1 Adult tickets for a parade are sold at three different prices, depending on the type of seats.

Children under the age of 12 and Senior Citizens aged 65 and above can enjoy 20% and 10% off the adult ticket prices respectively. Those who are between 12 and 64 years old (inclusive) will have to pay the full adult ticket price.

The number of tickets sold in each category for each group of people, together with the total cost of the tickets for each group, are given in the following table.

	Category 1 tickets (premier)	Category 2 tickets (sheltered)	Category 3 tickets (unsheltered)	Total cost
Adults (12 to 64 years old (inclusive))	150	120	200	\$18 000
Children (under 12 years old)	80	40	100	\$6 760
Senior Citizens (65 years old and above)	100	100	100	\$10 665

Write down and solve equations to find the price of an adult ticket for each of the ticket categories. [4]

2 Aden, a member of Raffles Art Club, is helping to design a graphic for the school's homecoming event. The main body of the graphic consists of N concentric circles C_1, C_2, $C_3, \ldots$, C_N with radii r, $2r$, $3r, \ldots$, Nr, where r is a constant and N is an even integer, as shown in Fig. 1.

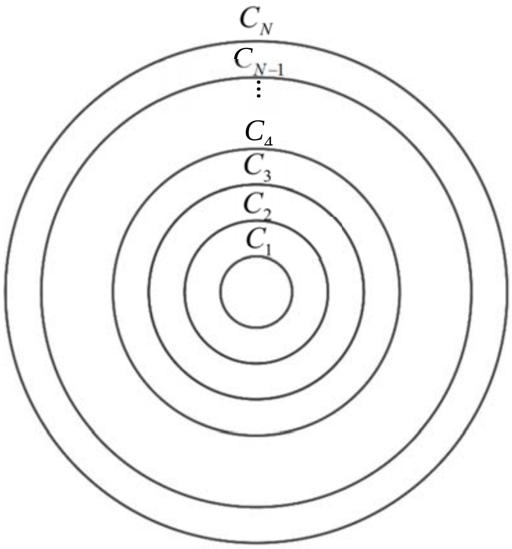


Fig. 1

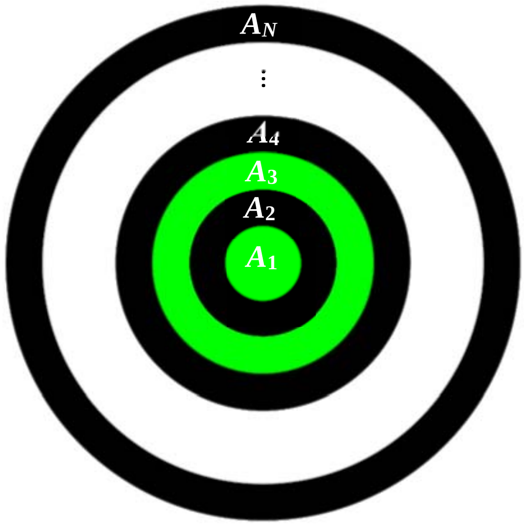


Fig. 2

The region enclosed by the circle C_1 is denoted by A_1 with area a_1 and the region between the circles C_{n-1} and C_n is denoted by A_n with area a_n for $n = 2, 3, 4, \ldots, N$, as shown in Fig. 2.

(i) Show that the sequence a_1, a_2, $a_3, \ldots$, a_N is an arithmetic progression. [4]

With the help of a graphic software, Aden fills A_1, A_2, $A_3, \ldots$, A_N with two of the school colours: green and black. A_n will be filled green if n is odd and will be filled black if n is even.

Aden wishes to create a better visual effect by having different intensities of green. He fills A_1 with the green colour that is the same as the school colour. He reduces the intensity of this green colour by ten percent to fill A_3; and reduces the intensity of the green colour used to fill A_3 by ten percent to fill A_5; and this process continues for all odd values of n. When he finishes filling up all the areas, Aden finds that the intensity of the green colour that fills A_{N-1} falls below one quarter of the intensity of the school colour for the first time.

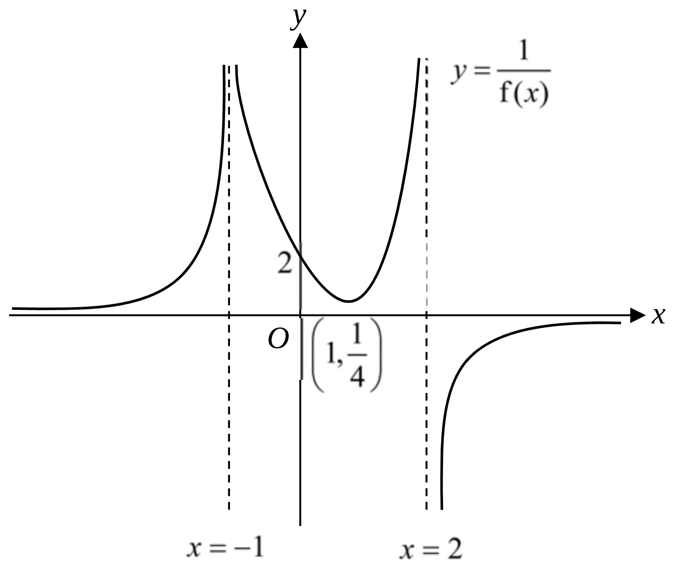
(ii) Find the value of N. [4]

[Turn over

3 **(a)** Find $\int_0^1 \frac{9x-4x^2}{9-4x^2}\,dx$, giving your answer in the form $p+q\ln 3+r\ln 5$ where p, q and r are rational numbers to be determined. [5]

(b) Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.
Hence, or otherwise, find $\int \sin x \sin 2x \sin 3x \,dx$. [4]

4 **(a)** The diagram below shows the curve of $y=\frac{1}{f(x)}$ where $f(x)$ is a polynomial. The curve has a minimum point at $\left(1,\frac{1}{4}\right)$ and cuts the y-axis at $(0,2)$. The lines $x=-1$ and $x=2$ are the vertical asymptotes and the line $y=0$ is the horizontal asymptote to the curve.



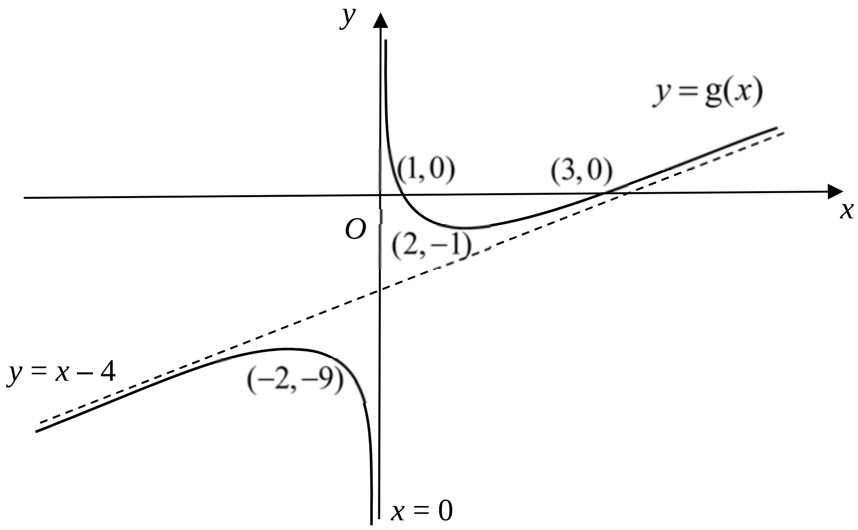
Sketch on separate diagrams, the graphs of

(i) $y=f(x)$, [3]

(ii) $y=f(|x|)$, [2]

labelling all relevant point(s).

4 **(b)** The diagram below shows the curve of $y = \mathrm{g}(x)$. The curve has a maximum point at $(-2, -9)$ and a minimum point at $(2, -1)$. The curve crosses the x-axis at $(1, 0)$ and $(3, 0)$. The line $x = 0$ is the vertical asymptote and the line $y = x - 4$ is the oblique asymptote to the curve.

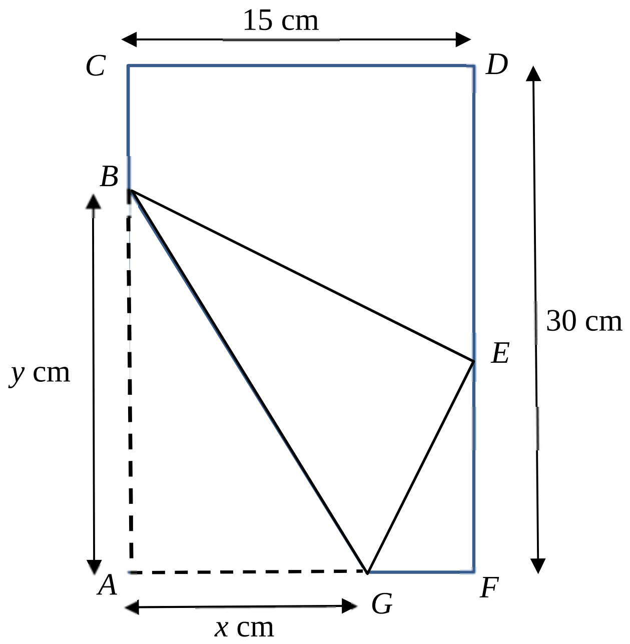


(i) Sketch the graph of $y = \mathrm{g}'(x)$, labelling all relevant point(s) and stating the equations of any asymptotes. [2]

(ii) Find the area bounded by the graph of $y = \mathrm{g}'(x)$, the lines $x = 1$, $x = 2$ and the x-axis. [2]

[Turn over

5



Tom has a rectangular piece of paper *ACDF* with length 30 cm and breadth 15 cm. He folds the lower left-hand corner, *A*, to reach the rightmost edge of the paper at *E*. After that he will cut out the triangle *EFG* and the trapezium *BCDE* to obtain a kite shaped figure *ABEG* with *AB* of length y cm and *AG* of length x cm.

