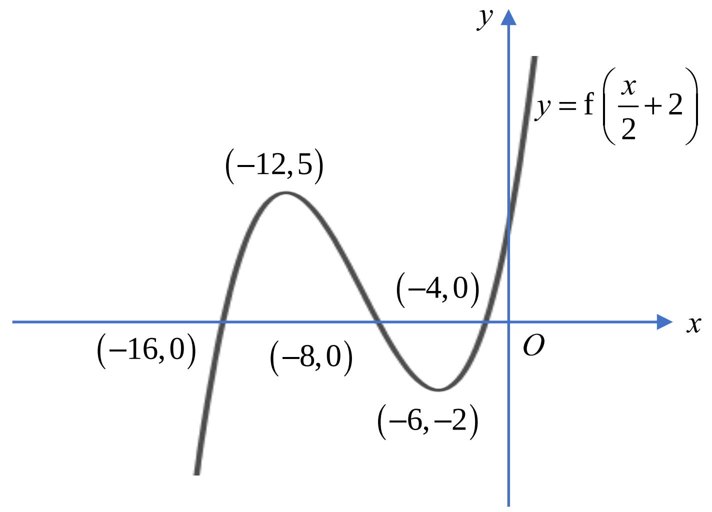
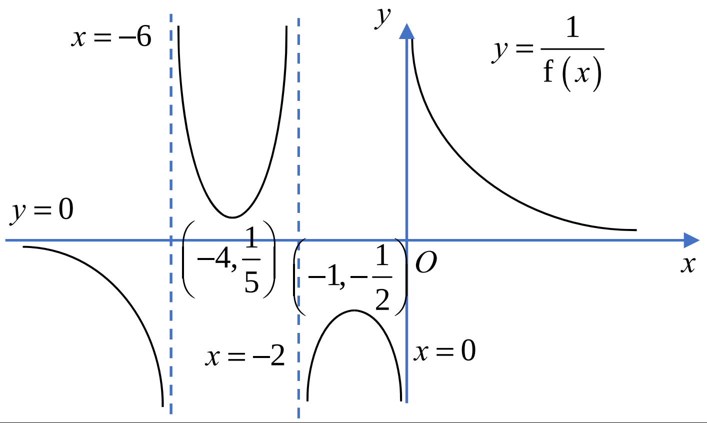
(iii) Use your answers to parts **(i)** and **(ii)** to explain which of $y = ax + b$ or $y = c\ln x + d$ is the better model. [2]

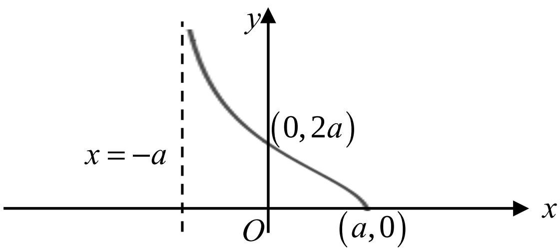
It is required to estimate the value of x for which $y = 17.2$.

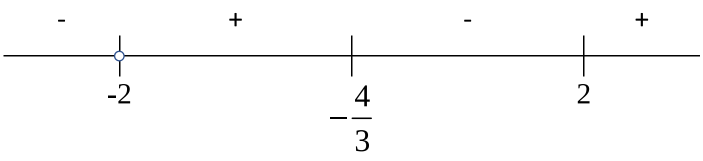
(iv) Explain why neither the regression line of x on y nor the regression line of $\ln x$ on y should be used. [1]

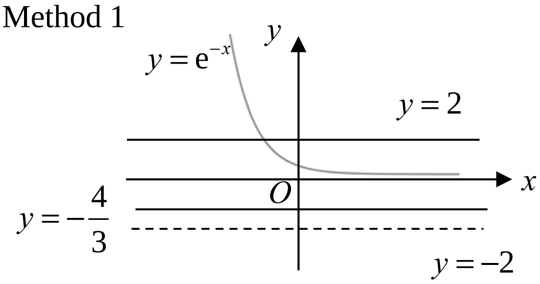
(v) Find the equation of a suitable regression line and use it to find the required estimate. [3]

[End of Paper]

Qn	Solutions
1	Let c, i and l be the amount Abel and Paula spent on a "circuit", "intervals" and "long run" session in minutes respectively. $23c + 40i + 61l = 12600$ $34c + 67(1.1)i + 75l = 17600.5$ $33c + 53i + 87l = 17725$ By GC, $c = 90$, $i = 65$ and $l = 130$ Abel, Mo and Paula spent 2600 minutes, 4790.5 minutes and 3445 minutes on "Intervals" respectively in the year.
2i	
ii	
3	$\frac{x^2}{64}+\frac{(y-6)^2}{36}=1$ $x^2=64\left(1-\frac{(y-6)^2}{36}\right)$

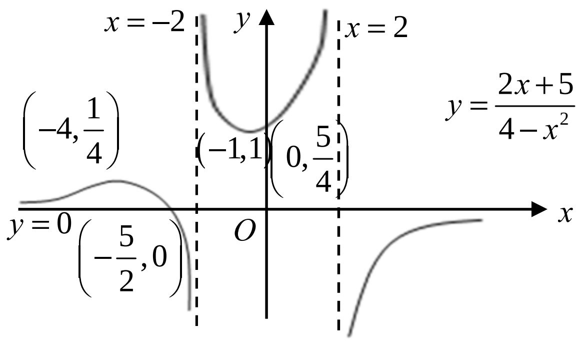
Qn	Solutions
	Volume of bowl $=\pi\int_{h}^{6}64\left(1-\frac{(y-6)^2}{36}\right)\mathrm{d}y$ $=64\pi\left[y-\frac{(y-6)^3}{108}\right]_h^6$ $=64\pi\left(6-h+\frac{(h-6)^3}{108}\right)$ $=384\pi-64\pi h+\frac{16(h-6)^3}{27}$ $384\pi-64\pi h+\frac{16(h-6)^3}{27}\geq 790$  From the graph, $0<x\leq 0.664$ Maximum height of the base is 0.664 units.
4i	 Optional: When x = 0, $a\cos 2t=0\Rightarrow t=\frac{\pi}{4}$ $y=2a\tan\left(\frac{\pi}{4}\right)=2a$ When y = 0, $2a\tan t=0\Rightarrow t=0$ $x=a\cos 0=a$ As $t\to\frac{\pi}{2}$ $,x\to -a, y\to\infty$ $\therefore x=-a$ is an asymptote.
ii	$x=3\cos(2t)$, $y=6\tan t$

