1 Without the use of a calculator, solve the inequality

$$\frac{x^2+2x-3}{\left(x^2-2x+11\right)\left(x+1\right)} \geq 0 .$$ [4]

2 The curve C has equation given by $y=\dfrac{x(x+3)}{x+2}$ where $x \neq -2$.

(i) Show that the gradient of C is always positive for $x \in \mathbb{R} \setminus \{-2\}$. [3]

(ii) Sketch the graph of C, indicating clearly the asymptotes and intercepts on the axes whenever applicable. [3]

3 **(i)** Show that $\mathrm{e}^{-r}-2\mathrm{e}^{-r+1}+\mathrm{e}^{-r+2}=\dfrac{(\mathrm{e}-1)^2}{\mathrm{e}^r}$. [1]

(ii) Hence find $\displaystyle\sum_{r=1}^{N}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}}$ in terms of N. [4]

(iii) Using your result in part **(ii)**, find $\displaystyle\sum_{r=9}^{N+1}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}}$ in terms of e. [2]

4 It is given that the function $y=\mathrm{f}(x)$ has the Maclaurin's series $1+x+bx^2+cx^3+\ldots$ and satisfies $\left(a-x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=y\left(1+2x-x^2\right)$, where a, b and c are real constants.

(i) Show that $a = 1$ and find the value of b and c. [7]

(ii) Given that the first four terms of this series are equal to the first four terms in the series expansion, in ascending powers of x, of $\dfrac{\mathrm{e}^x}{1-x^2}$, find the series expansion of $y=\dfrac{\mathrm{e}^{2x}}{1-x^2}$ up to and including the term in x^3. [3]

Do not use a graphic calculator in answering this question.

5 **(i)** By letting $z = a + b\mathrm{i}$, solve $z^2=\mathrm{i}$, giving your answers in exact form. [3]

(ii) Solve the equation $w^2+2w+(1-8\mathrm{i})=0$, giving the roots in Cartesian form. [3]

(iii) Hence, solve $(1-8\mathrm{i})v^2+2\mathrm{i}v-1=0$, giving your answers in Cartesian form. [3]

6 Relative to the origin O, the position vectors of points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The point C with position vector $\mathbf{c}$ lies on the line segment AB such that $AC : CB$ is $\lambda : 1-\lambda$. It is given that $\mathbf{b}$ is a unit vector, $|\mathbf{a}| = \frac{4}{3}$ and the angle formed between OA and OB is $120°$.

(i) Find the value of λ such that the points O, A and C form a right angle AOC. [4]

(ii) Find $\mathbf{m}$, the position vector of M, the midpoint of AC, in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

A circle is drawn with AC as its diameter and O is a point on the circumference of the circle drawn.

(iii) Determine if OB is a tangent to the circle described above. [3]

(iv) Give a geometrical interpretation of $|\mathbf{b} \bullet \mathbf{m}|$. Hence, explain $(\mathbf{b} \bullet \mathbf{m})\mathbf{b}$ in terms of its magnitude and direction. [2]

7 A curve is defined by the parametric equations:

$$x = a\cos^3 t,\ y = a\sin^3 t,\ 0 \le t < 2\pi \text{ where } a \text{ is a positive constant.}$$

(i) Sketch the curve, showing clearly the coordinates of the points where $t = 0, \frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$. [2]

(ii) Show that the equation of the normal at the point where $t = p$ is given by $y\sin p = x\cos p - a\cos 2p$. [4]

(iii) Find the value(s) of t when the normal at the point where $p = \frac{\pi}{3}$ meets the curve again. [3]

8 **(i)** Sketch the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{y}{b} - \frac{x}{a} = 1$, where $0 < b < a$ on a single diagram, labelling clearly any intersection between the curve and the line. [2]

(ii) Show that the area of the region R, bounded by the curve with equation $x = -\sqrt{a^2 - \frac{a^2y^2}{b^2}}$ and the line $\frac{y}{b} - \frac{x}{a} = 1$, where $0 < b < a$ is given by

$$-\frac{1}{2}ab + \int_0^b \sqrt{a^2 - \frac{a^2y^2}{b^2}}\,\mathrm{d}y.$$

Hence, by substituting $y = b\cos\theta$, find the exact value of the area in terms of a and b. [5]

(iii) Find the exact volume of the solid obtained when R is rotated 4 right angles about the y-axis, giving your answer in the form of ka^2b, where k is a constant. [3]

Do not use a graphic calculator in answering this question.

9 **(i)** It is given that $z = 3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ is a root of the equation $z^2 - 3z + 9 = 0$. Find in similar form, the other root of the equation. [1]

(ii) Show that $\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = 2\mathrm{i}\sin\theta$. [1]

(iii) Let the root found in **(i)** to be w_1 and $w_2 = 3\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}$. Hence, find the complex number $w_2 - w_1$, in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $\theta \in (-\pi\ ,\ \pi]$. [3]

(iv) Let the point *A* represent the complex number $w_2 - w_1$ on the Argand diagram. A perpendicular line is drawn from the point *A* to the real axis. The intersection point between this line and the real axis is *B*. Show that the area of the triangle *OAB* is $9\sin^2\left(\frac{2\pi}{9}\right)\sin\left(\frac{7\pi}{9}\right)$ square units, where *O* is the origin. [3]

10 Albert and Betty each took a study loan of \$100 000 from a bank on 1 January 2014 and both graduated on 31 December 2017. The bank only starts charging an annual interest rate of 5% on the outstanding loan at the end of each year from 2018 onwards. Albert pays the bank \$*x* on the 11th day of every month starting in January 2018. Let *n* be the number of years after 2017 that repayment of the study loan has begun.

(i) Find an expression for the amount of outstanding loan at the end of *n* years. [4]

(ii) Find the minimum value of *x* if Albert wishes to complete his repayment of the loan at the end of 2027, giving your answer to the nearest dollar. [3]

An investment fund pays out a constant *r* % dividend per annum on 31 December every year based on the amount of funds held to maturity from 1 January till 30 December of the same year.

On 1 January 2017, Betty decided to invest \$50 000 in the fund to finance her repayment of her study loan using the annual dividend payout. Her repayment is once per year which she uses the full amount of the annual dividend payout. The schedule of repayment is fixed on 11 January of each year, starting with effect from 2018.

Find the minimum value of *r* such that Betty will be able to complete her repayment of the loan at the end of 2027. [5]

[Note that the principal sum of Betty's investment remains unchanged at \$50 000 throughout the repayment period]

11 Joe wants to grow vegetables at his house balcony and designs an irrigation device, which takes the form of a cylindrical water tank, with a fixed cross sectional area A cm^2, and a small opening at the bottom of it.

It is known that the rate of change of the volume, V cm^3 , of water at any time t hours, dispensed from the tank, is proportional to the square root of the height of the water, h cm, above the opening.

(i) Show that the rate of change of the height of the water in the tank at any time t hours, is given by $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{k}{A}\sqrt{h},$ where k is a positive constant. [3]

(ii) Joe fills the water tank to an initial height of 81 cm with the opening closed. The water is then discharged from the opening and the height of the water level above the opening is decreasing at a rate of 0.3 cm/hr. By solving the differential equation in part **(i)**, find h in terms of t. [5]

(iii) Joe is planning for an overseas trip. He wants to make sure that he will be back before the water tank is empty. What is the maximum number of days, to the nearest integer, that Joe can be away? [2]

(iv) Given that the tank has a base radius of 20 cm, find the exact rate of change of the volume of water in the tank at the end of the fourth day. [4]

End of Paper

Section A: Pure Mathematics (40 marks)

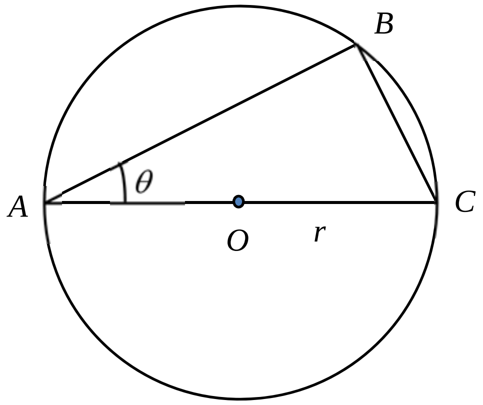
1 A curve C with equation $y=a(x-1)+\dfrac{b}{x+c}$, where a, b and c are real constants, undergoes in succession, the following transformations:

A: A reflection in the x-axis
B: A translation of 1 unit in the negative x-direction

The resulting curve with equation $y=\mathrm{f}(x)$ has the y-axis as one of its asymptotes.

Given that $\left(1,\dfrac{1}{6}\right)$ is a turning point of $y=\dfrac{1}{\mathrm{f}(x)}$, find the values of a, b and c. [5]

2



In the figure above, points A and C are fixed points on the circle which form the diameter passing through the centre O. The circle has a fixed radius r units. The variable point B moves along the circumference of the circle between A and C in the upper half of the circle. The chord AB makes an angle of θ radians with the diameter AC.

Use differentiation to find, the exact angle θ, such that S, the area of triangle ABC, is a maximum. [6]

3 **(i)** Using the R-formula, express $\sin\theta+m\cos\theta$, for $m>0$, in the form $R\sin(\theta+\alpha)$ where $R>0$ and $0<\alpha<\dfrac{\pi}{2}$ are constants to be determined in terms of m.