(i) Find the length of *EF* in terms of x. [2]

(ii) Show that $y = \dfrac{x\sqrt{15}}{\sqrt{2x-15}}$. [2]

(iii) Using differentiation, find the exact values of x and y which give the minimum area of the kite *ABEG*. [5]

6 **(a)** It is given that $e^y = \tan\left(x+\frac{\pi}{3}\right)$.

(i) Show that $\frac{dy}{dx} = e^y + e^{-y}$. [2]

(ii) Using differentiation, find the Maclaurin series for $\ln\left[\tan\left(x+\frac{\pi}{3}\right)\right]$ up to and including the term in x^2. [3]

(b) Let $f(x) = \frac{3x^2-4x+5}{(1+x)(1-x)^2}$.

(i) Express $f(x)$ in the form $\frac{A}{(1+x)}+\frac{B}{(1-x)}+\frac{C}{(1-x)^2}$ where *A*, *B* and *C* are constants to be determined. [2]

(ii) Hence find the expansion of $f(x)$ up to and including the term in x^4. [2]

(iii) Write down the coefficient of x^r in the expansion of $f(x)$ in terms of *r*. [1]

7 **Do not use a calculator in answering this question.**

(a) Find the roots of the equation $w^2(1-i)+4w+(10+10i)=0$, giving your answers in cartesian form $a+ib$. [3]

(b) It is given that $z=-\sqrt{3}+i$.

(i) Find an exact expression for z^5. Give your answer in the form $re^{i\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [2]

(ii) Find the three smallest positive whole number values of *n* for which $\frac{z^n}{iz*}$ is purely imaginary. [4]

(iii) Given that $\left|1+\frac{p}{z*}\right|=\sqrt{7}$, find exactly the possible values of the real number *p*. [3]

[Turn over

8 The function $\mathrm{f}(x)=a-(1+x)^{\frac{1}{3}}$ is defined for $0\le x\le 7$, where a is a positive real constant.

(i) Solve the inequality $\mathrm{f}(x)\ge a-1$. [2]

(ii) Show that the graph of $y=\mathrm{f}(x)$ has a negative gradient at all points on the graph. Find the range of f. [2]

Use $a=3$ in the rest of the question.

(iii) Find $\mathrm{f}^{-1}(x)$. [2]

(iv) Sketch on the same diagram the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$, showing clearly the geometrical relationship between the two graphs and the line $y=x$. [3]

(v) Show that the x-coordinate of the point of intersection of the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ satisfies the equation $x^3-9x^2+28x-26=0$.
Hence find the solution of the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [3]

9 **(a)** The variables x and y are related by the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{x}\mathrm{e}^{\sqrt{x}}$. Using the substitution $u=\sqrt{x}$, show that $y=k\int u^2\mathrm{e}^u\,\mathrm{d}u$ where k is a constant to be determined. Hence find the general solution of the differential equation. [5]

(b) Ms Frugal bought a second-hand oven which has a broken timer. Each time she bakes she uses the timer on her handphone. One day, she decides to make a loaf of bread. She takes the fermented dough out of her refrigerator and checks that the internal temperature of the dough is $4°\mathrm{C}$. She puts it into the oven which has been pre-heated to a constant temperature of $180°\mathrm{C}$ and forgets to set the timer on her handphone. She also does not note the time. At 10.06 am, she realizes that the timer has not been set. She checks her dough which now has an internal temperature of $80°\mathrm{C}$. Twenty minutes later, she checks the dough again and the internal temperature has risen to $130°\mathrm{C}$.

Newton's Law of Cooling states that the rate of increase of the temperature $\theta°\mathrm{C}$ of an object after t minutes is directly proportional to the difference in the temperatures of the object and its surrounding.

The dough has to be baked at a constant oven temperature of $180°\mathrm{C}$ for one and a half hour to cook through. By forming a differential equation, find the time (to the nearest minute) that Ms Frugal should remove the dough from the oven. [8]

10 Two children Hansel and Gretel are participating in the junior category of the Festival of Lights competition. They decided to set up a structure consisting of a pyramid with a rectangular base *OABC* and vertex *V*. The main power supply switch is positioned at *O* with coordinates $(0, 0, 0)$ and relative to *O*, the coordinates of *A* and *C* are $(1, 2, 1)$ and $(-1, 1, -1)$ respectively. The vertex *V* is fixed at a height $\sqrt{8}$ metres above the rectangular base *OABC* such that it is equidistant to the points *O*, *A*, *B* and *C*. All the surfaces of the pyramid, excluding the base *OABC*, are completely covered with LED (light emitting diodes) light strips so that it illuminates in the dark.

(i) Point *D* is the foot of the perpendicular from *V* to the rectangular base *OABC*. Show that the coordinates of *D* is $(0, 1.5, 0)$. [1]

(ii) Find a vector perpendicular to the rectangular base *OABC*. Hence, find the position vector of *V*, given that its **k**-component is positive. [4]

Hansel and Gretel decide to install two different colour display schemes on the pyramid and the control switch is to be installed inside the pyramid at a point *E* with position vector $\alpha\mathbf{i} + 1.5\mathbf{j} - \alpha\mathbf{k}$, where α is a constant.

(iii) Show that *VE* is perpendicular to the rectangular base *OABC* and explain why $-2 < \alpha < 0$. [3]

(iv) Find the distance between *E* and the surface *OVA* in terms of α, simplifying your answer. [4]

(v) Given that the ratio of the distance between *E* and the rectangular base *OABC* to the distance between *E* and the surface *OVA* is $10 : \sqrt{105}$, find the value of α. [2]

********* End of Paper *********

Pure Mathematics (40 marks)

1 When a solid turns into a gas without first becoming a liquid, the process is called sublimation. As a spherical mothball sublimes, its volume, in cm^3, decreases at a rate, in cm^3 per day, proportional to its surface area.

Show that the radius of the mothball decreases at a constant rate. [3]

[It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

2 The curve C has equation

$$y=\frac{(ax+1)(4x+b)}{2x+3}, \quad x\in \mathbb{R},\ x\neq -\frac{3}{2},$$

where a and b are constants. It is given that $y=2x-1$ is an asymptote of C.

(i) Find the values of a and b. [3]

(ii) Sketch C, labelling all relevant point(s) and stating the equations of any asymptotes. [4]

3 The position vectors of the points U and V with respect to the origin O are $\mathbf{u}$ and $\mathbf{v}$ respectively, where $\mathbf{u}$ and $\mathbf{v}$ are non-zero and non-parallel vectors.

(i) Show that $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})$ can be written as $k(\mathbf{u}\times\mathbf{v})$, where k is a constant to be determined, justifying your working. [2]

The vectors $\mathbf{u}$ and $\mathbf{v}$ are now given by

$$\mathbf{u}=\mathbf{i}-\mathbf{j}+2\mathbf{k} \text{ and } \mathbf{v}=a\mathbf{i}+b\mathbf{j},$$

where a and b are constants.

(ii) If $a=2$ and $b=3$, find a unit vector parallel to $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})$. [2]

(iii) If $\mathbf{v}$ is perpendicular to the vector $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, find $\mathbf{v}$ in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ and b. Interpret the solution geometrically. [3]

4 A curve C has parametric equations

$$x=\cot\theta,\ \ y=2\cos^2\theta,$$

where $0<\theta\leq\frac{1}{2}\pi.$

(i) The point P on C has parameter $\theta=\frac{1}{6}\pi$. Show that the gradient of the tangent at P is $\frac{\sqrt{3}}{4}$. Hence find the equation of the normal, L, at P. [3]

(ii) Sketch C and L on the same diagram, stating the coordinates of the points where the line L cuts the curve C and the x-axis respectively. [2]

(iii) Find the exact area of the region bounded by the curve C, the positive y-axis and the line $y=\frac{3}{2}$. [3]

(iv) Show that the Cartesian equation of C is given by $y=\frac{2x^2}{a+x^b}$, where a and b are constants to be determined. [2]

(v) The region bounded by the curve C, the line L, the positive x-axis and the line $x=\frac{1}{\sqrt{3}}$ is rotated through 2π radians about the x-axis. Find the volume of the solid formed. [3]

[Turn over

5 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_r = \begin{cases} 0 & \text{for } r = 2, \\ \mathrm{f}(r-1) - 2\mathrm{f}(r-3) & \text{for } r \text{ even, } r \neq 2, \\ \mathrm{f}(r) & \text{for } r \text{ odd.} \end{cases}$$

(i) Use the method of differences to find $\sum_{r=1}^{2n} u_r$. [3]

It is given that $\mathrm{f}(r) = \ln\left(\frac{r+1}{r}\right)$.