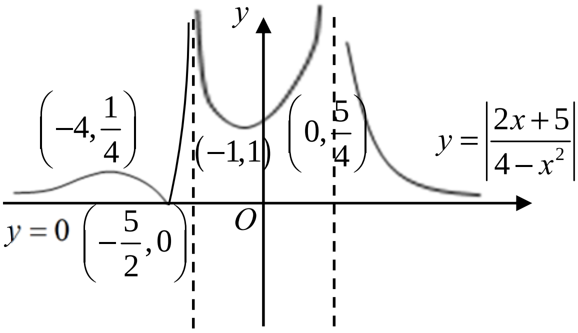
Qn	Solutions
	$\frac{\mathrm{d}x}{\mathrm{d}t}=-6\sin(2t)$, $\frac{\mathrm{d}y}{\mathrm{d}t}=6\sec^2 t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\sec^2 t}{\sin(2t)}$ When $t=\frac{\pi}{3}$, $x=-\frac{3}{2}$, $y=6\sqrt{3}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2^2}{\frac{\sqrt{3}}{2}}=-\frac{8}{\sqrt{3}}$ $y-6\sqrt{3}=-\frac{8}{\sqrt{3}}\left(x+\frac{3}{2}\right)$ $y=-\frac{8}{\sqrt{3}}x-4\sqrt{3}+6\sqrt{3}$ $y=-\frac{8}{\sqrt{3}}x+2\sqrt{3}$ Equation of tangent is $y=-\frac{8}{\sqrt{3}}x+2\sqrt{3}$
5	$\frac{7x+6}{x+2}\le 3x-1$ $\frac{(3x-1)(x+2)-7x-6}{x+2}\ge 0$ $\frac{3x^2+5x-2-7x-6}{x+2}\ge 0$ $\frac{3x^2-2x-8}{x+2}\ge 0$ $\frac{(x-2)(3x+4)}{x+2}\ge 0$  $-2<x\le-\frac{4}{3}$ or $x\ge 2$
	$\frac{7+6\mathrm{e}^x}{1+2\mathrm{e}^x}\le 3\mathrm{e}^{-x}-1$ $\frac{7\mathrm{e}^{-x}+6}{\mathrm{e}^{-x}+2}\le 3\mathrm{e}^{-x}-1$ Replace x with e^{-x}

Qn	Solutions
	Method 1  $\{x\in\mathbb{R} : x\le -\ln 2\}$ Method 2 $-2<\mathrm{e}^{-x}\le -\frac{4}{3}\left(\text{NAQ } \mathrm{e}^{-x}>0 \text{ for all } x\in\mathbb{R}\right)$ or $\mathrm{e}^{-x}\ge 2$ $-x\ge \ln 2$ $x\le -\ln 2$ $\{x\in\mathbb{R} : x\le -\ln 2\}$
6a	$\left(\dfrac{\sqrt{2}+\mathrm{i}\sqrt{2}}{1-\mathrm{i}\sqrt{3}}\right)^8=\left(\dfrac{2\mathrm{e}^{\frac{\pi}{4}\mathrm{i}}}{2\mathrm{e}^{-\frac{\pi}{3}\mathrm{i}}}\right)^8$ $=\left(\mathrm{e}^{\frac{7\pi}{12}\mathrm{i}}\right)^8$ $=\mathrm{e}^{\frac{14\pi}{3}\mathrm{i}}$ $=\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}}$ $=-\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}\mathrm{i}$
b	$2\mathrm{i}z^*-w=1+7\mathrm{i}\Rightarrow w=2\mathrm{i}z^*-1-7\mathrm{i}$ $z+2w=16-2\mathrm{i}$ $z+2\left(2\mathrm{i}z^*-1-7\mathrm{i}\right)=16-2\mathrm{i}$ Let $z=a+b\mathrm{i},\quad a,b\in\mathbb{R}$ $a+b\mathrm{i}+2\left(2\mathrm{i}\left(a-b\mathrm{i}\right)-1-7\mathrm{i}\right)=16-2\mathrm{i}$ $a+b\mathrm{i}+4a\mathrm{i}+4b-2-14\mathrm{i}=16-2\mathrm{i}$ $a+4b+\mathrm{i}\left(4a+b\right)=18+12\mathrm{i}$ $a+4b=18--(1)$ $4a+b=12--(2)$

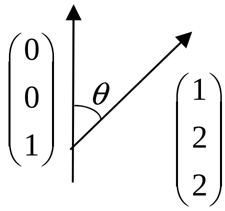
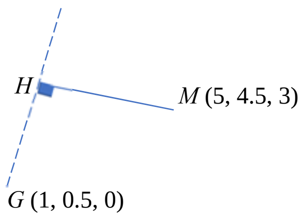
Qn	Solutions
	$(1)\times 4: 4a+16b=72 --(3)$ $(3)-(2): 15b=60 \Rightarrow b=4$ $a=18-4(4)=2$ $z=2+4\mathrm{i}$ $w=2\mathrm{i}(2-4\mathrm{i})-1-7\mathrm{i}=7-3\mathrm{i}$
7	$y=\tan^{-1}(ax+1)$ $\tan y = ax+1$ $\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=a \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2 y$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2a\cos y\sin y\frac{\mathrm{d}y}{\mathrm{d}x}=-a\sin 2y\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2a\cos 2y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-a\sin 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ When x = 0, $y=\tan^{-1}(1)=\frac{\pi}{4}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2\left(\frac{\pi}{4}\right)=\frac{1}{2}a$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-a\sin\left(\frac{\pi}{2}\right)\left(\frac{1}{2}a\right)=-\frac{1}{2}a^2$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0-a\sin\left(\frac{\pi}{2}\right)\left(-\frac{a^2}{2}\right)=\frac{a^3}{2}$ $y=\frac{\pi}{4}+\frac{1}{2}ax+\left(-\frac{a^2}{2}\right)\frac{x^2}{2!}+\left(\frac{a^3}{2}\right)\frac{x^3}{3!}+\ldots$ $=\frac{\pi}{4}+\frac{1}{2}ax-\frac{1}{4}a^2x^2+\frac{1}{12}a^3x^3+\ldots$ $\sqrt{9-x}=3\left(1-\frac{x}{9}\right)^{\frac{1}{2}}$ $=3\left(1+\frac{1}{2}\left(-\frac{x}{9}\right)+\ldots\right)$ $=3-\frac{x}{6}+\ldots$ $b\tan^{-1}(ax+1)=\frac{\pi b}{4}+\frac{1}{2}abx+\ldots$