Hence show that $2(\sin\theta+m\cos\theta)\sin(\theta-\alpha)=R(\cos 2\alpha-\cos 2\theta)$. [3]

(ii) Given that $\alpha=\dfrac{\pi}{2}$, evaluate $\displaystyle\int\frac{\cos 2\theta}{(\sin\theta+m\cos\theta)\sin(\theta-\alpha)}\,\mathrm{d}\theta$ in terms of m and θ. [4]

[Turn Over

4 **(i)** Solve the inequality $\ln(x-1) \le 0$. [2]

The function f is defined by

$$\mathrm{f}: x \mapsto \left|\ln(x-1)\right|, \qquad \text{for } x \in \mathbb{R},\ 1 < x \le 2.$$

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) Sketch, on the same diagram, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, giving the equations of any asymptotes and the coordinates of any points where the curves cross the axes. [3]

(iv) Hence solve the inequality $\mathrm{f}^{-1}(x) < \mathrm{f}(x)$. [1]

Another function g is defined by

$$\mathrm{g}: x \mapsto 1 + \frac{4}{4x^2+5} \text{ for } x \in \mathbb{R},\ x > 0.$$

(v) Using f^{-1}, find the exact value of a such that $\mathrm{fg}(a) = 3$. [2]

5 The line l_1 has equation $\mathbf{r} = \begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$ and the line l_2 has equation

$$\frac{x-4}{2} = \frac{y}{-3} = \frac{z+1}{4}.$$

(i) Show that l_1 and l_2 are skew lines. [3]

The line l_1 contains points A and B with coordinates (11, 6, 0) and (1, 6, 5) respectively with respect to the origin O. The plane p_1 which is parallel to l_1 has equation

$$\mathbf{r} \bullet \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} = 240.$$

(ii) Find the position vector of the point P on p_1 which has the shortest distance to the line l_1 and is equidistant from the points A and B. [5]

A plane p_2 contains the point P and is parallel to both l_1 and l_2.

(iii) Hence, find the Cartesian equation of p_2. [3]

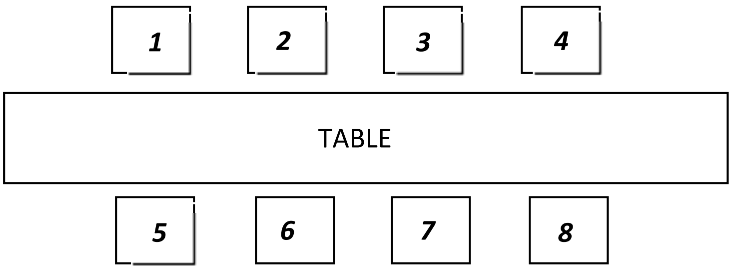
Section B: Statistics (60 marks)

6 A committee of eight people is to be chosen from 15 men and 7 women.

Find the number of ways in which the committee can be chosen if it consists of

at least 2 women. [3]

The chosen committee consists of 6 men (Allen, Ben, Calvin, Donald, Edwin and Felix) and 2 women (Gina and Hazel).
At a meeting, the committee members are seated at a rectangular table as shown in the diagram below, with seats labelled 1 to 8.



Find the number of possible seating arrangements if Gina and Hazel must be seated at any two of the corner seats labeled *1*, *4*, *5* or *8*. [2]

7 Oliver is practising for the upcoming target archery competition. During practices, Oliver shoots from distances ranging from 30m to 90m to the target. The probability, p, that he hits the bullseye is given by $p = \frac{2}{195}(95 - d)$, where d is the distance between the archer and the target in metres.

Each shot he made is assumed to be independent of any other shots made.

(i) Oliver shoots 18 arrows from a distance of 40 metres from the target. Find the probability that he hits the bullseye more than 6 times given that he hits the bullseye at most 10 times. [4]

(ii) Oliver shoots 18 arrows from a distance of x metres from the target. Find x such that Oliver has a 98% chance of hitting the bullseye at least twice. [3]

8 For two mutually exclusive events A and B, it is given that $\mathrm{P}(A)=0.65$ and

$\mathrm{P}(B\,|\,A')=\dfrac{2}{7}$.

(i) Show that $\mathrm{P}(B)=0.1$. [2]

For a third event C, it is given that $\mathrm{P}(A\cap C)=0.39$.

(ii) Find $\mathrm{P}(C'\,|\,A)$. [2]

It is given that B and C are independent and $\mathrm{P}(A'\cap B'\cap C)=0.15$.

(iii) Find $\mathrm{P}(B\cap C)$. [2]

(iv) Hence or otherwise, determine whether the events A and C are independent. [1]

9 Connie and Sally play a game using two six-sided dice. One of the dice is fair and each face is labelled with a digit from ‘1’ to ‘6’ respectively. The other die is biased such that the score, denoted by Y, has a probability distribution given as follows:

$$\mathrm{P}(Y=y)=\begin{cases} \dfrac{1}{6} & \text{for} \quad y=1, 3, 5 \\ \dfrac{1}{18}(y-1) & \text{for} \quad y=2, 4, 6 \\ 0 & \text{otherwise.} \end{cases}$$

Connie throws the two dice. Sally pays Connie \$5 if the difference between the scores on the fair and biased dice is more than 3. Both players receive nothing if the scores on the fair and biased dice are identical. Connie pays Sally \$3 for all other outcomes. Let X be Sally’s winnings after one game in dollars.

(i) Find Sally’s expected winnings in one game, leaving your answer in exact form. [4]

(ii) Find the probability that Sally’s total winnings in 50 independent games is at least \$65. [3]

10 **(a)** It is given that the regression line y on x for the following bivariate data is $y = 8 + 0.5x$.

x	20	22	24	26	28	30	32	34
y	16	21	a	24	22	24	27	20

Find a. [2]

(b) A botanist conducted an experiment to find out how the age of pine trees, x, in years, varies with their average height, y, in metres. The data collected were given below.

x (in years)	1	2	3	4	5	6	7	8	9	10
y (in metres)	2.74	3.38	3.75	4.08	4.30	4.48	4.51	4.68	4.72	4.75

(i) Draw a scatter diagram for the given data. [2]

The botanist felt that the data should be modelled by an equation of the form $y = a + bx$.

(ii) Give an interpretation, in this context, of the value of b. [1]

(iii) State, with a reason, which of the following models among A, B or C is the most appropriate for the given data.

A: $y = a - \dfrac{b}{x}$ B: $y = a + b\sqrt{x}$ C: $y = a + b\ln x$

Write down the equation of the least-squares regression line for the chosen model, stating clearly the values of a and b. [3]

(iv) Give two reasons why it would be reasonable to use your model to estimate the age of the pine tree when its height is 4.25 metres. [2]

11 A road named Spring Avenue has a speed limit of 40 km/h in a housing estate.

The residents were concerned that many vehicles travelled too fast along the road and they decided to set up a speed tracking device to monitor the speed of vehicles travelling along this road. The data generated from the device indicated that the mean speed of vehicles travelling through this road was 44.1 km/hour.

In an attempt to reduce the mean speed of vehicles travelling through Spring Avenue, life-size photographs of a police officer were put up next to the road. The speed, X km/hour of a sample of 100 randomly chosen vehicles was then measured and the following data obtained.

$$\sum x = 4327.0, \qquad \sum\left(x-\overline{x}\right)^2 = 925.71\,.$$

(i) Calculate the unbiased estimates of the population mean and variance of the speed of vehicles travelling along Spring Avenue. [2]

(ii) State an assumption that must be made about the sample in order to carry out a hypothesis test to investigate whether the desired reduction in mean speed had occurred. [1]

(iii) Given that the assumption that you stated in part **(ii)** is valid, carry out such a test, using the 5% level of significance. [5]

(iv) Explain what is meant by "5% level of significance" in the context of this question. [1]

(v) Subsequently, the residents detected that a measurement error has occurred when measuring the speed of the 100 randomly selected vehicles. To rectify the error, a multiplication of a positive constant k to each reading for the 100 randomly selected cars is recommended. Find the greatest possible value of k, to 3 significant figures, for the conclusion obtained in **(iii)** to remain the same. [3]

12 **(i)** Aquafresh mineral water is supplied in 1.5-litre bottles. The actual volume in millilitres, in a bottle may be modelled by a normal distribution with mean 1505 ml and standard deviation 10.2 ml.

(a) Calculate the probability that the volume of Aquafresh mineral water in a randomly selected bottle is more than 1480 ml. [1]

(b) The supplier requires that less than 10 per cent of bottles should contain less than 1480 ml of water.
Assuming that there has been no change in the value of the standard deviation, calculate the least mean volume in order to satisfy this requirement. Give your answer to one decimal place. [3]

(ii) Sparkling spring water is supplied in packs of six 0.5-litre bottles. The actual volume in a bottle may be modelled by a normal distribution with mean 508.5 ml and standard deviation 3.5 ml.

Find the probability that the volume of water in each of the 6 bottles from a randomly selected pack is more than 505 ml. [2]

(iii) Calculate the probability that the volume of 6 bottles of Sparkling spring water in a randomly selected pack differs from twice the volume of one randomly selected bottle of Aquafresh mineral water by less than 5.5 ml. [3]

(iv) The volume of tap water, V, used by a guest in a bathroom at a small hotel may be modelled by a random variable with mean 120 litres and standard deviation 65 litres. Give a numerical justification as to why V is unlikely to be normally distributed. [1]

Explain why $\overline{V}$, the mean of a random sample of 30 observations of V, may be assumed to be approximately normally distributed and state its distribution. [2]

End of Paper

St Andrew's Junior College
2018 Preliminary Examination
H2 Mathematics Paper 1 (9758/01) Solutions

1

$$\frac{x^2+2x-3}{\left(x^2-2x+11\right)\left(x+1\right)} \geq 0$$

$x^2-2x+11$

$$=x^2-2x+\left(\frac{-2}{2}\right)^2-\left(\frac{-2}{2}\right)^2+11$$

$=\left(x-1\right)^2-1+11$

$=\left(x-1\right)^2+10>0$ for all real values of x

Therefore, $\dfrac{x^2+2x-3}{\left(x^2-2x+11\right)\left(x+1\right)} \geq 0$

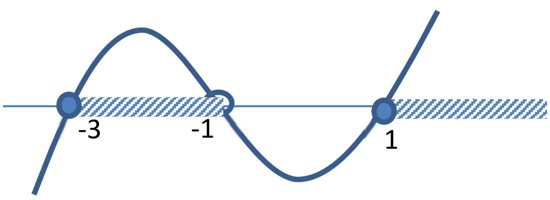
$$\Rightarrow \frac{x^2+2x-3}{\left(x+1\right)} \geq 0$$

$$\frac{\left(x+3\right)\left(x-1\right)}{\left(x+1\right)} \geq 0, \quad x \neq -1$$