(ii) Use your answer to part **(i)** to show that $\sum_{r=1}^{2n} u_r = -\ln 2 + 2\ln\left(1 + \frac{1}{2n-1}\right)$. Hence state the value of the sum to infinity. [2]

(iii) Find the smallest value of n for which $\sum_{r=1}^{2n} u_r$ is within 10^{-2} of the sum to infinity. [2]

(iv) By considering the graph of $y = \frac{1}{x}$ for $x > 0$, show, with the aid of a sketch, that

$$\frac{1}{2n} < u_{2n-1} < \frac{1}{2n-1}, \quad n \in ¢^{+}.$$ [3]

Statistics (60 marks)

6 For events A and B it is given that $\mathrm{P}(A) = 0.5$ and $\mathrm{P}(B') = 0.35$.

(a) Given that events A and B are not independent, find the range of values of $\mathrm{P}(A \cup B)$. [2]

(b) Given that $\mathrm{P}(A \mid B') = 0.6$, find

(i) $\mathrm{P}(A \cup B)$, [3]

(ii) $\mathrm{P}(A' \cup B')$. [1]

7 The speed of the surface of a tennis court, affected by a variety of factors, including the physical makeup of the court, can be measured using the Court Pace Index (CPI). The CPI of a court varies with every match played on it. A higher mean CPI for a court indicates a faster court surface.

For the rest of this question, assume that data is collected under similar conditions.

At a tennis club, the CPI of the main court is a continuous random variable X. Based on past readings, the standard deviation of X is k and the expected value of X is 33.8.

(i) A random sample of 50 readings is to be taken. When $k = 1.5$, estimate the probability that the mean value of X, for this sample, will be at most 0.5 units from 34. [2]

In January 2018, the main court was resurfaced with a new physical makeup.

(ii) The club management claims that the new physical makeup of the resurfaced main court increases its mean CPI. A random sample of 60 readings were taken from the resurfaced main court after January 2018 and the mean CPI of this sample is found to be 34.3.

A test is carried out, at the 5% level of significance, to determine whether the resurfaced main court has a faster court surface.

Find the set of possible values of k for which the result of the test should be accepted. You should state your hypotheses clearly. [3]

(iii) The club management later obtained information that the value of k is 0.8 and that X follows a normal distribution. Another random sample of 30 readings were taken from the resurfaced main court after January 2018 and the mean CPI of this new sample is found to be $\bar{x}$.

A test is carried out, at the 5% level of significance, to determine whether the mean CPI of the resurfaced main court has changed.

Find the set of possible values of $\bar{x}$ for which the result of the test should not be accepted. You should state your hypotheses clearly. [3]

[Turn over

8 In this question you should state clearly the values of the parameters of any normal distribution you use.

The masses in grams of apples have the distribution $N(80,\ 3^2)$ and the masses in grams of oranges have the distribution $N(100,\ 5^2)$.

(i) Find the probability that the total mass of 2 randomly selected oranges is less than 205 grams. [2]

(ii) Find the probability that the total mass of 4 randomly selected apples is more than three times the mass of 1 randomly selected orange. [3]

During the Community Health Week Event, visitors who take part in the activities are given gift boxes. Each gift box contains 3 apples and 2 oranges that are individually machine wrapped. The mass of wrapper used for each fruit is dependent on the mass of the fruit resulting in the mass of each wrapped fruit being 7% more than the mass of the fruit. The fruits are packed in a gift box and the mass of an empty gift box is normally distributed with mean 50 grams and standard deviation 2 grams. During packing, parts of the gift box is cut and removed. This process reduces the mass of each empty gift box by 10%.

(iii) The probability that the total mass of a randomly selected gift box is less than k grams is 0.8. Find the value of k.

Find the probability that, at a particular collection point, the 25^{th} gift box that is given is the 19^{th} gift box whose total mass is less than k grams. [5]

9 The age of a species of trees can be determined by counting its rings, but that requires either cutting a tree down or extracting a sample from the tree's core. A forester attempts to find a relationship between a tree's age and its diameter instead. Based on past records, the forester found data for the diameter and the age (determined by the counting of its rings) of 8 trees of the same species that had been cut down. The results are given in the following table.

Diameter, D (inches)	1.8	6.6	9.9	10.8	12.8	13.2	15.4	16.5
Age, t (years)	5	8	16	12	22	28	30	40

(i) Draw a scatter diagram for these values, labelling the axes clearly. [2]

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between D and t. Comment on whether a linear model would be appropriate. [2]

(iii) It was suggested that the tree's age can be modelled by one of the formulae

$$t = a + bD^2, \qquad t = ae^{bD},$$

where a and b are constants.

Find, correct to 4 decimal places, the product moment correlation coefficient between

(A) t and D^2,

(B) $\ln t$ and D.

Explain which of $t = a + bD^2$ and $t = ae^{bD}$ best models the age of this species of trees based on the given data. [2]

(iv) The forester wants to estimate the diameter of a tree given its age. Using a suitable regression line, find the required estimate of a tree that is 50 years old. Comment on the reliability of your estimate.

Explain why neither the regression line of t on D^2 nor the regression line of $\ln t$ on D should be used. [4]

[Turn over

10 Amy has four identical star shaped ornaments and six identical heart shaped ornaments.

(a) Find the number of ways in which Amy can arrange the ten ornaments in a line. [1]

The star shaped ornaments are coloured Red, Green, Yellow and Purple, and the heart shaped ornaments are then coloured Red, Green, Yellow, Purple, Black and White.

(b) Find the number of ways in which Amy can arrange the ten ornaments in a line such that exactly five heart shaped ornaments are next to each other. [3]

(c) Amy then decides to arrange the ten ornaments in a circular manner. Find the number of ways in which all ornaments of the same colour are next to each other. [2]

(d) Amy randomly distributes the ten ornaments into one group of four and two groups of three.

(i) Show that the probability that the black and white ornaments are in the same group is $\frac{4}{15}$. [3]

(ii) Find the probability that there are at least two heart shaped ornaments in the group of four given that the black and white ornaments are in the same group. [3]

11 The salad bar at a restaurant has n bowls each containing a different ingredient. Each customer to the restaurant is allowed only one visit to the salad bar. Jon visits the salad bar and makes a selection. At each bowl, he can take some or none of the contents, and he does not return to the bowl again. As the salad bar may not serve the same ingredients everyday, Jon does not know what ingredients will be served at each visit. On average, the probability that he takes some of the contents from each bowl is 0.4. For each visit to the salad bar, the number of different ingredients Jon takes is the random variable X.

(i) State, in the context of this question, an assumption needed to model X by a binomial distribution, and explain why the assumption may not hold. [2]

Assume now that X has the distribution $\mathrm{B}(n, 0.4)$.

(ii) Jon visits the salad bar on a randomly chosen day. The probability that Jon takes exactly 7 different ingredients is denoted by P. Determine the value of n that gives the greatest value of P. [2]

Jon's selection on any day is independent of his selection on any other day. During a promotional week, the salad bar serves 6 different ingredients (A, B, C, D, E, F) on Monday and Tuesday, 6 different ingredients (G, H, I, J, K, L) on Wednesday and Thursday, and all 12 different ingredients (A to L) on Friday, Saturday and Sunday.

(iii) If Jon visits the salad bar on Monday and Thursday of the promotional week, find the probability that he takes at least 3 different ingredients on each day. [2]

(iv) If Jon visits the salad bar on Saturday of the promotional week, find the probability that he takes at least 6 different ingredients. [1]

(v) Explain in context why the answer to part **(iv)** is greater than the answer to part **(iii)**. [1]

After the promotional week, the restaurant decides to charge customers \$3 for a visit to the salad bar, and an additional \$2 if they take more than $\dfrac{n}{2}$ different ingredients. Kai visits the salad bar after the promotional week and makes a selection. The number of different ingredients Kai takes is the random variable K. It is given that $\mathrm{P}(K \le k) = \left(\dfrac{k}{n}\right)^2,\ k = 0, 1, 2, ..., n$.

(vi) If Kai visits the salad bar on a particular day after the promotional week when $n = 9$, find the expected amount Kai pays for that particular visit. [3]

Kai visits the salad bar on a randomly chosen day after the promotional week.

(vii) Show that $\mathrm{P}(K = k) = \dfrac{2k-1}{n^2},\ k = 1, 2, ..., n$. [1]

(viii) Find, in terms of n, the expected number of different ingredients Kai takes on that day. [You may use the result $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$.] [2]

********* End of Paper *********

2018 RI Prelim Exam Paper 1

1 **[4]**	Let the cost of an adult ticket for Category 1, 2 and 3 be \$$x$, \$$y$ and \$$z$ respectively. $150x+120y+200z=18000$ $80(0.8x)+40(0.8y)+100(0.8z)=6760$ $100(0.9x)+100(0.9y)+100(0.9z)=10665$ $x=50, \quad y=40, \quad z=28.5$ The cost of a Category 1, 2 and 3 adult ticket is \$50, \$40 and \$28.50 respectively.