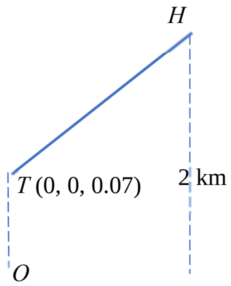
Qn	Solutions
	$\frac{\pi b}{4}=3 \Rightarrow b=\frac{12}{\pi}$ $\frac{1}{2}a\left(\frac{12}{\pi}\right)=-\frac{1}{6} \Rightarrow a=-\frac{\pi}{36}$
8ai	Equation of l is $\underset{\sim}{r}=\begin{pmatrix}1\\6\\-3\end{pmatrix}+s\begin{pmatrix}1\\-2\\2\end{pmatrix}, s\in \mathbb{R}$. Equation of p is $\underset{\sim}{r}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix}=6$. Substituting equation of l into equation of p, $\left(\begin{pmatrix}1\\6\\-3\end{pmatrix}+s\begin{pmatrix}1\\-2\\2\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix}=6$ $-34+20s=6 \Rightarrow s=2$ $\overrightarrow{OC}=\begin{pmatrix}1\\6\\-3\end{pmatrix}+2\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}$ $\therefore$ Position vector of C is $3\underset{\sim}{i}+2\underset{\sim}{j}+\underset{\sim}{k}$ (shown).
ii	Since l is perpendicular to line, $\begin{pmatrix}1\\-2\\2\end{pmatrix}$ is perpendicular to the line. Since p is perpendicular to line, $\begin{pmatrix}2\\-3\\6\end{pmatrix}$ is perpendicular to the line. Direction vector of line = $\begin{pmatrix}1\\-2\\2\end{pmatrix}\times\begin{pmatrix}2\\-3\\6\end{pmatrix}=\begin{pmatrix}-6\\-2\\1\end{pmatrix}$ Equation of line is $\underset{\sim}{r}=\begin{pmatrix}3\\2\\1\end{pmatrix}+t\begin{pmatrix}-6\\-2\\1\end{pmatrix}, t\in \mathbb{R}$.
bi	Let θ be the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Qn	Solutions
	$\lvert \underset{\sim}{a}\times\underset{\sim}{b}\rvert^2 = \left(\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\sin\theta\right)^2$ $= \lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2\sin^2\theta$ $= \lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2 - \lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2\cos^2\theta$ $= \lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2 - \left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)^2$ (shown) Alternatively $\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2 - \left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)^2 = \lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2 - \left(\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\theta\right)^2$ $= \lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2 - \lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2\cos^2\theta$ $= \lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2\left(1-\cos^2\theta\right)$ $= \left(\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\sin\theta\right)^2$ $= \lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert^2$ (shown)
ii	Area of $\Delta OAB = \frac{1}{2}\lvert\underset{\sim}{a}\times\underset{\sim}{b}\rvert$ $= \frac{1}{2}\sqrt{\lvert\underset{\sim}{a}\rvert^2\lvert\underset{\sim}{b}\rvert^2 - \left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)^2}$ $= \frac{1}{2}\sqrt{1^2(3)^2 - \left(\frac{1}{3}\right)^2}$ $= \frac{1}{2}\sqrt{\frac{80}{9}}$ $= \frac{2\sqrt{5}}{3}$ units^2
iii	$\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert$ is the length of projection of $\underset{\sim}{b}$ onto $\underset{\sim}{a}$.
9i	
ii	When y = 2, x = 0.82288 or -1.8229 Area formed $= \int_{-1.8229}^{0.82288} 2-\frac{2x+5}{4-x^2}\,\mathrm{d}x$ $= 1.95$ unit^2

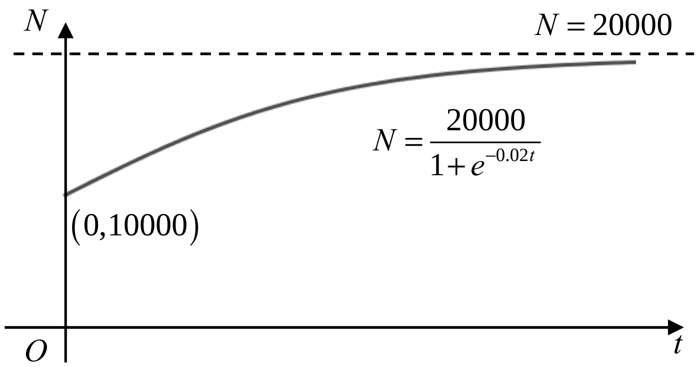
Qn	Solutions
iii	 From graph, $\left\{ k \in \mathbb{R} : 0 < k < \frac{1}{4} \text{ or } 1 < k < \frac{5}{4} \right\}$.
10ai	Gradient of $PQ = \tan(\pi - \theta) = -\tan\theta$ $y - 5 = -\tan\theta(x - 4)$ $y = (4 - x)\tan\theta + 5$
ii	When $x = 0$, $OQ = 4\tan\theta + 5$ When $y = 0$, $OP = 4 + 5\cot\theta$ $OP + OQ = 4\tan\theta + 5\cot\theta + 9$ Let $OP + OQ = s$ $\frac{\mathrm{d}s}{\mathrm{d}\theta} = 4\sec^2\theta - 5\operatorname{cosec}^2\theta$ At stationary value, $4\sec^2\theta - 5\operatorname{cosec}^2\theta = 0$ $4\tan^2\theta - 5 = 0$ $\tan\theta = \frac{\sqrt{5}}{2} \left(\text{Since } 0 < \theta < \frac{\pi}{2} \right)$ $s = 2\sqrt{5} + 2\sqrt{5} + 5 = 4\sqrt{5} + 5$ $\frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2} = 8\sec^2\theta\tan\theta + 10\operatorname{cosec}^2\theta\cot\theta$ Since θ is in the first quadrant, $\frac{\mathrm{d}^2 s}{\mathrm{d}\theta^2} > 0$. The stationary value is a minimum value.
b	$\frac{1}{2}ab = 100 \Rightarrow b = \frac{200}{a}$ Method 1