Multiplying both sides by $\left(x+1\right)^2$

$\left(x+3\right)\left(x-1\right)\left(x+1\right) \geq 0$

Since $x \neq -1$,



Hence,
$-3 \leq x < -1$ or $x \geq 1$

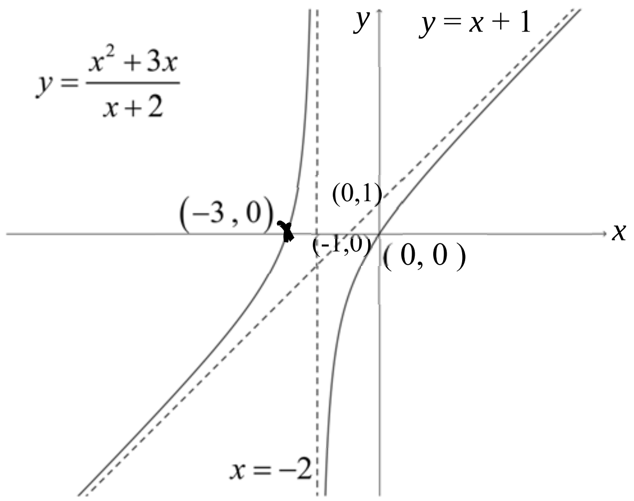
2 (i)

$$y=\frac{x^2+3x}{x+2}=x+1-\frac{2}{x+2}$$

$$\frac{\mathrm{dy}}{\mathrm{dx}}=1+\left(\frac{2}{(x+2)^2}\right)>0 \text{ for all } x \in ¡$$

since $(x+2)^2>0$ for all x, $x \neq -2$.

Alternatively:

	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+2)(2x+3)-(x^2+3x)(1)}{(x+2)^2}$ $=\dfrac{x^2+4x+6}{(x+2)^2}$ $=\dfrac{(x+2)^2+2}{(x+2)^2}>0$ Since $(x+2)^2\geq 0$ for all $x\in\mathbb{R}$, $(x+2)^2+2>0$ for all $x\in\mathbb{R}$ and $(x+2)^2>0$ for all $x\in\mathbb{R}\setminus\{2\}$, Hence C has a positive gradient for all $x\in\mathbb{R}$.
2 (ii)	 Oblique asymptote: $y = x + 1$ Vertical asymptote: $x=-2$
3 (i)	$\mathrm{e}^{-r}-2\mathrm{e}^{-r+1}+\mathrm{e}^{-r+2}$ $=\mathrm{e}^{-r}\left(1-2\mathrm{e}+\mathrm{e}^2\right)$ $=\dfrac{\left(\mathrm{e}-1\right)^2}{\mathrm{e}^r}$
3 (ii)	$\displaystyle\sum_{r=1}^{N}\frac{\left(\mathrm{e}-1\right)^2}{\mathrm{e}^{r+1}}$ $\displaystyle=\sum_{r=1}^{N}\frac{\left(\mathrm{e}-1\right)^2}{\mathrm{e}^r\mathrm{e}^1}$ $\displaystyle=\frac{1}{\mathrm{e}}\sum_{r=1}^{N}\frac{\left(\mathrm{e}-1\right)^2}{\mathrm{e}^r}$ $\displaystyle=\frac{1}{\mathrm{e}}\sum_{r=1}^{N}\left(\mathrm{e}^{-r}-2\mathrm{e}^{-r+1}+\mathrm{e}^{-r+2}\right)$

	$=\frac{1}{e}[e^{-1}-2e^{0}+e^{1}$ $+e^{-2}-2e^{-1}+e^{0}$ $+e^{-3}-2e^{-2}+e^{-1}$ $+e^{-4}-2e^{-3}+e^{-2}$ $+\dots\dots\dots\dots\dots\dots\dots\dots$ $+e^{-N+2}-2e^{-N+3}+e^{-N+4}$ $+e^{-N+1}-2e^{-N+2}+e^{-N+3}$ $+e^{-N}-2e^{-N+1}+e^{-N+2}]$ $=\frac{1}{e}\left(e-1+e^{-N}-e^{-N+1}\right)$ $=1-\frac{1}{e}+e^{-N-1}-e^{-N}$
3 (iii)	$\sum_{r=9}^{N+1}\frac{(e-1)^2}{e^{r+1}}=\sum_{r=1}^{N+1}\frac{(e-1)^2}{e^{r+1}}-\sum_{r=1}^{8}\frac{(e-1)^2}{e^{r+1}}$ $=1-\frac{1}{e}+e^{-N-2}-e^{-N-1}-\left(1-\frac{1}{e}+e^{-9}-e^{-8}\right)$ $=e^{-N-2}-e^{-N-1}-\frac{1}{e^9}+\frac{1}{e^8}$
4 (i)	$y=f(x)$ $=1+x+bx^2+cx^3+\dots$ $=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots$ Comparing, $f(0)=1,\ f'(0)=1,\ \frac{f''(0)}{2!}=b,\ \frac{f'''(0)}{3!}=c$ When $x=0, y=1, \frac{dy}{dx}=1, \frac{d^2y}{dx^2}=2b, \frac{d^3y}{dx^3}=6c.$ $\left(a-x^2\right)\frac{dy}{dx}=y\left(1+2x-x^2\right)$ $\Rightarrow\left(a-0^2\right)(1)=(1)\left(1+2(0)-0^2\right)$ $\Rightarrow a=1$ (Shown) When $a=1$, we have $\left(1-x^2\right)\frac{dy}{dx}=y\left(1+2x-x^2\right)$ Differentiate w.r.t. x,

$$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}=y\left(2-2x\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+2x-x^2\right)$$

$$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=y\left(2-2x\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+4x-x^2\right)\ldots\ (1)$$

Differentiate w.r.t. x,

$$\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2y+\frac{\mathrm{d}y}{\mathrm{d}x}\left(2-2x\right)$$

$$+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(1+4x-x^2\right)+\frac{\mathrm{d}y}{\mathrm{d}x}\left(4-2x\right)$$

$$\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2y+\frac{\mathrm{d}y}{\mathrm{d}x}\left(6-4x\right)+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(1+6x-x^2\right)\ldots\ (2)$$

substitute $x=0, y=1, \frac{\mathrm{d}y}{\mathrm{d}x}=1$ into (1) and (2),

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2(1)+1=3$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2(1)+6(1)+3=7$$

Hence, from the given expansion,

$$\frac{\mathrm{f}''(0)}{2!}=b$$

$$2b=3$$

$$\Rightarrow b=\frac{3}{2}$$

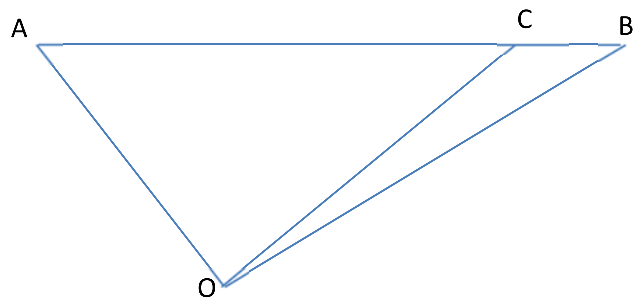
$$\frac{\mathrm{f}'''(0)}{3!}=c$$

$$\mathrm{f}'''(0)=6c=7$$

$$\Rightarrow c=\frac{7}{6}$$

Hence, the expansion is $y=1+x+\frac{3}{2}x^2+\frac{7}{6}x^3+\ldots$

4 (ii)	$y=\dfrac{\left(\mathrm{e}^{x}\right)^{2}}{1-x^{2}}$ $=\dfrac{\mathrm{e}^{x}}{\left(1-x^{2}\right)}(e^{x})$ Using (i) and Standard Series of e^{x} $=\left(1+x+\dfrac{3}{2}x^{2}+\dfrac{7}{6}x^{3}+\ldots\right)\left(1+x+\dfrac{x^{2}}{2}+\dfrac{x^{3}}{3!}\cdots\right)$ $\approx 1+x+\dfrac{x^{2}}{2}+\dfrac{x^{3}}{3!}+x+x^{2}+\dfrac{1}{2}x^{3}+\dfrac{3}{2}x^{2}+\dfrac{3}{2}x^{3}+\dfrac{7}{6}x^{3}$ $=1+2x+3x^{2}+\dfrac{10x^{3}}{3}$
5 (i)	Let $\left(a+b\mathrm{i}\right)^{2}=\mathrm{i}$ $\mathrm{a}^{2}-b^{2}+2ab\mathrm{i}=\mathrm{i}$ Comparing real and imaginary parts, $a^{2}-b^{2}=0$ and $2ab=1$ $a=b$ or $-b$ $\quad ab=\dfrac{1}{2}$ At $a=b$, $a^{2}=\dfrac{1}{2}$ $a=\pm\dfrac{1}{\sqrt{2}}$ Hence, $b=\pm\dfrac{1}{\sqrt{2}}$ For $a=-b$, $a^{2}=-\dfrac{1}{2}$ has no solutions since $a\in\mathbb{R}$ Hence the square roots are: $\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}\mathrm{i}$ or $-\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{2}}\mathrm{i}$
5 (ii)	Given $w^{2}+2w+\left(1-8\mathrm{i}\right)=0$, $w=\dfrac{-2\pm\sqrt{4-4\left(1-8\mathrm{i}\right)}}{2}$ $=\dfrac{-2\pm2\sqrt{1-1+8\mathrm{i}}}{2}$ $=-1\pm2\sqrt{2}\left(\sqrt{\mathrm{i}}\right)$ For $w=-1\pm2\sqrt{2}\left(\sqrt{\mathrm{i}}\right)$,