2(i) **[4]**	$a_n=\pi(nr)^2-\pi[(n-1)r]^2 \quad$ for $n=2,3,...,N$ $=\pi[nr-(n-1)r][nr+(n-1)r]$ $=\pi(r)(2nr-r)$ $=\pi r^2(2n-1)$ $a_1=\pi r^2=\pi r^2(2\times1-1)$ and it follows the form $A_n=\pi r^2(2n-1)$ when $n=1$. Therefore, $A_n=\pi r^2(2n-1)$ for $n=1, 2, 3, K, N$. The difference between any two consecutive terms (for $n=2,3,...,N$) $=a_n-a_{n-1}$ $=[\pi r^2(2n-1)]-\{\pi r^2[2(n-1]-1\}$ $=2\pi r^2$ Since $2\pi r^2$ is independent of n, it is a constant. Since the difference between every two consecutive terms is constant, the sequence a_1, a_2, a_3, K, a_N is an arithmetic progression. (shown)
(ii) **[4]**	The intensity of green colour in area $A_1, A_3, A_5, \ldots, A_n, \ldots, A_{N-1}$, where n is an odd number, is: $1, \ 0.9, \ 0.9^2 \ \ldots, \ 0.9^{\frac{N-2}{2}}$, which is a geometric progression $0.9^{\frac{N-2}{2}}<0.25$ $\frac{N-2}{2}>\frac{\ln 0.25}{\ln 0.9}$ $N>28.315 \quad$ (5s.f.) Since N is an even integer, $N=30$. Alternative 1 Since $\frac{N-2}{2}$ is an integer and $\frac{N-2}{2}>\frac{\ln 0.25}{\ln 0.9}=13.158 \quad$ (5s.f.), then $\frac{N-2}{2}=14 \Rightarrow N=30$. Alternative 2

X	Y1			
26	0.2824			
27	0.2679			
28	0.2542			
29	0.2411			
30	0.2288			
31	0.217			
32	0.2059			
33	0.1953			
34	0.1853			
35	0.1758			
36	0.1668			

X=26

From GC, (N is an even integer)

when $N = 28, \quad 0.9^{\frac{N-2}{2}} = 0.2542 > 0.25$;

when $N = 30, \quad 0.9^{\frac{N-2}{2}} = 0.2288 < 0.25$.

$\therefore N = 30$

Alternative 3

Let the intensity of the green-colour regions form a geometric progression of the form $x,\ 0.9x,\ 0.9^2 x,\ \ldots, 0.9^{n-1}x,\ \ldots$ where the intensity for region n is given by $0.9^{n-1}x$. We want

$$0.9^{n-1}x < 0.25x$$

$$n-1 > \frac{\ln 0.25}{\ln 0.9}$$

$$n > 14.158 \quad (5\text{s.f.})$$

The least value of n is 15. Hence the first region that satisfies the condition is the 15th odd region, A_{29}, and N is 30.

3(a)
[5]

$$\int_0^1 \frac{9x-4x^2}{9-4x^2}\,dx$$

$$= \int_0^1 \left(1 + \frac{9x-9}{9-4x^2}\right) dx$$

$$= 1 + 9\int_0^1 \frac{x}{9-4x^2}\,dx - 9\int_0^1 \frac{1}{9-4x^2}\,dx$$

$$= 1 + 9\left[-\frac{1}{8}\ln\left(9-4x^2\right) - \frac{1}{12}\ln\left(\frac{3+2x}{3-2x}\right)\right]_0^1$$

$$= 1 + \frac{9}{8}\ln\frac{9}{5} - \frac{3}{4}\ln 5$$

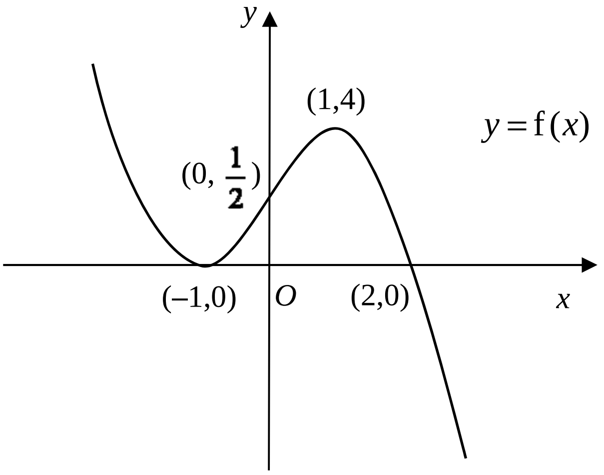
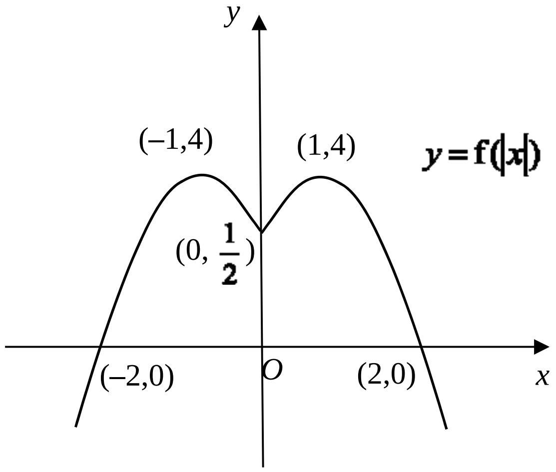
$$= 1 + \frac{9}{4}\ln 3 - \frac{15}{8}\ln 5$$

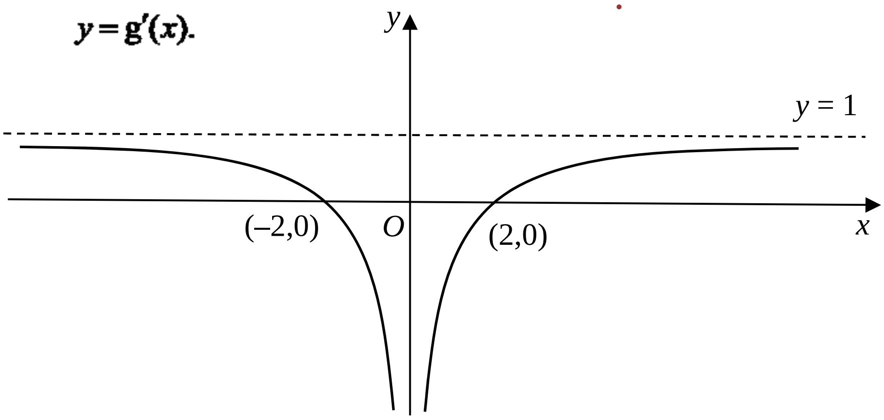
3(b)
[4]

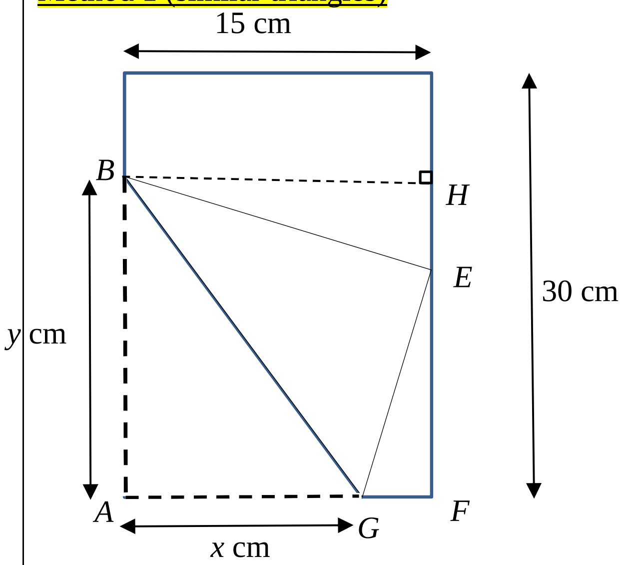
$$\begin{aligned} \sin 3x &= \sin(2x+x) \\ &= \sin 2x \cos x + \cos 2x \sin x \\ &= 2\sin x \cos^2 x + (1-2\sin^2 x)\sin x \\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x \\ &= 3\sin x - 4\sin^3 x \end{aligned}$$

$$\int \sin x \sin 2x \sin 3x\,dx = \int \sin x(2\sin x\cos x)(3\sin x - 4\sin^3 x)\,dx$$

	$= 6\int \sin^3 x \cos x \, dx - 8\int \sin^5 x \cos x \, dx$ $= \frac{3}{2}\sin^4 x - \frac{4}{3}\sin^6 x + c$ Alternatively $\int \sin x \sin 2x \sin 3x \, dx = -\frac{1}{2}\int (\cos 4x - \cos 2x)\sin 2x \, dx$ $= -\frac{1}{2}\int (\sin 2x \cos 4x - \sin 2x \cos 2x) \, dx$ $= -\frac{1}{2}\int (\sin 2x(2\cos^2 2x - 1) - \sin 2x \cos 2x) \, dx$ $= -\int \sin 2x \cos^2 2x \, dx + \frac{1}{2}\int \sin 2x \, dx + \frac{1}{4}\int \sin 4x \, dx$ $= -\int \sin 2x \cos^2 2x \, dx + \frac{1}{2}\int \sin 2x \, dx + \frac{1}{4}\int \sin 4x \, dx$ $= \frac{1}{6}\cos^3 2x - \frac{1}{4}\cos 2x - \frac{1}{16}\cos 4x + c$