Qn	Solutions
	$c^2 = a^2 + b^2$ $\quad = a^2 + \frac{200^2}{a^2}$ $2c\frac{\mathrm{d}c}{\mathrm{d}t} = 2a\frac{\mathrm{d}a}{\mathrm{d}t} - \frac{2(200)^2}{a^3}\frac{\mathrm{d}a}{\mathrm{d}t}$ When $a = 20, c = \sqrt{20^2 + \frac{200^2}{20^2}} = \sqrt{500}$ $\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{1}{2\sqrt{500}}\left(2(20)(3) - \frac{2(200)^2}{20^3}(3)\right)$ $\quad = \frac{90}{20\sqrt{5}}$ $\quad = \frac{9}{2\sqrt{5}}$ (or 2.01) <u>Method 2</u> $c = \sqrt{a^2 + \frac{200^2}{a^2}}$ $\frac{\mathrm{d}c}{\mathrm{d}a} = \frac{1}{2}\left(a^2 + \frac{200^2}{a^2}\right)^{-\frac{1}{2}}\left(2a + 200^2\left(-2a^{-3}\right)\right)$ When $a = 20$, $\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{\mathrm{d}c}{\mathrm{d}a}.\frac{\mathrm{d}a}{\mathrm{d}t}$ $\quad = \frac{1}{2}\left(20^2 + \frac{200^2}{20^2}\right)^{-\frac{1}{2}}\left(40 + 200^2\left(\frac{-2}{20^3}\right)\right)(3)$ $\quad = 2.01$
11i	When $t = 0$, $\overrightarrow{OG} = \begin{pmatrix} 1 \\ 0.5 \\ 0 \end{pmatrix}$ The coordinates of G are (1, 0.5, 0). The flight path is a straight line starting from the point (1, 0.5, 0) moving in the direction of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.
ii	Let θ be the angle the flight path makes with the vertical.

Qn	Solutions
	$\cos\theta = \dfrac{\begin{pmatrix}1\\2\\2\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{3} = \dfrac{2}{3}$  $\theta = 48.190°\ \left(\text{or } 0.84107\right)$ Required angle = $90^{\circ} - 48.190^{\circ} = 41.8^{\circ}$ (or $\frac{\pi}{2} - 0.84107 = 0.730$) The acute angle the flight path makes with the horizontal is $41.8°$.
iii	<u>Method 1</u> $\overrightarrow{OH} = \begin{pmatrix}1\\0.5\\0\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix}$ $\overrightarrow{MH} = \begin{pmatrix}-4\\-4\\-3\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix}$  $\left(\begin{pmatrix}-4\\-4\\-3\end{pmatrix} + t\begin{pmatrix}1\\2\\2\end{pmatrix}\right)\cdot\begin{pmatrix}1\\2\\2\end{pmatrix} = 0$ $-18 + 9t = 0 \Rightarrow t = 2$ <u>Method 2</u> $\overrightarrow{GM} = \begin{pmatrix}5\\4.5\\3\end{pmatrix} + \begin{pmatrix}1\\0.5\\0\end{pmatrix} = \begin{pmatrix}4\\4\\3\end{pmatrix}$ Projection of $\overrightarrow{GM}$ on $\overrightarrow{GH}$ $= \left[\begin{pmatrix}4\\4\\3\end{pmatrix}\cdot\frac{1}{\sqrt{3}}\begin{pmatrix}1\\2\\2\end{pmatrix}\right]\frac{1}{\sqrt{3}}\begin{pmatrix}1\\2\\2\end{pmatrix}$ $= 2\begin{pmatrix}1\\2\\2\end{pmatrix}$ The helicopter will be closest to M 2 min after takeoff.
iv	Equation for flight path of eagle is

Qn	Solutions
	$\underset{\sim}{r}=\begin{pmatrix}2\\4.5\\2\end{pmatrix}+s\begin{pmatrix}3\\0\\1\end{pmatrix}, s\in\mathbb{R}$ Suppose the flight paths intersect, $\begin{pmatrix}1\\0.5\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}2\\4.5\\2\end{pmatrix}+s\begin{pmatrix}3\\0\\1\end{pmatrix}$ $1+t=2+3s---(1)$ $0.5+2t=4.5----(2)$ $2t=2+s---(3)$ From (2) and (3), t = 2 and s = 2. Substitute into (1): LHS = -4 and RHS = 1 The flight paths will not intersect.
v	When z = 2, t = 1. $\overrightarrow{OH}=\begin{pmatrix}2\\2.5\\2\end{pmatrix}$  Let top of control tower be T. $\overrightarrow{TH}=\begin{pmatrix}2\\2.5\\1.93\end{pmatrix}$ $TH=3.7383$ The controller is able to sight the helicopter.
12i	When $N\to K$, $\frac{\mathrm{d}N}{\mathrm{d}t}\to 0$. K is the population size where there is no change in the population. OR K is the population size that the population will eventually approach.