	At $\mathrm{i}=\left(\pm\frac{1}{\sqrt{2}}\pm\frac{1}{\sqrt{2}}\mathrm{i}\right)^2$, $\sqrt{\mathrm{i}}=\sqrt{\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}\right)^2}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}$ $w=-1\pm2\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\mathrm{i}\right)$ $=-1\pm(2+2\mathrm{i})$ $=-1+2+2\mathrm{i}$ or $-1-2-2\mathrm{i}$ $=1+2\mathrm{i}$ or $-3-2\mathrm{i}$
5 (iii)	$(1-8\mathrm{i})v^2+2\mathrm{i}v-1=0$ $(1-8\mathrm{i})+\frac{2\mathrm{i}}{v}-\frac{1}{v^2}=0$ $-\frac{1}{v^2}+\frac{2\mathrm{i}}{v}+(1-8\mathrm{i})=0$ $\frac{\mathrm{i}^2}{v^2}+\frac{2\mathrm{i}}{v}+(1-8\mathrm{i})=0$ $\left(\frac{\mathrm{i}}{v}\right)^2+2\left(\frac{\mathrm{i}}{v}\right)+(1-8\mathrm{i})=0$ Comparing with $w^2+2w+(1-8\mathrm{i})=0$ in (ii), we replace w in (ii) as $\left(\frac{\mathrm{i}}{v}\right)$ in (iii) for both roots in (i), $\frac{\mathrm{i}}{v}=1+2\mathrm{i}$ and $\frac{\mathrm{i}}{v}=-3-2\mathrm{i}$ $v=\frac{\mathrm{i}}{1+2i}=\frac{(1-2\mathrm{i})\mathrm{i}}{1+2^2}=\frac{1}{5}(2+\mathrm{i})$ and $v=\frac{\mathrm{i}}{-3-2\mathrm{i}}=\frac{(-3+2\mathrm{i})\mathrm{i}}{3^2+2^2}=\frac{1}{13}(-2-3\mathrm{i})$
6 (i)	Since $AC:CB$ is $\lambda:1-\lambda$, 

By Ratio Theorem,

$$\mathbf{c}=\frac{\lambda\mathbf{b}+(1-\lambda)\mathbf{a}}{\lambda+1-\lambda}$$
$$=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$$

Since OC is perpendicular to OA,

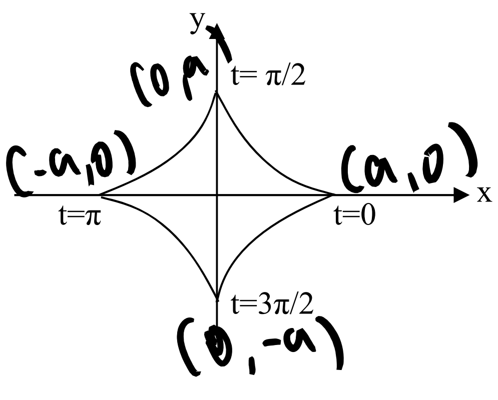
$$\mathbf{c}\bullet\mathbf{a}=0$$
$$\left[\lambda\mathbf{b}+(1-\lambda)\mathbf{a}\right]\bullet\mathbf{a}=0$$
$$\lambda\mathbf{b}\bullet\mathbf{a}+(1-\lambda)\mathbf{a}\bullet\mathbf{a}=0$$
$$\lambda\left[\frac{4}{3}\times1\times\cos120^\circ\right]+(1-\lambda)|\mathbf{a}|^2=0,\ \text{since } |\mathbf{a}|=\frac{4}{3}$$
$$\lambda\left[\frac{4}{3}\times\left(-\frac{1}{2}\right)\right]+(1-\lambda)\left(\frac{4}{3}\right)^2=0$$
$$\left(-\frac{2}{3}\right)\lambda+\frac{16}{9}(1-\lambda)=0$$
$$-\frac{2}{3}\lambda+\frac{16}{9}-\frac{16}{9}\lambda=0$$
$$\frac{22}{9}\lambda=\frac{16}{9}$$
$$\lambda=\frac{16}{22}=\frac{8}{11}$$

6(ii)

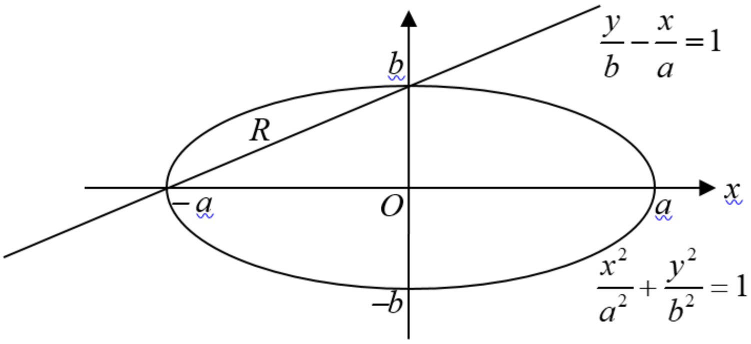
$$\mathbf{c}=\frac{3}{11}\mathbf{a}+\frac{8}{11}\mathbf{b}$$

By Mid-point Theorem, find $\overrightarrow{OM}=\mathbf{m}$, where M is the mid-point of AC.

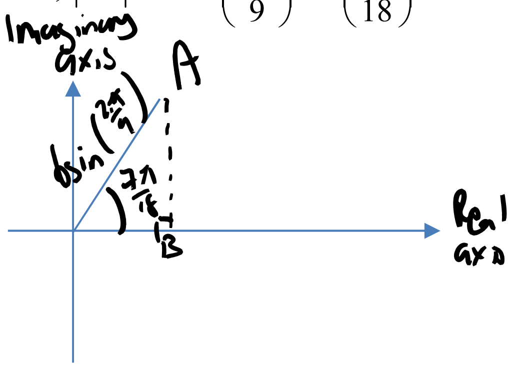
$$\overrightarrow{OM}=\mathbf{m}$$
$$=\frac{\mathbf{c}+\mathbf{a}}{2}$$
$$=\frac{1}{2}\left[\frac{3}{11}\mathbf{a}+\frac{8}{11}\mathbf{b}+\mathbf{a}\right]$$
$$=\frac{1}{2}\left(\frac{14}{11}\mathbf{a}+\frac{8}{11}\mathbf{b}\right)$$
$$=\frac{7}{11}\mathbf{a}+\frac{4}{11}\mathbf{b}$$

6 (iii)	$\mathbf{m}\bullet\mathbf{b}=\left(\frac{7}{11}\mathbf{a}+\frac{4}{11}\mathbf{b}\right)\bullet\mathbf{b}$ $=\frac{7}{11}\mathbf{a}\bullet\mathbf{b}+\frac{4}{11}\mathbf{b}\bullet\mathbf{b}$ $=\frac{7}{11}\mathbf{a}\bullet\mathbf{b}+\frac{4}{11}\lvert\mathbf{b}\rvert^2$ $=\frac{7}{11}\left(-\frac{2}{3}\right)+\frac{4}{11}(1)^2$ $=-\frac{2}{33}\neq 0$ Since the vector **b** is not perpendicular to **m**, where *OM* is the radius of the circle, *OB* is not a tangent to the circle.
6 (iv)	$\lvert\mathbf{b}\bullet\mathbf{m}\rvert$ is the length of projection of **m** on **b**. $(\mathbf{b}\bullet\mathbf{m})\mathbf{b}$ is a vector with magnitude $\lvert\mathbf{b}\bullet\mathbf{m}\rvert$, which is the length of projection of **m** on **b**. Moreover, it is in the opposite direction of **b** as $\mathbf{m}\bullet\mathbf{b}=-\frac{2}{33}<0$.
7 (i)	$x=a\cos^3 t$, $y=a\sin^3 t$ $t=0,\quad x=a\cos^3 0=a$, $y=a\sin^3 0=0$ $t=\frac{\pi}{2}, x=a\cos^3\frac{\pi}{2}=0$, $y=a\sin^3\frac{\pi}{2}=a$ $t=\pi\ , x=a\cos^3\pi=-a$, $y=a\sin^3\pi=0$ $t=\frac{3\pi}{2}, x=a\cos^3\frac{3\pi}{2}=0$, $y=a\sin^3\frac{3\pi}{2}=-a$ 

7 (ii)	$\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$ $\frac{dy}{dt} = 3a\sin^2 t(\cos t)$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}}$ $= \frac{3a\sin^2 t(\cos t)}{3a\cos^2 t(-\sin t)}$ $\frac{dy}{dx} = \frac{-\sin t}{\cos t}$ Gradient of normal $= \frac{\cos t}{\sin t}$ At $t = p$, the equation of the normal is: $y - a\sin^3 p = \frac{\cos p}{\sin p}(x - a\cos^3 p)$ $y\sin p - a\sin^4 p = x\cos p - a\cos^4 p$ $y\sin p = x\cos p - a\cos^4 p + a\sin^4 p$ $y\sin p = x\cos p + a(\sin^4 p - \cos^4 p)$ $= x\cos p + a(\sin^2 p - \cos^2 p)(\sin^2 p + \cos^2 p)$ $= x\cos p + a\left(-\cos 2p\right)(1)$ $= x\cos p - a\cos 2p$ (Shown)
7 (iii)	At $p = \frac{\pi}{3}$, $y\sin\frac{\pi}{3} = x\cos\frac{\pi}{3} - a(\cos\frac{2\pi}{3})$ $\frac{\sqrt{3}}{2}y = \frac{1}{2}x - a\left(-\frac{1}{2}\right)$ $\frac{\sqrt{3}}{2}y = \frac{1}{2}x + \frac{1}{2}a$ $\sqrt{3}y = x + a$ Since the normal meets the curve again, $\sqrt{3}(a\sin^3 t) = (a\cos^3 t) + a$ $\sqrt{3}\sin^3 t = \cos^3 t + 1,\ a \neq 0$ Using GC, $t = 2.32$ or 3.14 (3 s.f)