4a (i) [3]	
(ii) [2]	

(b) **[4]**	$y = g'(x)$.  Required area = $\int_1^2 g'(x)\, dx = -[g(x)]_1^2$ $= -[g(2) - g(1)]$ (from graph in question) $= -[-1 - 0] = 1$

5(i) **[2]**	Let the length of *EF* be *w* $w^2 = x^2 - (15 - x)^2$ $w^2 = x^2 - (15^2 - 30x - x^2)$ $\Rightarrow w^2 = 30x - 15^2$ $\Rightarrow w = \sqrt{15}(2x - 15)^{\frac{1}{2}}$
(ii) **[2]**	Method 1 (similar triangles)  Triangle *BHE* is similar to triangle *EFG* Therefore $\frac{BE}{EG} = \frac{BH}{EF} \Rightarrow \frac{y}{x} = \frac{15}{\sqrt{15}(2x-15)^{\frac{1}{2}}}$ $\Rightarrow y = \frac{x\sqrt{15}}{\sqrt{(2x-15)}}$

	Method 2 (Pythagoras Theorem) $BH^2+HE^2=BE^2$ $15^2+(y-EF)^2=y^2$ $225+(y-\sqrt{15}\sqrt{(2x-15)})^2=y^2$ $225+y^2-2y\sqrt{15}\sqrt{(2x-15)}+15(2x-15)=y^2$ $-2y\sqrt{15}\sqrt{(2x-15)}+30x=0$ $y=\dfrac{x\sqrt{15}}{\sqrt{(2x-15)}}$
(iii) **[5]**	Area of kite $ABEG$ $=xy$ Method 1 (find stationary point) $\therefore \quad A=\dfrac{x^2\sqrt{15}}{\sqrt{2x-15}}$ $\Rightarrow A^2=\dfrac{15x^4}{2x-15}$ $\Rightarrow 2A\dfrac{\mathrm{d}A}{\mathrm{d}x}=\dfrac{60x^3(2x-15)-2(15x^4)}{(2x-15)^2}$ For stationary value, $\dfrac{\mathrm{d}A}{\mathrm{d}x}=0$ $\Rightarrow 30x^3\left[2(2x-15)-x\right]=0$ $\Rightarrow \quad x=0$ (reject) or $x=10$ When $x=10$, $y=\dfrac{10\sqrt{15}}{\sqrt{5}}=10\sqrt{3}$ Method 2 (find stationary point) $\therefore \quad A=\dfrac{x^2\sqrt{15}}{\sqrt{2x-15}}=\sqrt{15}x^2(2x-15)^{-\frac{1}{2}}$ $\therefore \quad \dfrac{\mathrm{d}A}{\mathrm{d}x}=2\sqrt{15}x(2x-15)^{-\frac{1}{2}}+\sqrt{15}x^2\left[-\dfrac{1}{2}(2x-15)^{-\frac{3}{2}}(2)\right]$ $\Rightarrow \quad \dfrac{\mathrm{d}A}{\mathrm{d}x}=x\sqrt{15}(2x-15)^{-\frac{3}{2}}\left[2(2x-15)-x\right]$ $\Rightarrow \quad \dfrac{\mathrm{d}A}{\mathrm{d}x}=x\sqrt{15}(2x-15)^{-\frac{3}{2}}[3x-30]$ For stationary value, $\dfrac{\mathrm{d}A}{\mathrm{d}x}=0$ $\Rightarrow \quad x\sqrt{15}(2x-15)^{-\frac{3}{2}}[3x-30]=0$ $\Rightarrow \quad x=0$ (reject) or $x=10$ When $x=10$, $y=\dfrac{10\sqrt{15}}{\sqrt{5}}=10\sqrt{3}$

Method 1 (Determine nature of stationary point)

Using Second Derivative Test

$$\frac{d^2A}{dx^2}=\sqrt{15}\left(-\frac{3}{2}\right)(2x-15)^{-\frac{5}{2}}\left[3x^2-30x\right]+\sqrt{15}(2x-15)^{-\frac{3}{2}}[6x-30]$$

$=10.3923>0$

Therefore $x=10$, $y=10\sqrt{3}$ gives the minimum area of the kite.

Method 2 (Determine nature of stationary point)

Using first derivative test

	$x=10^-$	$x=10$	$x=10^+$
x	+ve	+ve	+ve
$3(x-10)$	–ve	0	+ve
$\frac{dA}{dx}=x\sqrt{15}(2x-15)^{-\frac{3}{2}}[3x-30]$	–ve	0	+ve

We conclude that $x=10$, $y=10\sqrt{3}$ gives the minimum area of the kite.

6a(i) [2]

$$e^y=\tan\left(x+\frac{\pi}{3}\right)$$

Differentiate with respect to x

$$e^y\frac{dy}{dx}=\sec^2\left(x+\frac{\pi}{3}\right)$$

$$e^y\frac{dy}{dx}=1+\tan^2\left(x+\frac{\pi}{3}\right)$$

$$e^y\frac{dy}{dx}=1+e^{2y}$$

$$\frac{dy}{dx}=e^y+e^{-y}$$

(ii) [3]

$$\frac{dy}{dx}=e^y+e^{-y}$$

$$\frac{d^2y}{dx^2}=e^y\frac{dy}{dx}-e^{-y}\frac{dy}{dx}$$

When $x=0$, $e^y=\tan\frac{\pi}{3}\Rightarrow y=\ln\left(\sqrt{3}\right)=\frac{1}{2}\ln 3$

$$\frac{dy}{dx}=e^{\frac{1}{2}\ln 3}+e^{-\frac{1}{2}\ln 3}\Rightarrow\frac{dy}{dx}=\sqrt{3}+\frac{1}{\sqrt{3}}=\frac{4}{\sqrt{3}}$$

$$\frac{d^2y}{dx^2}=\frac{4}{\sqrt{3}}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)=\frac{8}{3}$$

$$\ln\left[\tan\left(x+\frac{\pi}{3}\right)\right]=\frac{1}{2}\ln 3+\frac{4}{\sqrt{3}}x+\frac{\frac{8}{3}}{2!}x^2+\ldots=\frac{1}{2}\ln 3+\frac{4}{\sqrt{3}}x+\frac{4}{3}x^2$$

b(i) **[2]**	$\frac{3x^2-4x+5}{(1+x)(1-x)^2}=\frac{A}{(1+x)}+\frac{B}{(1-x)}+\frac{C}{(1-x)^2}$ $3x^2-4x+5=A(1-x)^2+B(1+x)(1-x)+C(1+x)$ When $x=1$, $3-4+5=2C \Rightarrow C=2$ $x=-1$, $3+4+5=4A \Rightarrow A=3$ $x=0$, $5=A+B+C \Rightarrow B=0$
(ii) **[2]**	$\frac{3}{(1+x)}+\frac{2}{(1-x)^2}=3(1+x)^{-1}+2(1-x)^{-2}$ $=3\left(1-x+x^2-x^3+x^4+\cdots\right)$ $+2\left(1+2x+\frac{-2(-3)}{2!}x^2+\frac{-2(-3)(-4)}{3!}(-x)^3+\frac{-2(-3)(-4)(-5)}{4!}(-x)^4+\cdots\right)$ $=3\left(1-x+x^2-x^3+x^4+\cdots\right)+2\left(1+2x+3x^2+4x^3+5x^4+\cdots\right)$ $=5+x+9x^2+5x^3+13x^4+\cdots$
(iii) **[1]**	The coefficient of x^r is $=3(-1)^r+2(r+1)$

7(a) **[3]**	$w^2(1-\mathrm{i})+4w+(10+10\mathrm{i})=0$ $w=\frac{-4\pm\sqrt{4^2-4(1-\mathrm{i})(10+10\mathrm{i})}}{2(1-\mathrm{i})}$ $=\frac{-4\pm\sqrt{16-40(1-\mathrm{i})(1+\mathrm{i})}}{2(1-\mathrm{i})}$ $=\frac{-4\pm\sqrt{16-40(2)}}{2(1-\mathrm{i})}$ $=\frac{-4\pm\sqrt{-64}}{2(1-\mathrm{i})}$ $=\frac{-4\pm 8\mathrm{i}}{2(1-\mathrm{i})}\times\frac{1+\mathrm{i}}{1+\mathrm{i}}$ $=(-1\pm 2\mathrm{i})(1+\mathrm{i})$ $w=(-1+2\mathrm{i})(1+\mathrm{i})$ or $w=(-1-2\mathrm{i})(1+\mathrm{i})$ $=-1-\mathrm{i}+2\mathrm{i}+2\mathrm{i}^2$ $=-1-\mathrm{i}-2\mathrm{i}-2\mathrm{i}^2$ $=-3+\mathrm{i}$ $=1-3\mathrm{i}$
(b) **(i)** **[2]**	Given $z=-\sqrt{3}+\mathrm{i}$, $\lvert z\rvert=2$, $\arg(z)=\frac{5\pi}{6}$ Method 1 : $\lvert z^5\rvert=2^5=32$ $\arg(z^5)=5\arg(z)-4\pi=5\left(\frac{5\pi}{6}\right)-4\pi=\frac{\pi}{6}$ $\therefore z^5=32e^{\mathrm{i}\frac{\pi}{6}}$