Qn	Solutions
ii	$$\frac{\mathrm{d}N}{\mathrm{d}t}=rN\left(1-\frac{N}{K}\right)$$ $$=rN\left(\frac{K-N}{K}\right)$$ $$\frac{K}{N(K-N)}\frac{\mathrm{d}N}{\mathrm{d}t}=r$$ $$\int\frac{1}{N}+\frac{1}{K-N}\,\mathrm{d}N=rt+C$$ $$\ln N-\ln\lvert K-N\rvert=rt+C$$ $$\frac{N}{\lvert K-N\rvert}=A\mathrm{e}^{rt}$$ $$\frac{N}{K-N}=D\mathrm{e}^{rt}$$ $$N=KD\mathrm{e}^{rt}-ND\mathrm{e}^{rt}$$ $$N+ND\mathrm{e}^{rt}=KD\mathrm{e}^{rt}$$ $$N=\frac{KD\mathrm{e}^{rt}}{1+D\mathrm{e}^{rt}}$$ $$=\frac{K}{\frac{1}{D}\mathrm{e}^{-rt}+1}$$ Let $B=\frac{1}{D}$ $$N=\frac{K}{1+B\mathrm{e}^{-rt}}$$ Alternatively $$\frac{1}{N(K-N)}\frac{\mathrm{d}N}{\mathrm{d}t}=\frac{r}{K}$$ $$\int\frac{1}{\left(K/2\right)^2-\left(N-K/2\right)^2}\,\mathrm{d}N=\int\frac{r}{K}\,\mathrm{d}t$$ $$\frac{1}{2\left(\frac{K}{2}\right)}\ln\left\lvert\frac{\frac{K}{2}+\left(N-\frac{K}{2}\right)}{\frac{K}{2}-\left(N-\frac{K}{2}\right)}\right\rvert=\frac{r}{K}t+C$$ $$\frac{1}{K}\ln\frac{N}{\lvert K-N\rvert}=\frac{r}{K}t+C$$

Qn	Solutions
	$\ln\frac{N}{\lvert K-N\rvert}=rt+CK$ $\frac{N}{\lvert K-N\rvert}=De^{rt}$
iii	When N = 10000, $\frac{dN}{dt}=10000r\left(1-\frac{10000}{K}\right)=100$ $100r-\frac{1000000r}{K}=1$ When N = 15000, $\frac{dN}{dt}=15000r\left(1-\frac{15000}{K}\right)=75$ $200r-\frac{3000000r}{K}=1$ By GC, $r=\frac{1}{50},\frac{r}{K}=10^{-6}\Rightarrow K=20000$ When t = 0, $N=\frac{20000}{1+B}=10000\Rightarrow B=1$ $N=\frac{20000}{1+e^{-0.02t}}$ 

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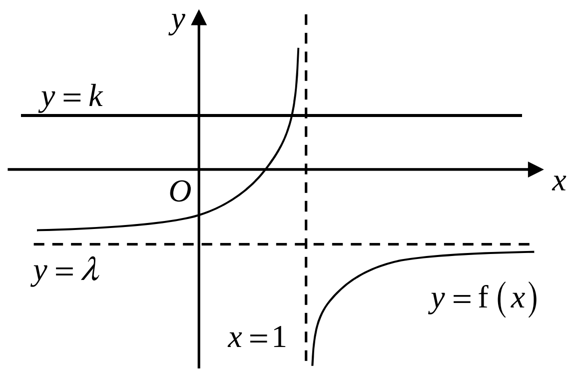
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>1i</td><td>



Since <u>every</u> horizontal line $y = k$, <u>$k \in \mathrm{R_f}$</u>, cuts the graph of $y=\mathrm{f}\left(x\right)$ exactly once, <u>f is one-one</u>.
Hence, f^{-1} exists.
</td></tr>
<tr><td>ii</td><td>

$\lambda+\frac{1}{1-x}=2x$
$\lambda-\lambda x+1=2x-2x^2$
$2x^2-\left(\lambda+2\right)x+\lambda+1=0$

$\left[-\left(\lambda+2\right)\right]^2-4\left(2\right)\left(\lambda+1\right)\geq 0$
$\lambda^2+4\lambda+4-8\lambda-8\geq 0$
$\lambda^2-4\lambda-4\geq 0$
$\lambda\leq -0.82843$ or $\lambda \geq 4.8284$

Since $\lambda<0$, $\left\{\lambda \in \mathbb{R} : \lambda \leq -0.828\right\}$.
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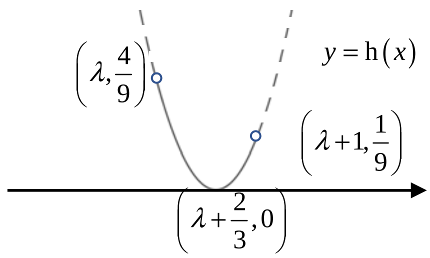
Method A:
$\mathrm{R_g}$ = $\left(\lambda, \lambda+1\right)$
Using $\mathrm{R_g}$ as the domain of h,



(Observe that the coordinates of the minimum point of $\left(x-\left(\lambda+\frac{2}{3}\right)\right)^2$ are $\left(\lambda+\frac{2}{3},0\right)$.)
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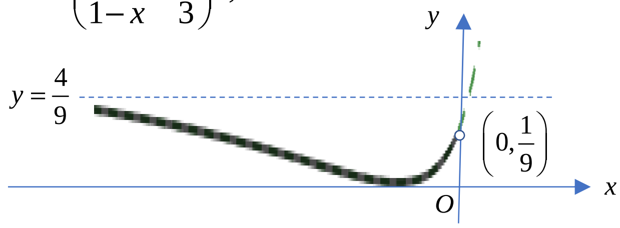
</table>

Qn	Solution
	$R_{hg} = \left[0, \frac{4}{9}\right)$. Method B: $\text{hg} : x \text{ a } \left(\frac{1}{1-x} - \frac{2}{3}\right)^2, x < 0$  $R_{hg} = \left[0, \frac{4}{9}\right)$.
2i	$\cos\left(\frac{2n-1}{2}\right) - \cos\left(\frac{2n+1}{2}\right)$ $= -2\sin\left(\frac{1}{2}\left(\frac{2n-1}{2} + \frac{2n+1}{2}\right)\right)$ $\sin\left(\frac{1}{2}\left(\frac{2n-1}{2} - \frac{2n+1}{2}\right)\right)$ $= -2\sin\left(\frac{4n}{4}\right)\sin\left(-\frac{2}{4}\right)$ $= -2\sin(n)\left(-\sin\left(\frac{1}{2}\right)\right)$ $= 2\sin\left(\frac{1}{2}\right)\sin(n)$ Alternatively, $2\sin\left(\frac{1}{2}\right)\sin(n) = -(\cos P - \cos Q)$ $= 2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$ Let $\frac{P+Q}{2} = n$ and $\frac{P-Q}{2} = \frac{1}{2}$. $P+Q = 2n$ $P-Q = 1$ $2P = 2n+1 \Rightarrow P = \frac{2n+1}{2}$ $2Q = 2n-1 \Rightarrow Q = \frac{2n-1}{2}$