8 (i)	
8 (ii)	Area of R = Area of quadrant-Area of triangle $=-\int_0^b x\,\mathrm{d}y-\frac{1}{2}ab$ $=-\int_0^b -\sqrt{a^2-\frac{a^2y^2}{b^2}}\,\mathrm{d}y-\frac{1}{2}ab$ $=-\frac{1}{2}ab+\int_0^b \sqrt{a^2-\frac{a^2y^2}{b^2}}\,\mathrm{d}y$ $y=b\cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}\theta}=-b\sin\theta$ Area= $-\frac{1}{2}ab+\int_0^b \sqrt{a^2-\frac{a^2y^2}{b^2}}\,\mathrm{d}y$ $=-\frac{1}{2}ab+\int_{\frac{\pi}{2}}^0 \sqrt{a^2-\frac{a^2b^2\cos^2\theta}{b^2}}(-b\sin\theta)\,\mathrm{d}\theta$ $=-\frac{1}{2}ab+\int_{\frac{\pi}{2}}^0 -ab\sin^2\theta\ \mathrm{d}\theta$ $=-\frac{1}{2}ab-ab\int_{\frac{\pi}{2}}^0\left(\frac{1-\cos 2\theta}{2}\right)\ \mathrm{d}\theta$ $=-\frac{1}{2}ab-ab\left[\frac{1}{2}\theta-\frac{\sin 2\theta}{4}\right]_{\frac{\pi}{2}}^0$ $=-\frac{1}{2}ab-(ab)\left[-\left(\frac{\pi}{4}\right)\right]$ $=ab\left[\frac{\pi}{4}-\frac{1}{2}\right]\text{units}^2$

8 (iii)	Volume generated $=\pi\int_0^b x^2\,\mathrm{d}y-\frac{1}{3}\pi a^2 b$ $=\pi\int_0^b\left(a^2-\frac{a^2y^2}{b^2}\right)\mathrm{d}y-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2y-\frac{a^2y^3}{3b^2}\right]_0^b-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2b-\frac{a^2b^3}{3b^2}\right]-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2b-\frac{a^2b}{3}\right]-\frac{1}{3}\pi a^2 b$ $=\pi\left(\frac{2}{3}a^2b\right)-\frac{1}{3}\pi a^2 b$ $=\frac{1}{3}\pi a^2 b$ units^3
9(i)	Since $z^2-3z+9=0$ has all real coefficients, given that $z=3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ is a root of the equation, $z=3\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$ is the other root of the equation.
9(ii)	$\mathrm{e}^{\mathrm{i}\theta}-\mathrm{e}^{-\mathrm{i}\theta}$ $=(\cos\theta+\mathrm{i}\sin\theta)-[\cos(-\theta)+\mathrm{i}\sin(-\theta)]$ $=(\cos\theta+\mathrm{i}\sin\theta)-(\cos\theta-\mathrm{i}\sin\theta)$ $=2\mathrm{i}\sin\theta$
9 (iii)	Since $w_1=3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$, $w_2=3\mathrm{e}^{\mathrm{i}\frac{\pi}{9}}$

	$w_2 - w_1$ $= 3\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{9}\right)} - 3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$ $= 3\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{9}\right)} - 3\mathrm{e}^{\mathrm{i}\left(-\frac{3\pi}{9}\right)}$ $= 3\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{9}-\frac{\pi}{9}\right)} - 3\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{9}-\frac{\pi}{9}\right)}$ $= 3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{9}\right)}\left[\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{9}\right)} - \mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{9}\right)}\right]$ $= 3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{9}\right)}\left[2\mathrm{i}\sin\left(\frac{2\pi}{9}\right)\right]$ $= 6\sin\left(\frac{2\pi}{9}\right)\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{9}+\frac{\pi}{2}\right)}$ $= 6\sin\left(\frac{2\pi}{9}\right)\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{18}\right)}$
9 (iv)	At point *B*, $\lvert OB\rvert = 6\sin\left(\frac{2\pi}{9}\right)\cos\left(\frac{7\pi}{18}\right)$  Hence, Area of triangle *OAB*

$$= \frac{1}{2}|OB||OA|\sin\left(\frac{7\pi}{18}\right)$$

$$= \frac{1}{2}\left[6\sin\left(\frac{2\pi}{9}\right)\cos\left(\frac{7\pi}{18}\right)\right]\left[6\sin\left(\frac{2\pi}{9}\right)\right]\sin\left(\frac{7\pi}{18}\right)$$

$$= \frac{36}{2}\sin^2\left(\frac{2\pi}{9}\right)\sin\left(\frac{7\pi}{18}\right)\cos\left(\frac{7\pi}{18}\right)$$

$$= \frac{36}{2}\sin^2\left(\frac{2\pi}{9}\right)\left[\frac{\sin\left(\frac{14\pi}{18}\right)}{2}\right]$$

$$= 9\sin^2\left(\frac{2\pi}{9}\right)\sin\left(\frac{7\pi}{9}\right)$$

10 (i)

Year	n	Amount of loan at the beginning of nth year	Amount of loan at the end of nth year
2018	1	100000	$(100000-12x)(1.05)$
2019	2	$(100000-12x)(1.05)$	$\left[(100000-12x)(1.05)-12x\right](1.05)$ $=100000(1.05)^2-12x(1.05)^2$ $-12x(1.05)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
	n		$100000(1.05)^n-12x(1.05)^n$ $-12x(1.05)^{n-1}-\ldots$ $-12x(1.05)$

Amount of loan left at the end of n years

$$=100000(1.05)^n-12x(1.05)^n-12x(1.05)^{n-1}-\ldots-12x(1.05)$$

$$= 100000(1.05)^n-12x(1.05)\left[1+1.05+1.05^2+\ldots+1.05^{n-1}\right]$$

$$= 100000(1.05)^n-12x(1.05)\left[\frac{1.05^n-1}{1.05-1}\right]$$

$$= 100000(1.05)^n-252x(1.05^n-1)$$

10 (ii)

$$(1.05)^{10}(100000)-252x(1.05^{10}-1)\le 0$$

$$x\left[252(1.05^{10}-1)\right]\ge(1.05)^{10}(100000)$$

$$x\ge 1027.81$$

Hence, minimum value of x = \$1028 (to the nearest dollar)

10 (iii)

Amount of pay-out per year

$=50000\left(\dfrac{r}{100}\right)$

$=500r$

Year	n	Amt left on 1 Jan of nth year	Amt owed on 31 Dec of nth year
2018	1	$100000-500r$	$[100000-500r](1.05)$
2019	2	$[100000-500r](1.05)$ $-500r$	$[100000-500r](1.05)^2$ $-500r(1.05)$
…			
2027	10		$[100000-500r](1.05)^{10}$ $-500r(1.05)^9$ $-500r(1.05)^8 \ldots$ $-500r(1.05)^2-500r(1.05)$ $=100000(1.05)^{10}$ $-500r(1.05)\left[1+1.05+\cdots+1.05^9\right]$

Amount owed on 31 December of 10th year

$=100000(1.05)^{10}-500r(1.05)^{10}-500r(1.05)^9$

$-500r(1.05)^8\ldots-500r(1.05)^2-500r(1.05)$

$=100000(1.05)^{10}-500r(1.05)\left[1+1.05+\cdots+1.05^9\right]$

$$=100000(1.05)^{10}-500r(1.05)\left[\frac{1\left(1.05^{10}-1\right)}{1.05-1}\right]$$

$$=100000(1.05)^{10}-500r(1.05)\left[\frac{\left(1.05^{10}-1\right)}{0.05}\right]$$

$=100000(1.05)^{10}-10500r\left(1.05^{10}-1\right)$

For her to have completed paying,

$100000(1.05)^{10}-10500r\left(1.05^{10}-1\right)\leq 0$

$r \geq 24.668=24.7$ (to 3 sf)

11 (i)

$V = Ah$

$$\frac{dV}{dt}=A\frac{dh}{dt}$$

On the other hand,

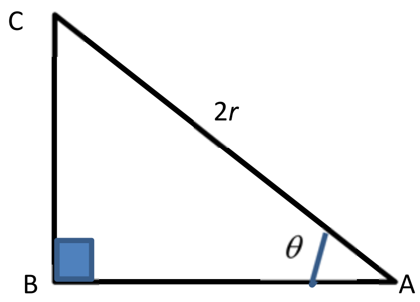
	$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V_{in}}{\mathrm{d}t}-\frac{\mathrm{d}V_{out}}{\mathrm{d}t}$ $=0-k\sqrt{h}$, where k is a positive constant $=-k\sqrt{h}$ Equating the expressions, $-k\sqrt{h}=A\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-k\sqrt{h}}{A}$
11 (ii)	$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-k\sqrt{h}}{A}$ $\int\frac{1}{\sqrt{h}}\,\mathrm{d}h=-\frac{k}{A}\int\mathrm{d}t$ $2\sqrt{h}=-\frac{k}{A}t+C$---(I) When t = 0, h =81, $\therefore C=18.$ $\Rightarrow 2\sqrt{h}=-\frac{k}{A}t+18$ When t = 0, $\frac{\mathrm{d}h}{\mathrm{d}t}=-0.3$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-k\sqrt{h}}{A}$=-0.3 --- (II) $-\frac{k}{A}(9)=-0.3$ $\frac{k}{A}=\frac{1}{30}$ $\therefore 2\sqrt{h}=-\frac{1}{30}t+18$ $\sqrt{h}=-\frac{1}{60}t+9$ --- (III) $\Rightarrow h=(9-\frac{1}{60}t)^2$
11 (iii)	If Joe is to be back *before* the water tank to be emptied, $h\geq 0$ $(9-\frac{1}{60}t)^2\geq 0$ Equivalently, using (III),