Method 2 :

$$z^5 = \left(2e^{i\frac{5\pi}{6}}\right)^5$$
$$= 2^5 e^{i\left(\frac{25\pi}{6} - 4\pi\right)}$$
$$= 32e^{i\frac{\pi}{6}}$$

(ii)
[4]

$$\frac{z^n}{iz^*} = \frac{2^n e^{i\frac{5n\pi}{6}}}{e^{i\frac{\pi}{2}} 2e^{-i\frac{5\pi}{6}}}$$
$$= 2^{n-1} e^{i\left(\frac{5n\pi}{6} - \frac{\pi}{2} + \frac{5\pi}{6}\right)}$$
$$= 2^{n-1} e^{i\frac{(5n+2)\pi}{6}}$$

Alternatively : Just determine $\arg\left(\frac{z^n}{iz^*}\right)$

$$n\arg(z) - \arg(i) - \arg(z^*)$$
$$= n\left(\frac{5\pi}{6}\right) - \frac{\pi}{2} + \frac{5\pi}{6} \qquad \text{since } \arg(z^*) = -\arg(z) = -\frac{5\pi}{6}$$
$$= \left(\frac{5n+2}{6}\right)\pi$$

For $\frac{z^n}{iz^*}$ to be purely imaginary, $\cos\left(\frac{5n+2}{6}\right)\pi = 0$

$$\left(\frac{5n+2}{6}\right)\pi = (2k+1)\frac{\pi}{2} \quad , \; k \in ¢$$
$$5n + 2 = 3(2k+1)$$
$$n = \frac{6k+1}{5}$$

$\therefore$ The three smallest positive whole number values of n are 5,11,17.

OR

$$\frac{z^n}{iz^*} = \frac{-iz^{n+1}}{|z|^2} = -2^{n-1} i e^{i\frac{5(n+1)\pi}{6}}$$

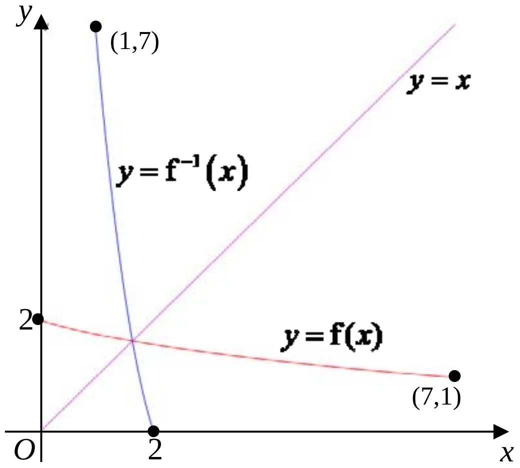
For $\frac{z^n}{iz^*}$ to be purely imaginary,

$$\sin\left(\frac{5n+5}{6}\right)\pi = 0 \quad , \; k \in ¢$$
$$\left(\frac{5n+5}{6}\right)\pi = k\pi \quad , \; k \in ¢$$
$$5n = 6k - 5$$
$$n = \frac{6k}{5} - 1$$

$\therefore$ The three smallest positive whole number values of n are 5,11,17.

(iii)	$\left\|1+\frac{p}{z^*}\right\|=\sqrt{7} \Leftrightarrow \left\|\frac{z^*+p}{z^*}\right\|=\sqrt{7}$ $\frac{\left\|-\sqrt{3}-\mathrm{i}+p\right\|}{\left\|z^*\right\|}=\sqrt{7}$ $\left\|\left(-\sqrt{3}+p\right)-\mathrm{i}\right\|=2\sqrt{7}$ since $\left\|z^*\right\|=\left\|z\right\|=2$ $\left(-\sqrt{3}+p\right)^2+\left(-1\right)^2=\left(2\sqrt{7}\right)^2$ $\left(-\sqrt{3}+p\right)^2=27$ $-\sqrt{3}+p=\pm\sqrt{27}$ $p=\sqrt{3}\pm 3\sqrt{3}$ $\therefore p=4\sqrt{3}$ or $-2\sqrt{3}$

8(i) **[2]**	$a-\left(1+x\right)^{\frac{1}{3}}\ge a-1$ $\left(1+x\right)^{\frac{1}{3}}\le 1$ $x\le 0$ But $0\le x\le 7$, Solution is $x=0$
(ii) **[2]**	$\mathrm{f}'\left(x\right)=-\frac{1}{3}\left(1+x\right)^{-\frac{2}{3}}<0$ Thus, the graph of $y=\mathrm{f}\left(x\right)$ has a negative gradient at all points on the graph Since f is continuous and decreasing over $0\le x\le 7$, $\mathrm{f}\left(7\right)\le \mathrm{f}\left(x\right)\le \mathrm{f}\left(0\right)$ $a-2\le \mathrm{f}\left(x\right)\le a-1$ $\therefore R_{\mathrm{f}}=\left[a-2,a-1\right]$
(iii) **[2]**	$y=3-\left(1+x\right)^{\frac{1}{3}}$ $1+x=\left(3-y\right)^3$ $x=\left(3-y\right)^3-1$ $\mathrm{f}^{-1}\left(x\right)=\left(3-x\right)^3-1$

(iv) **[3]**	
(v) **[3]**	From the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, we see that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ exactly where $\mathrm{f}^{-1}(x) = x$. $(3-x)^3 - 1 = x$ $27 - 27x + 9x^2 - x^3 - 1 = x$ $x^3 - 9x^2 + 28x - 26 = 0$ (shown) From GC, $x = 1.62$ (3sf)

9(a) **[5]**	$u = \sqrt{x} \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{2\sqrt{x}} \Rightarrow \mathrm{d}x = 2u\,\mathrm{d}u$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x}e^{\sqrt{x}}$ $y = \int \sqrt{x}e^{\sqrt{x}}\,\mathrm{d}x$ $= \int ue^u \cdot 2u\,\mathrm{d}u$ $= 2\int u^2e^u\,\mathrm{d}u$ $= 2\left(u^2e^u - 2\int ue^u\,\mathrm{d}u\right)$ $= 2u^2e^u - 4\left(ue^u - \int e^u\,\mathrm{d}u\right)$ $= 2u^2e^u - 4ue^u + 4e^u + c$ $= 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + c$
9(b) **[8]**	$\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = k(180-\theta)$ where k is positive $\int \dfrac{1}{180-\theta}\,\mathrm{d}\theta = \int k\,\mathrm{d}t$ $-\ln(180-\theta) = kt + c,\quad$ where $\theta < 180$ When $t = 0, \theta = 4$, $c = -\ln 176$ $\therefore -\ln(180-\theta) = kt - \ln 176$ Let t_0 be the time elapsed from $4°\mathrm{C}$ to $80°\mathrm{C}$.

When $t = t_0, \theta = 80$,

$$-\ln 100 = kt_0 - \ln 176 \Rightarrow kt_0 = \ln\frac{44}{25} \qquad \text{K (1)}$$

When $t = t_0 + 20, \theta = 130$,

$$-\ln 50 = k(t_0 + 20) - \ln 176 \Rightarrow k(t_0 + 20) = \ln\frac{88}{25} \qquad \text{K (2)}$$

By (1) and (2), $\dfrac{t_0}{t_0 + 20} = \dfrac{\ln\frac{44}{25}}{\ln\frac{88}{25}}$

$t_0 = 16.312$ (5s.f)

Remaining number of minutes in oven after 10.06am

$= 90 - 16.312$

$= 73.688$ mins

Thus the time that Ms Frugal should take out the dough is 11.20am.

Alternatively

$\dfrac{d\theta}{dt} = k(180 - \theta)$ where k is positive

$\displaystyle\int \frac{1}{180 - \theta}\, d\theta = \int k\, dt$

$-\ln(180 - \theta) = kt + c, \quad$ where $\theta < 180$

When $t = 0, \theta = 80$,

$c = -\ln 100$

$\therefore -\ln(180 - \theta) = kt - \ln 100$

When $t = 20, \theta = 130$,

$-\ln 50 = k(20) - \ln 100$

$k = \dfrac{\ln 2}{20}$

$\therefore -\ln(180 - \theta) = \left(\dfrac{\ln 2}{20}\right)t - \ln 100$

When $\theta = 4$,

$-\ln 176 = \left(\dfrac{\ln 2}{20}\right)t - \ln 100$

$t = -16.312$

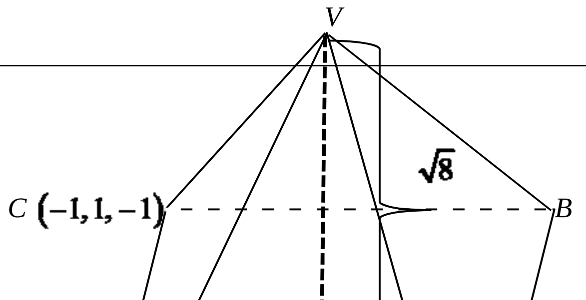
Remaining number of minutes in oven after 10.06am

$= 90 - 16.312$

$= 73.688$ mins

Thus the time that Ms Frugal should take out the dough is 11.20am.