Qn	Solution
	$2\sin\left(\frac{1}{2}\right)\sin(n)=\cos Q-\cos P$ $=\cos\left(\frac{2n-1}{2}\right)-\cos\left(\frac{2n+1}{2}\right)$
ii	$\sum_{n=1}^{N}\sin(n)$ $=\sum_{n=1}^{N}\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left(2\sin\left(\frac{1}{2}\right)\sin(n)\right)$ $=\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\sum_{n=1}^{N}\left(\cos\left(\frac{2n-1}{2}\right)-\cos\left(\frac{2n+1}{2}\right)\right)$ $=\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left(\begin{array}{cccc} & \cos\left(\frac{1}{2}\right) & - & \cancel{\cos\left(\frac{3}{2}\right)} \\ + & \cancel{\cos\left(\frac{3}{2}\right)} & - & \cancel{\cos\left(\frac{5}{2}\right)} \\ & & \text{M} & \\ + & \cancel{\cos\left(\frac{2N-3}{2}\right)} & - & \cancel{\cos\left(\frac{2N-1}{2}\right)} \\ + & \cancel{\cos\left(\frac{2N-1}{2}\right)} & - & \cos\left(\frac{2N+1}{2}\right) \end{array}\right)$ $=\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\left(\cos\left(\frac{1}{2}\right)-\cos\left(\frac{2N+1}{2}\right)\right)$ $=\frac{1}{2}\cot\left(\frac{1}{2}\right)-\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\cos\left(\frac{2N+1}{2}\right)$
iii	Since $\frac{2N+1}{2}$ is not equal to odd multiples of $\frac{\pi}{2}$ for all positive integers *N*, $\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\cos\left(\frac{2N+1}{2}\right)\neq 0.$ Hence, the series will not equal $\frac{1}{2}\cot\left(\frac{1}{2}\right)$ for all $N\in ¢^{+}$.
iv	$\sum_{n=10}^{25}\sin(n)$ $=\sum_{n=1}^{25}\sin(n)-\sum_{n=1}^{9}\sin(n)$ $=\frac{1}{2}\cot\left(\frac{1}{2}\right)-\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\cos\left(\frac{51}{2}\right)-\left(\frac{1}{2}\cot\left(\frac{1}{2}\right)-\frac{1}{2}\operatorname{cosec}\left(\frac{1}{2}\right)\cos\left(\frac{19}{2}\right)\right)$ $=-2.01$

<table>
<tr><td>Qn</td><td>Solution</td></tr>
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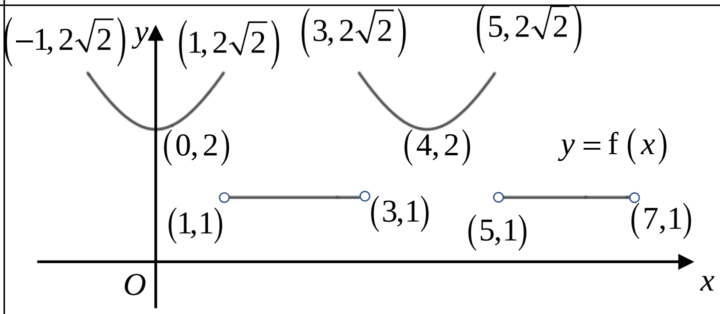
<tr><td>3i</td><td>


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<tr><td>ii</td><td>

$\int_3^5 \mathrm{f}(x)\,\mathrm{d}x$

$=\int_{-1}^{1} 2\sqrt{1+x^2}\,\mathrm{d}x$

$=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+\tan^2\theta}\sec^2\theta\,\mathrm{d}\theta$

$=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} |\sec\theta|\sec^2\theta\,\mathrm{d}\theta$

$x=\tan\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$

When $x=-1, \theta=-\frac{\pi}{4}$

When $x=1, \theta=\frac{\pi}{4}$

Since $\sec\theta>0$ when $-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}$, $|\sec\theta|=\sec\theta$

$\int_3^5 \mathrm{f}(x)\,\mathrm{d}x=2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,\mathrm{d}\theta$
</td></tr>
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$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,\mathrm{d}\theta$

$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^2\theta\sec\theta\,\mathrm{d}\theta$

$=[\tan\theta\sec\theta]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec\theta\tan^2\theta\,\mathrm{d}\theta$

$=\left(\sqrt{2}+\sqrt{2}\right)-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec\theta\left(\sec^2\theta-1\right)\mathrm{d}\theta$

$=2\sqrt{2}-\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,\mathrm{d}\theta+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec\theta\,\mathrm{d}\theta$

$2\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\sec^3\theta\,\mathrm{d}\theta=2\sqrt{2}+\left[\ln|\sec\theta+\tan\theta|\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$=2\sqrt{2}+\ln\left(\sqrt{2}+1\right)-\ln\left(\sqrt{2}-1\right)$

$\int_3^5 \mathrm{f}(x)\,\mathrm{d}x=2\sqrt{2}+\ln\left(\sqrt{2}+1\right)-\ln\left(\sqrt{2}-1\right)$
</td></tr>
</table>

<table>
<tr><th>Qn</th><th>Solution</th></tr>
<tr><td>4i</td><td>

$$\frac{20}{2}\left(2\left(2\right)+19t\right)=60$$
$$40+190t=60$$
$$t=\frac{2}{19}$$

If method A’s claim is realistic, we will be able to recover more than 100% of the actual metal content.