	$\sqrt{h}=-\frac{1}{60}t+9\ \geq 0$ $-\frac{t}{60}\geq -9$ $\frac{t}{60}\leq 9$ $t\ \leq\ 540$ Number of days he can be away $\leq \frac{540}{24}=22.5$ Maximum number of days = 22.
11 (iv)	At the end of 4^{th} day, 96 hours have lapsed. $h=(9-\frac{1}{60}\cdot 96)^2=(7.4)^2$ **<u>Method (1):</u>** $A=400\pi$ given that the radius is 20 cm. Using $\frac{k}{A}=\frac{1}{30}$ $k=\frac{400\pi}{30}$ $\Rightarrow k=\frac{40}{3}\pi$ $\frac{\mathrm{d}V}{\mathrm{d}t}=-k\sqrt{h}$ $=-\frac{40\pi}{3}(7.4)$ $=-\frac{296}{3}\pi\ \text{cm}^3/\text{hour}$ The water is being delivered at a rate of $-\frac{296}{3}\pi$ cm^3/hour. **<u>Method (2):</u>** $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\cdot\frac{\mathrm{d}h}{\mathrm{d}t}$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\mathrm{A}=400\pi$ $\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-k\sqrt{h}}{A}=-\frac{1}{30}(7.4)$ $\frac{\mathrm{d}V}{\mathrm{d}t}=(400\pi)(-\frac{1}{30}(7.4))=-\frac{296}{3}\pi\ \text{cm}^3/\text{hour}.$

St Andrew's Junior College
2018 Preliminary Examination
H2 Mathematics Paper 2 (9758/02) Solutions

Section A

Qn	**Solution**
1	$y=a(x-1)+\dfrac{b}{x+c}$ $\downarrow$ A (replace y with $-y$) $-y=a(x-1)+\dfrac{b}{x+c}$ $y=-a(x-1)-\dfrac{b}{x+c}$ $\downarrow$ B (replace x with $x+1$) $y=-a(x+1-1)-\dfrac{b}{x+1+c}$ $y=-ax-\dfrac{b}{x+1+c}$ Since $x=0$ is a vertical asymptote, $1+c=0$ $c=-1$ Therefore, $y=-ax-\dfrac{b}{x}=\mathrm{f}(x)$. Since $\left(1,\dfrac{1}{6}\right)$ is a turning point on $y=\dfrac{1}{\mathrm{f}(x)}$, $(1,6)$ is a turning point on $y=\mathrm{f}(x)$. $-a-b=6$ --- (1) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-a+\dfrac{b}{x^2}$ Since when $x=1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$, $-a+b=0$ --- (2) Solving (1), (2) using a GC, $a=-3,\ b=-3$ Therefore, $a=-3,\ b=-3$ and $c=-1$
2	Given that AC is the diameter, $\angle ABC=\dfrac{\pi}{2}$.



$AB = 2r\cos\theta$, $BC = 2r\sin\theta$

Let S be the area of triangle ABC.

$$S = \frac{1}{2}(AB)(BC)$$

$$= \frac{1}{2}(2r\cos\theta)(2r\sin\theta)$$

$$= 2r^2\sin\theta\cos\theta$$

$$= r^2\sin 2\theta$$

$$\frac{\mathrm{d}S}{\mathrm{d}\theta} = r^2(2\cos 2\theta)$$

For stationary values of S, $\frac{\mathrm{d}S}{\mathrm{d}\theta} = 0$

Since $2r^2 \neq 0$,

$\cos 2\theta = 0$

Since $0 < \theta < \frac{\pi}{2} \Rightarrow 0 < 2\theta < \pi$

Hence,

$$2\theta = \frac{\pi}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$\frac{\mathrm{d}^2S}{\mathrm{d}\theta^2} = 2r^2(-2\sin 2\theta) = -4r^2\sin 2\theta$$

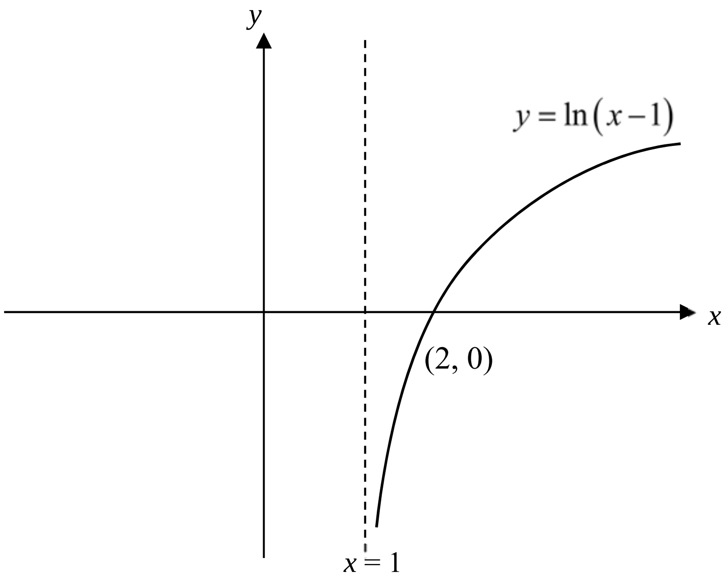
At $\theta = \frac{\pi}{4}$,

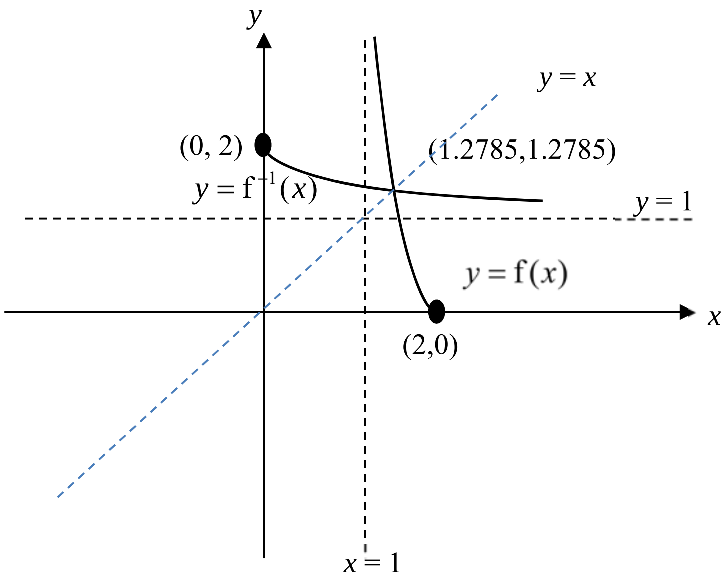
$$\frac{\mathrm{d}^2S}{\mathrm{d}\theta^2} = -4r^2\sin 2\left(\frac{\pi}{4}\right)$$

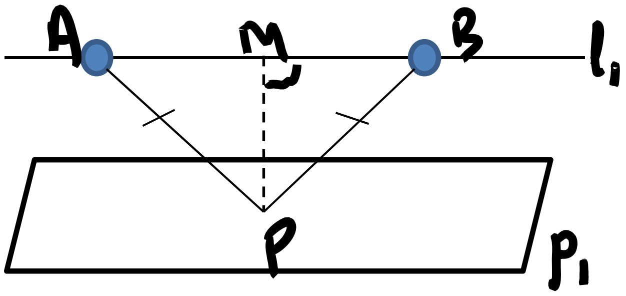
$$= -4r^2 < 0$$

Hence S is maximum when $\theta = \frac{\pi}{4}$.

3 (i)	$\sin\theta + m\cos\theta = \sqrt{1+m^2}\,\sin\left(\theta + \tan^{-1} m\right)$ $\therefore R = \sqrt{1+m^2}, \alpha = \tan^{-1} m$
	$2(\sin\theta + m\cos\theta)\sin(\theta-\alpha)$ $= R\left[2\sin(\theta+\alpha)\sin(\theta-\alpha)\right]$ $= R\left[-(\cos 2\theta - \cos 2\alpha)\right]$ $= R\left(\cos 2\alpha - \cos 2\theta\right)$
3 (ii)	$\int \frac{\cos 2\theta}{(\sin\theta + m\cos\theta)\sin(\theta-\alpha)}\,\mathrm{d}\theta$ $= 2\int \frac{\cos 2\theta}{2(\sin\theta + m\cos\theta)\sin(\theta-\alpha)}\,\mathrm{d}\theta$ $= \frac{2}{\sqrt{1+m^2}}\int \frac{\cos 2\theta}{(\cos(2\alpha)-\cos(2\theta))}\mathrm{d}\theta$ $= \frac{-2}{\sqrt{1+m^2}}\int \frac{\cos 2\theta}{(\cos 2\theta + 1)}\mathrm{d}\theta$ since $\cos(2\alpha) = -1$ when $\alpha = \frac{\pi}{2}$ $= \frac{-2}{\sqrt{1+m^2}}\int\left[\frac{2\cos^2\theta - 1}{2\cos^2\theta}\right]\mathrm{d}\theta$ $= \frac{-2}{\sqrt{1+m^2}}\int\left[1 - \frac{1}{2\cos^2\theta}\right]\mathrm{d}\theta$ $= \frac{-2}{\sqrt{1+m^2}}\left[\int 1\,\mathrm{d}\theta - \frac{1}{2}\int \sec^2\theta\,\mathrm{d}\theta\right]$ $= \frac{-2}{\sqrt{1+m^2}}\left[\theta - \frac{1}{2}\tan\theta\right] + C,$ where C is an arbitrary constant

4(i)	 From the graph, the solution is $1<x\leq 2$
	Alternative $\ln(x-1)\leq 0$ $0<x-1\leq 1$ $1<x\leq 2$ Since $\ln(x-1)$ is defined for $x>1$, the solution is $1<x\leq 2$.
4 (ii)	Let $y=\lvert\ln(x-1)\rvert$ Since $1<x\leq 2$, $y=-\ln(x-1)$ $-y=\ln(x-1)$ $x-1=e^{-y}$ $x=1+e^{-y}$ $\mathrm{f}^{-1}(x)=1+e^{-x}$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=[0,\infty)$ Alternative: $y=\lvert\ln(x-1)\rvert$ $\pm y=\ln(x-1)$ $e^{\pm y}=x-1$ $e^{-y}=x-1$ (since $y\geq 0$ and $1<x\leq 2$) $\therefore x=e^{-y}+1$ $\mathrm{f}^{-1}(x)=1+e^{-x}$ $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=[0,\infty)$

4 (iii)	
4 (iv)	Point of intersection of $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}(x)$ is (1.28, 1.28) $\mathrm{f}^{-1}(x) < \mathrm{f}(x)$ From the graph, $1 < x \le 1.27$.
4 (v)	$\mathrm{fg}(a) = 3$ $\mathrm{g}(a) = \mathrm{f}^{-1}(3)$ $1 + \dfrac{4}{4a^2+5} = 1 + e^{-3}$ $\dfrac{4}{4a^2+5} = \dfrac{1}{e^3}$ $4e^3 = 4a^2 + 5$ $a^2 = \dfrac{4e^3-5}{4}$ $a = \sqrt{\dfrac{4e^3-5}{4}}$ or $a = -\sqrt{\dfrac{4e^3-5}{4}}$ (reject since $D_{\mathrm{fg}} = D_g =$ ¡$^+$)
5 (i)	$l_1:\ \mathbf{r} = \begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},\ \lambda \in$ ¡ $l_2: \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}, \mu \in$ ¡ **Condition 1:** $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \neq k \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ for any real values of k.