10(i)
[1]



Since V is equidistant to the points O, A, B and C, and point D is the foot of perpendicular from V to the rectangular base $OABC$, by symmetry D is the centre of the rectangular $OABC$. In other words, D is the midpoint of AC.

$$\overrightarrow{OD} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right) = \frac{1}{2}\left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 1.5 \\ 0 \end{pmatrix}$$

$\therefore$ Co-ordinates of D is $(0, 1.5, 0)$. (shown)

(ii)
[4]

A vector perpendicular to $OABC$ $= \overrightarrow{OA} \times \overrightarrow{OC} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 0 \\ 3 \end{pmatrix}$

Since the **k**-component of V is positive, $\overrightarrow{DV} = \frac{\sqrt{8}}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$

$$\overrightarrow{DV} = \overrightarrow{OV} - \overrightarrow{OD}$$

$$\overrightarrow{OV} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1.5 \\ 0 \end{pmatrix}$$

$$\overrightarrow{OV} = \begin{pmatrix} -2 \\ 1.5 \\ 2 \end{pmatrix}$$

(iii)
[2]

Method 1:

$$\overrightarrow{VE} = \overrightarrow{OE} - \overrightarrow{OV} = \begin{pmatrix} \alpha \\ 1.5 \\ -\alpha \end{pmatrix} - \begin{pmatrix} -2 \\ 1.5 \\ 2 \end{pmatrix} = \begin{pmatrix} \alpha+2 \\ 0 \\ -\alpha-2 \end{pmatrix} = (\alpha+2)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Since $(\alpha+2)$ is a constant, $\overrightarrow{VE}$ is parallel to $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, a vector normal to the plane $OABC$.

Thus VE is perpendicular to the rectangular base $OABC$ (shown).

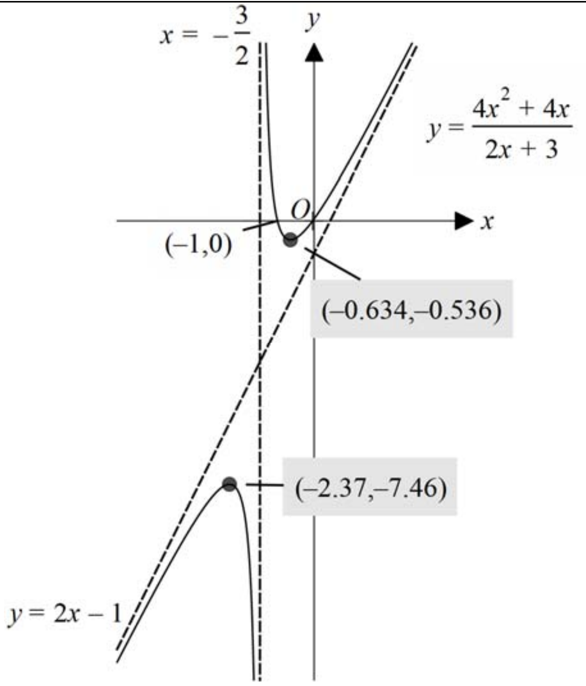
	Method 2: $\overrightarrow{VE}\,.\,\overrightarrow{OA}=(\alpha+2)\begin{pmatrix}1\\0\\-1\end{pmatrix}.\begin{pmatrix}1\\2\\1\end{pmatrix}=(\alpha+2)(1-1)=0$ $\overrightarrow{VE}\,.\,\overrightarrow{OC}=(\alpha+2)\begin{pmatrix}1\\0\\-1\end{pmatrix}.\begin{pmatrix}-1\\1\\-1\end{pmatrix}=(\alpha+2)(-1+1)=0$ Since $\overrightarrow{VE}$ is perpendicular to $\overrightarrow{OA}$ and $\overrightarrow{OC}$, which are two non-parallel vectors parallel to plane *OABC*, *VE* is perpendicular to the rectangular base *OABC* (shown). Method 1: $\overrightarrow{VE}=(\alpha+2)\begin{pmatrix}1\\0\\-1\end{pmatrix}=\left(\frac{\alpha+2}{2}\right)\overrightarrow{VD}$ Since *E* is between *V* & *D*, $0<\frac{\alpha+2}{2}<1 \Rightarrow -2<\alpha<0$. Method 2: Equation of line VD: $\mathbf{r}=\begin{pmatrix}0\\1.5\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\-1\end{pmatrix},\ \lambda\in\mathbb{R}$ We have $\overrightarrow{OV}$ and $\overrightarrow{OD}$ with $\lambda=-2$ and $\lambda=0$ respectively. $\overrightarrow{OE}=\begin{pmatrix}\alpha\\1.5\\-\alpha\end{pmatrix}=\begin{pmatrix}0\\1.5\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\-1\end{pmatrix}$ satisfies the equation of line VD with $\lambda=\alpha$. Since *E* is between *V* and *D*, $-2<\alpha<0$.
(iv) [4]	Normal of plane $OVA=\begin{pmatrix}1\\2\\1\end{pmatrix}\times\begin{pmatrix}-2\\3/2\\2\end{pmatrix}=\begin{pmatrix}2.5\\-4\\5.5\end{pmatrix}=\frac{1}{2}\begin{pmatrix}5\\-8\\11\end{pmatrix}$ Distance of E to plane $OVA=\dfrac{\left\lvert\overrightarrow{OE}.\begin{pmatrix}5\\-8\\11\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}5\\-8\\11\end{pmatrix}\right\rvert}=\dfrac{\left\lvert\begin{pmatrix}\alpha\\1.5\\-\alpha\end{pmatrix}.\begin{pmatrix}5\\-8\\11\end{pmatrix}\right\rvert}{\sqrt{210}}$

	$= \dfrac{\lvert -6\alpha - 12 \rvert}{\sqrt{210}}$ $= \dfrac{6\alpha + 12}{\sqrt{210}}$ since $-2 < \alpha < 0$
(v) **[2]**	Distance between E and $OABC$ $= \left\lvert \overrightarrow{ED} \right\rvert = \left\lvert \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rvert = -\alpha\sqrt{2}$ (since $\alpha < 0$) Given $\sqrt{105}\left(-\alpha\sqrt{2}\right) = 10\left(\dfrac{6\alpha + 12}{\sqrt{210}}\right) \Rightarrow -210\alpha = 60\alpha + 120 \Rightarrow \alpha = -\dfrac{4}{9}$

<u>2018 RI Prelim Exam Paper 2</u>

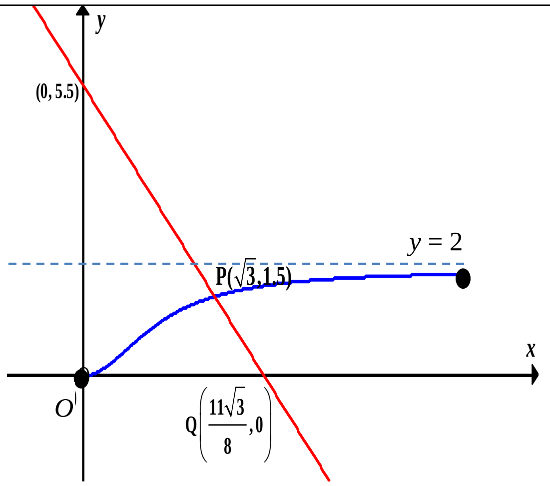
Pure Mathematics (40 marks)

1 **[3]**	Let V and A be the volume and surface area of the spherical mothball respectively. $\dfrac{\mathrm{d}V}{\mathrm{d}t} = kA,\ k<0 \quad \text{L L L L L} \quad (1)$ $V = \dfrac{4}{3}\pi r^3$ $\dfrac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \dfrac{\mathrm{d}r}{\mathrm{d}t} = A\dfrac{\mathrm{d}r}{\mathrm{d}t} \quad \text{L L L} \quad (2)$ $(1)=(2) \Rightarrow \dfrac{\mathrm{d}r}{\mathrm{d}t} = k < 0$ $\therefore$ The radius decreases at a constant rate (shown).