</td></tr>
<tr><td>ii</td><td>

$$\frac{10\left(1-r^{10}\right)}{1-r}=60$$
$$1-r^{10}=6-6r$$
$$r^{10}-6r+5=0$$

If r = 1, we will recover 100% after 10 reactions, instead of 60%.

By GC, r = 0.879

</td></tr>
<tr><td>iii</td><td>

<table>
<tr><th>n</th><th>Before reaction</th><th>After reaction</th></tr>
<tr><td>1</td><td>100</td><td>0.7(100)</td></tr>
<tr><td>2</td><td>0.7(100) + 15</td><td>$0.7^2(100) + 15(0.7)$</td></tr>
<tr><td>3</td><td>$0.7^2(100) + 15(0.7)$
+15</td><td>$0.7^3(100) + 15(0.7)^2$
+15(0.7)</td></tr>
</table>

Amount left after 3 reactions = 52.15 kg

</td></tr>
<tr><td>iv</td><td>

Amount left after n reactions

$$=\left(0.7\right)^n\left(100\right)+15\left(0.7+0.7^2+\ldots+0.7^{n-1}\right)$$
$$=\left(0.7\right)^n\left(100\right)+\frac{15\left(0.7\right)\left(1-0.7^{n-1}\right)}{1-0.7}$$
$$=\left(0.7\right)^n\left(100\right)+35\left(1-0.7^{n-1}\right)$$
$$=70\left(0.7\right)^{n-1}+35-35\left(0.7\right)^{n-1}$$
$$=35\left(0.7\right)^{n-1}+35$$

When $n\to\infty,\ 20\left(0.7\right)^{n-1}\to 0.$

Amount left at the end of a reaction in the long run is 35 kg.

</td></tr>
<tr><td>5i</td><td>12! = 479001600</td></tr>
<tr><td>ii</td><td>

$2!\times 5!\times 6!=172800$

</td></tr>
<tr><td></td><td>

$1\times 7!\times {}^{8}\mathrm{P}_4=8467200$

</td></tr>
</table>

Qn	Solution
	$1-\frac{{}^6C_0\times{}^6C_6+{}^6C_1\times{}^6C_5}{{}^{12}C_6}=\frac{887}{924}\left(\text{or } 0.960\right)$
6i	$\bar{x}=2$
ii	The probability that a frog carries the genetic trait is constant for all frogs **in a box.** The presence of the genetic trait in a frog is independent among the frogs **in a box**.
iii	$n=6,\ p=\frac{1}{3}$
	$X\sim \mathrm{B}\left(6,\frac{1}{3}\right)$ $\mathrm{P}(X=2)=0.32922$ Let Y be the number of boxes that contain exactly 2 frogs with the genetic trait. $Y\sim \mathrm{B}(10,0.32922)$ $\mathrm{P}(Y>3)=1-\mathrm{P}(Y\le 2)=0.691$
7i	Let $y=x-27$ Unbiased estimate of population mean $=\bar{x}$ $=\bar{y}+27$ $=\frac{\sum y}{n}+27$ $=\frac{12}{50}+27$ $=27.24$ Unbiased estimate of population variance $=s^2$ $=\frac{1}{n-1}\left(\sum y^2-\frac{(\sum y)^2}{n}\right)$ $=\frac{1}{49}\left(36.4-\frac{12^2}{50}\right)$ $=\frac{33.52}{49}$ $=0.68408$ $=0.684$ (3 s.f)
ii	Let X cm be the length of a steel rod. Let μ cm be the population mean length of steel rods.

Qn	Solution
	$H_0 : \mu = 27$ $H_1 : \mu \neq 27$ Level of significance: 2.5% Test Statistic: since $n = 50$ is sufficiently large, by Central Limit Theorem, $\bar{X}$ is approximately normal. When H_0 is true, $Z = \dfrac{\bar{X} - 27}{\frac{S}{\sqrt{50}}} \sim N(0,1)$ approximately Computation: $\bar{x} = 27.24$, $s = \sqrt{0.68408} = 0.82709$, $p - \text{value} = 0.040185 = 0.0402$ (3 s.f) Conclusion : Since $p - \text{value} = 0.0402 > 0.025$, H_0 is not rejected at 2.5% level of significance. Hence there is insufficient evidence that the population mean length of steel rods produced differs from 27.0 cm. Since *n* = 50 is large enough, by Central Limit Theorem, sample mean lengths of steel rods is approximately normal. Hence, there is no need for the population distribution of lengths of steel rods to be known.
iii	There is a probability of 0.025 that the test will conclude that the population mean length of steel rods produced differs from 27.0 cm when it is actually 27.0 cm.
	$H_0 : \mu = l$ $H_1 : \mu > l$ Level of significance: 5% Test Statistic: since $n = 50$ is sufficiently large, by Central Limit Theorem, $\bar{X}$ is approximately normal. When H_0 is true, $Z = \dfrac{\bar{X} - l}{\frac{3}{\sqrt{50}}} \sim N(0,1)$ approximately . Computation: To reject H_0, $z \geq 1.6449$ $\dfrac{23.8 - l}{\frac{3}{\sqrt{50}}} \geq 1.6449$ $l \leq 23.102$ $\{l \in \mathbb{R}^+ : l \leq 23.1\}$