	Hence lines l_1 and l_2 are not parallel to each other. **Condition 2:** Suppose both lines intersect, $\begin{pmatrix} 11-2\lambda \\ 6 \\ \lambda \end{pmatrix} = \begin{pmatrix} 4+2\mu \\ -3\mu \\ -1+4\mu \end{pmatrix}$ $\begin{cases} 11-2\lambda = 4+2\mu \\ 6=-3\mu \\ \lambda = -1+4\mu \end{cases}$ $\Rightarrow \begin{cases} 2\mu+2\lambda = 7 ---(1) \\ 6=-3\mu ---(2) \\ -4\mu+\lambda=-1---(3) \end{cases}$ Using G.C. to solve (1) and (3), $\mu = \frac{9}{10},\ \lambda = \frac{13}{5}$ Checking with (2), $-3\mu = -3\left(\frac{9}{10}\right)$ $= -\frac{27}{10} \neq 6$ Hence, the two lines do not intersect at any unique points. Combining both conditions (1) and (2), lines l_1 and l_2 are skew lines.
5 (ii)	$\overrightarrow{OA} = \begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix}$ Since P is on p_1 such that it is equidistant from A and B, ABP forms an isosceles triangle.  Let the mid-point of AB be M. By mid-point theorem,

$$\overrightarrow{OM} = \frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$$

$$= \frac{\begin{pmatrix} 11 \\ 6 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} 6 \\ 6 \\ \frac{5}{2} \end{pmatrix}$$

To find the foot of perpendicular, P, of M on p_1,

$$\overrightarrow{OP} = \begin{pmatrix} 6 \\ 6 \\ \frac{5}{2} \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}, \text{ for some } s \in \mathbb{R}$$

$$= \begin{pmatrix} 6+4s \\ 6+3s \\ \frac{5}{2}+8s \end{pmatrix}$$

Given p_1 : $\mathbf{r} \bullet \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} = 240$

Since P lies on p_1,

$$\begin{pmatrix} 6+4s \\ 6+3s \\ \frac{5}{2}+8s \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} = 240$$

$$4(6+4s)+3(6+3s)+8\left(\frac{5}{2}+8s\right) = 240$$

$$24+16s+18+9s+20+64s = 240$$

$$62+89s = 240$$

$$89s = 178$$

$$s = \frac{178}{89} = 2$$

	Hence, $\overrightarrow{OP} = \begin{pmatrix} 6+4(2) \\ 6+3(2) \\ \frac{5}{2}+8(2) \end{pmatrix}$ $= \begin{pmatrix} 14 \\ 12 \\ \frac{37}{2} \end{pmatrix}$
5 (iii)	Normal vector of p_2 $= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ $= \begin{pmatrix} 0-(-3) \\ -(-8-2) \\ 6-0 \end{pmatrix}$ $= \begin{pmatrix} 3 \\ 10 \\ 6 \end{pmatrix}$ Since the plane p_2 contains the point P, $\mathbf{r} \bullet \begin{pmatrix} 3 \\ 10 \\ 6 \end{pmatrix} = \begin{pmatrix} 14 \\ 12 \\ \frac{37}{2} \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 10 \\ 6 \end{pmatrix}$ $= 273$ The Cartesian equation of p_2 is: $3x + 10y + 6z = 273$

Section B

6	Case 1: Committee only has 1 woman

<table>
<tr><td></td><td>
No. of ways = $^{15}C_7\,{}^{7}C_1 = 45045$
Case 2: Committee has no women
No. of ways = $^{15}C_8 = 6435$
Hence, total number of ways
= $^{22}C_8 - 45045 - 6435 = 268290$

<u>ALT</u>

Case 1: Committee only has 2 women

No. of ways = $^{15}C_6\,{}^{7}C_2 = 105105$

Case 2: Committee only has 3 women

No. of ways = $^{15}C_5\,{}^{7}C_3 = 105105$

Case 3: Committee only has 4 women
No. of ways = $^{15}C_4\,{}^{7}C_4 = 47775$

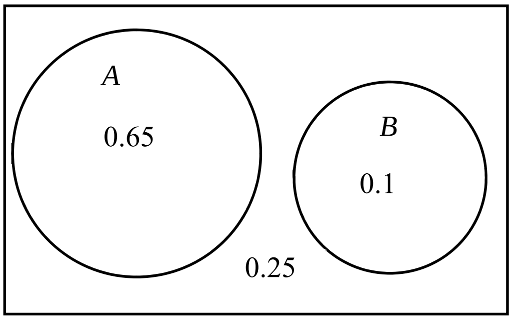
Case 4: Committee only has 5 women
No. of ways = $^{15}C_3\,{}^{7}C_5 = 9555$

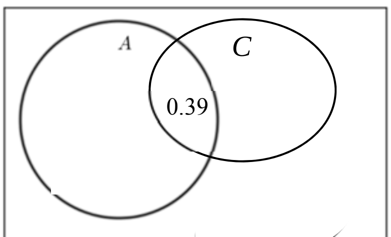
Case 5: Committee only has 6 women
No. of ways = $^{15}C_2\,{}^{7}C_6 = 735$

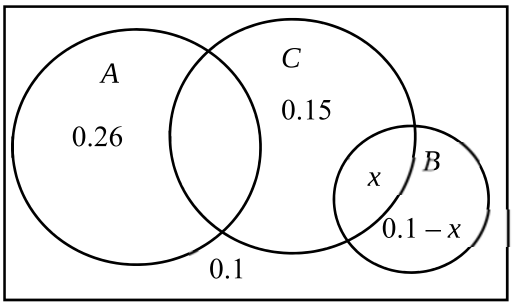
Case 6: Committee only has 7 women
No. of ways = $^{15}C_1\,{}^{7}C_7 = 15$

Hence, total number of ways
= $105105 + 105105 + 47775 + 9555 + 735 + 15$
= 268290
</td></tr>
<tr><td></td><td>
Number of ways to choose the 4 corner seats = $^{4}C_2$
Number of ways to arrange the remaining 6 committee members = 6!
Number of ways to arrange the Gina and Hazel = 2!
Hence, total number of ways = $^{4}C_2 \times 2! \times 6! = 8640$
</td></tr>
</table>

7 (i)	Let W denote the number of shots that hits the bullseye 40m away from the target out of 18 shots. $W \sim B(18, \frac{2}{195}(95-40))$ $W \sim B(18, \frac{22}{39})$ $\mathrm{P}\left(W > 6 \middle\vert W \le 10\right)$ $= \frac{\mathrm{P}\left([W > 6] \cap [W \le 10]\right)}{\mathrm{P}(W \le 10)}$ $= \frac{\mathrm{P}\left(7 \le W \le 10\right)}{\mathrm{P}(W \le 10)}$ $= \frac{\mathrm{P}\left(W \le 10\right) - \mathrm{P}\left(W \le 6\right)}{\mathrm{P}(W \le 10)}$ $= \frac{0.51941}{0.56103}$ $= 0.926$ (to 3 s.f.)

7(ii)	Let Y denote the number of shots that hits the bullseye x m away from the target out of 18 shots. $Y \sim B(18, \frac{2}{195}(95-x))$ $Y \sim B\left(18, \left(\frac{190}{195}-\frac{2}{195}x\right)\right)$ $P(Y \geq 2) = 0.98$ $1 - P(Y=0) - P(Y=1) = 0.98$ $P(Y=0) + P(Y=1) = 0.02$ $\left[1-\left(\frac{190}{195}-\frac{2}{195}x\right)\right]^{18}$ $+\left\{{}^{18}C_1 \times \left(\frac{190}{195}-\frac{2}{195}x\right)^1 \times \left[1-\left(\frac{190}{195}-\frac{2}{195}x\right)\right]^{17}\right\} = 0.02$ $\left(\frac{1}{39}+\frac{2}{195}x\right)^{18} + \left[{}^{18}C_1 \times \left(\frac{190}{195}-\frac{2}{195}x\right)^1 \times \left(\frac{1}{39}+\frac{2}{195}x\right)^{17}\right] = 0.02$ Using the GC, $x = 67.3$ m (to 3 s.f.)
8 (i)	 $P(B \mid A') = \frac{2}{7}$ $\frac{P(B \cap A')}{P(A')} = \frac{2}{7}$ $P(B \cap A') = \frac{2}{7}(1-0.65)$ $= 0.1$ $P(B) = P(B \cap A')$ $= 0.1$ (shown)
8 (ii)	



	$\mathrm{P}(C'\mid A)$ $=\dfrac{\mathrm{P}(A\cap C')}{\mathrm{P}(A)}$ $=\dfrac{\mathrm{P}(A)-\mathrm{P}(A\cap C)}{\mathrm{P}(A)}$ $=\dfrac{0.65-0.39}{0.65}$ $=0.4$
8 (iii)	 Let $\mathrm{P}(B\cap C)=x$ When *B* and *C* are independent, $\mathrm{P}(B\cap C)=\mathrm{P}(B)\times\mathrm{P}(C)$ $x=0.1\times(0.39+0.15+x)$ $x=0.054+0.1x$ $0.9x=0.054$ $x=0.06$ $\therefore \mathrm{P}(B\cap C)=0.06$
8 (iv)	$\mathrm{P}(C)=0.39+0.15+0.06=0.6$ Since $\mathrm{P}(A)\times\mathrm{P}(C)=(0.65)(0.6)=0.39=\mathrm{P}(A\cap C)$ (or $\mathrm{P}(C')=1-\mathrm{P}(C)=0.4=\mathrm{P}(C'\mid A)$) *A* and *C* are independent.
9 (i)	Let *X* be the amount of winnings Sally gets after one game in dollars.

Let the random variables F be the scores on the fair dice.

$$\begin{aligned}&\mathrm{P}(X=-5)\\&=\mathrm{P}(|F-Y|>3)\\&=\mathrm{P}[F=5,Y=1]+\mathrm{P}[F=6,Y=1]+\\&\mathrm{P}[F=6,Y=2]+\mathrm{P}[F=1,Y=5]\\&+\mathrm{P}[F=1,Y=6]+\mathrm{P}[F=2,Y=6]\\&=\left(\frac{1}{6}\times\frac{1}{6}\right)+\left(\frac{1}{6}\times\frac{1}{6}\right)+\left(\frac{1}{6}\times\frac{1}{18}\right)\\&+\left(\frac{1}{6}\times\frac{1}{6}\right)+\left(\frac{1}{6}\times\frac{5}{18}\right)+\left(\frac{1}{6}\times\frac{5}{18}\right)\\&=\frac{5}{27}\end{aligned}$$

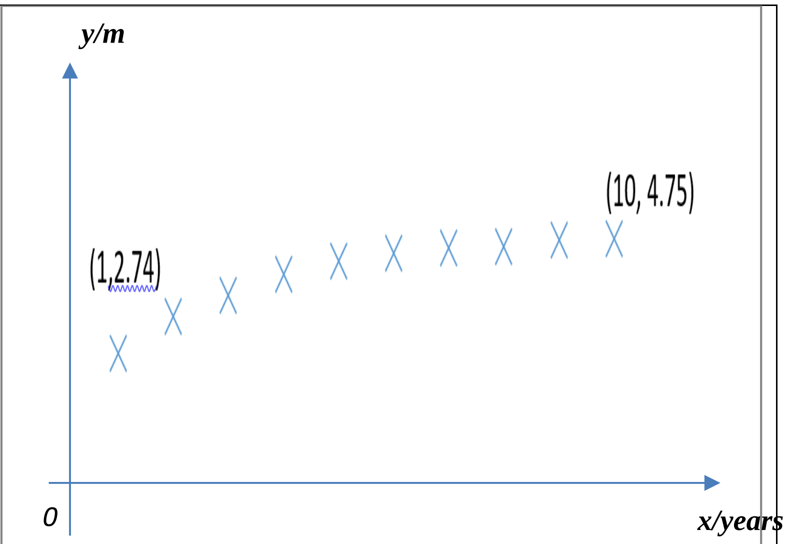
$$\begin{aligned}&\mathrm{P}(X=0)\\&=\mathrm{P}[F=1,Y=1]+\mathrm{P}[F=2,Y=2]+\\&\mathrm{P}[F=3,Y=3]+\mathrm{P}[F=4,Y=4]\\&+\mathrm{P}[F=5,Y=5]+\mathrm{P}[F=6,Y=6]\\&=\frac{1}{6}\left(\frac{1}{6}+\frac{1}{18}+\frac{1}{6}+\frac{3}{18}+\frac{1}{6}+\frac{5}{18}\right)\\&=\frac{1}{6}\end{aligned}$$

$$\begin{aligned}\mathrm{P}(X=3)&=1-\mathrm{P}(X=0)-\mathrm{P}(X=-5)\\&=1-\frac{5}{27}-\frac{1}{6}=\frac{35}{54}\end{aligned}$$

x	-5	0	3
$\mathrm{P}(X=x)$	$\frac{5}{27}$	$\frac{1}{6}$	$\frac{35}{54}$

$$\begin{aligned}&\mathrm{E}(X)\\&=\sum_{\text{all }x}x\mathrm{P}(X=x)\\&=-5\left(\frac{5}{27}\right)+0\left(\frac{1}{6}\right)+(3)\left(\frac{35}{54}\right)\\&=\frac{55}{54}\end{aligned}$$

9(ii)	$\mathrm{E}(X^2)$ $=\sum_{all\ x} x^2\mathrm{P}(X=x)$ $=(-5)^2\left(\frac{5}{27}\right)+(0)^2\left(\frac{1}{6}\right)+(3)^2\left(\frac{35}{54}\right)$ $=\frac{565}{54}$ $\mathrm{Var}(X)$ $=\mathrm{E}(X^2)-[\mathrm{E}(X)]^2$ $=\frac{565}{54}-\left(\frac{55}{54}\right)^2$ $=\frac{27485}{2916}$ Since the number of games of 50 is sufficiently large, by Central Limit Theorem, $X_1+X_2+\dots+X_{50}\sim \mathrm{N}\left(50\left(\frac{55}{54}\right),50\left(\frac{27485}{2916}\right)\right)$ approximately $\mathrm{P}(X_1+X_2+\dots+X_{50}\geq 65)$ $=0.25839$ $=0.258$ (to 3 sf)
10 (a)	$\bar{x}=\frac{20+22+24+...+34}{8}=27$ and $\bar{y}=\frac{16+21+a+24+22+24+27+20}{8}$ $=\frac{154+a}{8}$. Since $(\bar{x},\bar{y})$ lies on the regression line, $\frac{154+a}{8}=8+0.5(27)$ $a=18$
(b) (i)	

	
(ii)	The height of the tree increases by b metres every year.
(iii)	Let the product moment correlation coefficient for this model be r. For A, $r_A = 0.958$, For B, $r_B = 0.972$ and for C, $r_C = 0.996$ Since $r_C = 0.996$ is closest to 1, the most appropriate model is Model C. Equation required is $y = 2.78 + 0.903\ln x$ $a = 2.78$ and $b = 0.903$
(iv)	Since $y = 4.25$ is within the given range of values [2.74, 4.75], the estimate obtained via interpolation is reliable. Since $r_C = 0.996$ is close to 1 which suggests a strong linear correlation between $\ln x$ and y , it would be reasonable to use Model C.
11 (i)	Let X be the random variable " speed of a vehicle travelling along Spring Avenue, measured in km/hour". The unbiased estimate of population mean = $\bar{x}$ = $\frac{\sum x}{100} = \frac{4327.0}{100} = 43.27$ The unbiased estimate of population variance,

	$s^2 = \frac{\sum(x-\bar{x})^2}{n-1} = \frac{925.71}{99} = 9.3506 = 9.35$ (to 3 sf)
(ii)	The 100 vehicles are selected independently.
(iii)	Let μ be the population mean speed Test H_0: $\mu = 44.1$ H_1: $\mu < 44.1$ Under H_0, $\bar{X} \sim N(44.1, \frac{9.3506}{100})$ approximately by Central Limit Theorem, since sample size 100 is sufficiently large. Carry out one-tailed *z*-test at 5% significance level. $\bar{x} = 43.27$ gives rise to test statistic $z = -2.71$ and *p*-value 0.00332. Since p-value $\leq$ 0.05, we reject H_0 and conclude that at the 5% significance level, there is sufficient evidence that the mean speed of vehicles has been reduced after life sized police photos have been erected along Spring Avenue.
(iv)	5% significance level refers to the probability of 0.05 of wrongly concluding that the mean speed of vehicles is less than 44.1 km/hour when it is in fact 44.1 km/hour.
(v)	New unbiased estimate of population mean $= 43.27k$ New unbiased estimate of population variance $= \frac{30857}{3300}k^2$ Test H_0: $\mu = 44.1$ H_1: $\mu < 44.1$ Under H_0, $\bar{X} \sim N(44.1, \frac{\frac{30857}{3300}k^2}{100})$ approximately by Central Limit Theorem, since sample size 100 is sufficiently large. To have the same conclusion, i.e. to reject H_0, $z_{cal} \leq -1.64485$ $\frac{(43.27k)-44.1}{(k\sqrt{\frac{30857}{3300}})/10} \leq -1.64485$ $43.2k - 44.1 \leq -\frac{1.64485\sqrt{\frac{30857}{3300}}}{10}k$ $k \leq 1.01$ Since k > 0, $0 < k \leq 1.01$ Greatest possible value of k is 1.01.

12 (i)(a)	Let X be the random variable " volume of Aquafresh mineral water in a bottle in ml" $X \sim \text{N}(1505, 10.2^2)$ $\text{P}(X > 1480) = 0.993$ from G.C.
(i)(b)	Let μ_1 be the new mean volume. $\text{P}(X < 1480) < 0.10$ $\text{P}(Z < \frac{1480-\mu_1}{10.2}) < 0.10$ $\frac{1480-\mu_1}{10.2} < -1.28155$ $\mu_1 > 1493.07$ The least mean volume is 1493.1 ml.
(ii)	Let Y be the random variable " volume of Sparkling spring water in a bottle in ml" $Y \sim \text{N}(508.5, 3.5^2)$ P(the volume of water in each of the 6 bottles from a randomly selected pack is more than 505 ml) $= [\text{P}(Y > 505)]^6$ $= [0.84134]^6$ $= 0.355$ (3 sig figures)
(iii)	Let $T = Y_1 + Y_2 + ... + Y_6 - 2X$ $T \sim N(41, 489.66)$ $P(\lvert T \rvert < 5.5) = P(-5.5 < T < 5.5) = 0.0365$
(iv)	Let V be the volume of water used by a guest in a bathroom in litres. $V \sim \text{N}(120, 65^2)$ $\text{P}(V < 0) = 0.0324$ which is impossible as the volume of water used cannot be negative.
	Since sample size 30 is sufficiently large, at least 20, Central Limit Theorem can be applied. $\therefore \overline{V} \sim N(120, \frac{65^2}{30})$ approximately.