Qn	Solution
8	Let the probability of obtaining a ‘6’ be k. $5\left(\frac{1}{3}k\right)+k=1$ $\frac{8}{3}k=1\Rightarrow k=\frac{3}{8}$
i	P(obtaining a ‘6’) = $\frac{1}{4}\times\frac{3}{8}+\frac{3}{4}\times\frac{1}{6}$ $=0.21875\left(\text{or }\frac{7}{32}\right)$
iia	P(obtaining n ‘6’s) = $\frac{1}{4}\times\left(\frac{3}{8}\right)^n+\frac{3}{4}\times\left(\frac{1}{6}\right)^n$
b	$\dfrac{\frac{1}{4}\times\left(\frac{3}{8}\right)^n}{\frac{1}{4}\times\left(\frac{3}{8}\right)^n+\frac{3}{4}\times\left(\frac{1}{6}\right)^n}>0.95$ <table><tr><td>n</td><td>Probability</td></tr><tr><td>4</td><td>0.89521</td></tr><tr><td>5</td><td>0.95055</td></tr></table> Least n is 5.
iii	Let \$$X$ be the winnings of the player. <table><tr><td>x</td><td>$-2k$</td><td>$2-k$</td><td>4</td></tr><tr><td>P(X = x)</td><td>$\left(\frac{25}{32}\right)^2=\frac{625}{1024}$</td><td>$2\left(\frac{7}{32}\right)\left(\frac{25}{32}\right)=\frac{175}{512}$</td><td>$\left(\frac{7}{32}\right)^2=\frac{49}{1024}$</td></tr></table> $\mathrm{E}(X)=-2k\left(\frac{625}{1024}\right)+(2-k)\left(\frac{175}{512}\right)+4\left(\frac{49}{1024}\right)$ $0=-\frac{625k}{512}+\frac{175}{256}-\frac{175k}{512}+\frac{49}{256}$ $\frac{25}{16}k=\frac{7}{8}$ $k=0.56$ <table><tr><td>x</td><td>-1.12</td><td>1.44</td><td>4</td></tr><tr><td>P(X = x)</td><td>$\frac{625}{1024}$</td><td>$\frac{175}{512}$</td><td>$\frac{49}{1024}$</td></tr></table> $\mathrm{Var}(X)=2.24$ The variance of the player’s winnings is $\$^2 2.24$.

Qn	Solution

Alternatively,
Let \$$A$ be the winnings from randomly choosing one dice from the bag and rolled once.

Let \$$B$ be the winnings of the player.

$B = A_1 + A_2$

$$\mathrm{E}(B) = \mathrm{E}(A_1 + A_2)$$
$$= 2\mathrm{E}(A)$$

Given that the expected winnings is \$0,

$\mathrm{E}(B) = 0 \Rightarrow 2\mathrm{E}(A) = 0 \Rightarrow \mathrm{E}(A) = 0$

a	$-k$	2
P(A = a)	$\frac{25}{32}$	$\frac{7}{32}$

$$\frac{25}{32}(-k) + \frac{7}{32}(2) = 0$$
$\Rightarrow k = 0.56$ (shown)

$$\mathrm{Var}(B) = \mathrm{Var}(A_1 + A_2)$$
$$= 2\mathrm{Var}(A)$$

a	-0.56	2
P(A = a)	$\frac{25}{32}$	$\frac{7}{32}$

By GC, Var(A) = 1.12
Var(B) = 2.24
The variance of the player's winnings is \$^{2}2.24.

9i

Let X g be the mass of a randomly chosen Cameo Apple.

$X \sim \mathrm{N}(\mu, \sigma^2)$

$\mathrm{P}(X > 125) = 0.081$	$\mathrm{P}(X < 90) = 0.147$
$\mathrm{P}\left(Z > \frac{125-\mu}{\sigma}\right) = 0.081$	$\mathrm{P}\left(Z > \frac{90-\mu}{\sigma}\right) = 0.147$
$\frac{125-\mu}{\sigma} = 1.3984$	$\frac{90-\mu}{\sigma} = -1.0494$

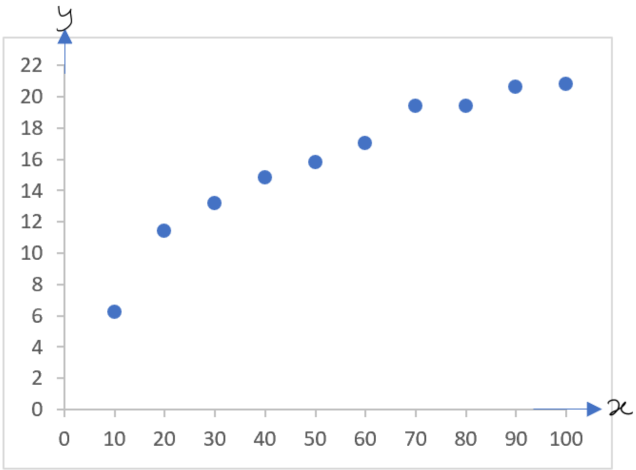
By GC,

$\mu = 105.00, \quad \sigma = 14.298$

$\approx 105 \qquad \approx 14.3$

ii

Let C g be the mass of a randomly bag of Cameo apples.

Qn	Solution
	$C = X_1 + X_2 + \ldots + X_8$ $C \sim \mathrm{N}(840, 1635.92)$ $\mathrm{P}(C < 845) = 0.54919 \approx 0.549 \text{ (3sf)}$
iii	Let J g be the mass of a randomly chosen bag of Jazz apples. $Y = \frac{7.30}{1000}(J_1 + J_2 + J_3) - \frac{8.50}{1000}(2C)$ $\mathrm{E}(Y) = \frac{7.30}{1000}(3)(700) - \frac{8.50}{1000}(2)(840)$ $= 1.05$ $\mathrm{Var}(Y)$ $= \left(\frac{7.30}{1000}\right)^2 (3)(35)^2 + \left[\frac{8.50}{1000}(2)\right]^2 (1635.92)$ $= 0.66862$ $Y \sim \mathrm{N}(1.05, 0.66862)$ $\mathrm{P}(Y > 1) = 0.524$
10 i	
iia	$r = 0.9541$ (4 d.p)
b	$r = 0.9942$ (4 d.p)
iii	From the scatter diagram, as x increases, y increases by decreasing amounts. In addition, the product moment correlation coefficient between $\ln x$ and y, 0.9942, is closer to 1 as compared to that between x and y, 0.9541. Hence $y = c + d \ln x$ is the better model.
iv	Since x is the independent variable, neither the regression line of x on y nor the regression line of $\ln x$ on y should be used to estimate the value of x when $y = 17.2$.
v	Equation of regression line of y on $\ln x$ is

Qn	Solution
	$$y = 6.3074\ln x - 8.1904$$ $$y = 6.31\ln x - 8.19$$ When $y = 17.2$, $17.2 = -8.1904 + 6.3074\ln x$ $x = 56.008$ $= 56.0$