2(i) **[3]**	$y = \dfrac{(ax+1)(4x+b)}{2x+3} = (2x-1) + \dfrac{c}{2x+3}$ $\therefore \dfrac{(ax+1)(4x+b)}{2x+3} = \dfrac{(2x-1)(2x+3)+c}{2x+3}$ $(ax+1)(4x+b) = (2x-1)(2x+3)+c$ By comparing coefficients of $x^2 : a=1,$ $x : ab+4 = 6-2 \Rightarrow ab=0 \Rightarrow b=0$
(ii) **[4]**	

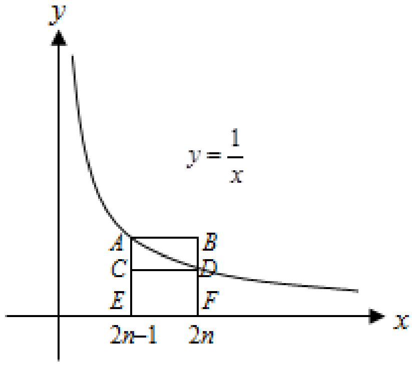
3(i) **[2]**	$(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})=(\mathbf{u}\times\mathbf{v})-2(\mathbf{u}\times\mathbf{u})+(\mathbf{v}\times\mathbf{v})-2(\mathbf{v}\times\mathbf{u})$ $=(\mathbf{u}\times\mathbf{v})+2(\mathbf{u}\times\mathbf{v})$ $(\because \mathbf{u}\times\mathbf{u}=\mathbf{0},\ \mathbf{v}\times\mathbf{v}=\mathbf{0},\ \mathbf{v}\times\mathbf{u}=-(\mathbf{u}\times\mathbf{v}))$ $=3(\mathbf{u}\times\mathbf{v})$ $\therefore k=3$
(ii) **[2]**	$\mathbf{u}\times\mathbf{v}=\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}2\\3\\0\end{pmatrix}=\begin{pmatrix}-6\\4\\5\end{pmatrix}$ A unit vector parallel to $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-2\mathbf{u})$ is $=\pm\frac{1}{\sqrt{77}}\begin{pmatrix}-6\\4\\5\end{pmatrix}$.
(iii) **[3]**	Since $\mathbf{v}$ is perpendicular to the vector $\mathbf{i}-2\mathbf{j}+3\mathbf{k}$, $\begin{pmatrix}a\\b\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\3\end{pmatrix}=0$ $a-2b=0$ $a=2b$ $\mathbf{v}=\begin{pmatrix}2b\\b\\0\end{pmatrix}=b\begin{pmatrix}2\\1\\0\end{pmatrix}$ The solution represents the position vector of any point on the line which passes through the origin and is parallel to the vector $2\mathbf{i}+\mathbf{j}$.

4(i) **[3]**	$x=\cot\theta\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}\theta}=-\mathrm{cosec}^2\theta$ $y=2\cos^2\theta\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}\theta}=-4\sin\theta\cos\theta$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-4\sin\theta\cos\theta}{-\mathrm{cosec}^2\theta}=4\sin^3\theta\cos\theta$ When $\theta=\frac{\pi}{6}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=4\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{3}}{4}$ (shown) At P, $x=\sqrt{3}$ $y=\frac{3}{2}$. Equation of L, the normal at P, is $y-\frac{3}{2}=-\frac{4}{\sqrt{3}}\left(x-\sqrt{3}\right)$ or $y=-\frac{4}{\sqrt{3}}x+\frac{11}{2}$

(ii) [2]	
(iii) [3]	$\text{Area} = \int_0^{\frac{3}{2}} x\,\mathrm{d}y$ $y=0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \cot\theta \frac{\mathrm{d}y}{\mathrm{d}\theta}\mathrm{d}\theta$ $y = \frac{3}{2} \Rightarrow \cos\theta = \frac{\sqrt{3}}{2}\ (\because 0<\theta\le\frac{\pi}{2}) \Rightarrow \theta = \frac{\pi}{6}$ $= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{\cos\theta}{\sin\theta}(4\cos\theta(-\sin\theta))\mathrm{d}\theta$ $= -4\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \cos^2\theta\mathrm{d}\theta$ $= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\mathrm{d}\theta$ $= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ Alternatively, use Cartesian form of the equation (found in **(iv)**) $\text{Area} = (\sqrt{3})(1.5) - \int_0^{\sqrt{3}} \frac{2x^2}{1+x^2}\,\mathrm{d}x$ $= \frac{3\sqrt{3}}{2} - \int_0^{\sqrt{3}} \frac{2(1+x^2-1)}{1+x^2}\,\mathrm{d}x$ $= \frac{3\sqrt{3}}{2} - \int_0^{\sqrt{3}} 2 - \frac{2}{1+x^2}\,\mathrm{d}x$ $= \frac{3\sqrt{3}}{2} - \left[2x - 2\tan^{-1}x\right]_0^{\sqrt{3}}$ $= \frac{3\sqrt{3}}{2} - \left[2\sqrt{3} - \frac{2\pi}{3}\right]$ $= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$
(iv) [2]	$x = \cot\theta \Rightarrow x^2 = \cot^2\theta$ $y = 2\cos^2\theta \Rightarrow y = 2 - 2\sin^2\theta \Rightarrow \operatorname{cosec}^2\theta = \frac{2}{2-y}$

	$\text{cosec}^2\theta = 1+\cot^2\theta$ $\frac{2}{2-y} = 1+x^2$ $2 = 2+2x^2-y-x^2y$ $y = \frac{2x^2}{1+x^2},\quad x>0 \qquad (\because 0<\theta\le\frac{\pi}{2}\Rightarrow x>0)$
(v) **[2]**	Volume of solid $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \pi y^2\,\mathrm{d}x + \frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3\sqrt{3}}{8}\right)$ $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \pi\left(\frac{2x^2}{1+x^2}\right)^2\,\mathrm{d}x + \frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3\sqrt{3}}{8}\right)$ $= 6.17\quad \text{units}^3$ Alternatively Volume of solid $= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \pi y^2\,\mathrm{d}x + \frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3\sqrt{3}}{8}\right)$ $= \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \pi(2\cos^2\theta)^2\left(-\text{cosec}^2\theta\,\mathrm{d}\theta\right) + \frac{1}{3}\pi\left(\frac{3}{2}\right)^2\left(\frac{3\sqrt{3}}{8}\right)$ $= 6.17\quad \text{units}^3$

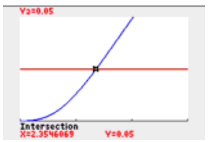
5(i) **[3]**	$\sum_{r=1}^{2n} u_r = \mathrm{f}(1)$ $+0$ $+\mathrm{f}(3)$ $+\mathrm{f}(3)-2\mathrm{f}(1)$ $+\mathrm{f}(5)$ $+\mathrm{f}(5)-2\mathrm{f}(3)$ $+\mathrm{f}(7)$ $+\mathrm{f}(7)-2\mathrm{f}(5)$ $+\ldots$ $+\mathrm{f}(2n-3)$ $+\mathrm{f}(2n-3)-2\mathrm{f}(2n-5)$ $+\mathrm{f}(2n-1)$ $+\mathrm{f}(2n-1)-2\mathrm{f}(2n-3)$ $= -\mathrm{f}(1)+2\mathrm{f}(2n-1)$
(ii) **[2]**	$\sum_{r=1}^{2n} u_r = -\mathrm{f}(1)+2\mathrm{f}(2n-1)$ $= -\ln\left(\frac{2}{1}\right)+2\ln\left(\frac{(2n-1)+1}{2n-1}\right)$ $= -\ln 2+2\ln\left(1+\frac{1}{2n-1}\right)$ (shown)

	As $n \to \infty$, $\frac{1}{2n-1} \to 0$, $1+\frac{1}{2n-1} \to 1$, $\ln\left(1+\frac{1}{2n-1}\right) \to 0$, $\sum_{r=1}^{2n} u_r \to -\ln 2$. Sum to infinity is $-\ln 2$.
(iii) **[2]**	$\left\lvert \sum_{r=1}^{2n} u_r - (-\ln 2) \right\rvert < 10^{-2}$ $2\ln\left(1+\dfrac{1}{2n-1}\right) < 0.01$ $\left(\because 1+\frac{1}{2n-1} > 1 \Rightarrow \ln\left(1+\frac{1}{2n-1}\right) > 0\right)$ From GC, $n > 100.25$ Or <table><tr><th>n</th><th>$2\ln\left(1+\frac{1}{2n-1}\right)$</th></tr><tr><td>100</td><td>0.010025 > 0.01</td></tr><tr><td>101</td><td>0.0099256 < 0.01</td></tr></table>$\therefore$ Smallest value of n = 101
(iv) **[3]**	From the graph of $y = \dfrac{1}{x}$ for $x > 0$,  Area of rectangle CDFE $< \int_{2n-1}^{2n} \dfrac{1}{x}\,dx <$ Area of rectangle ABFE $$(1)\left(\frac{1}{2n}\right) < [\ln x]_{2n-1}^{2n} < (1)\left(\frac{1}{2n-1}\right)$$ $$\frac{1}{2n} < \ln\frac{2n}{2n-1} < \frac{1}{2n-1}$$ $\ln\dfrac{2n}{2n-1} = \ln\dfrac{(2n-1)+1}{2n-1} = u_{2n-1}$ since $2n-1$ is odd, $\therefore \dfrac{1}{2n} < u_{2n-1} < \dfrac{1}{2n-1}$ (shown).

Statistics (60 marks)

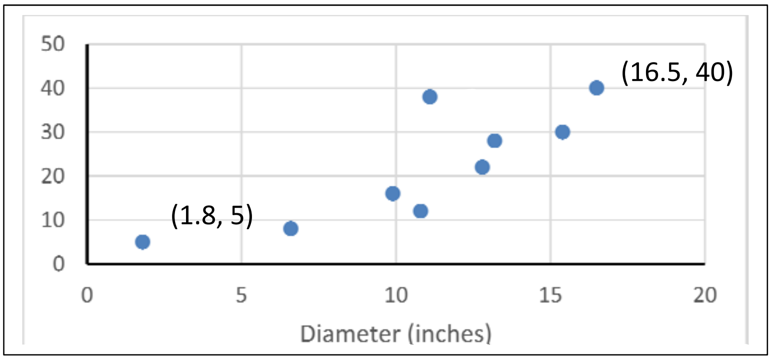
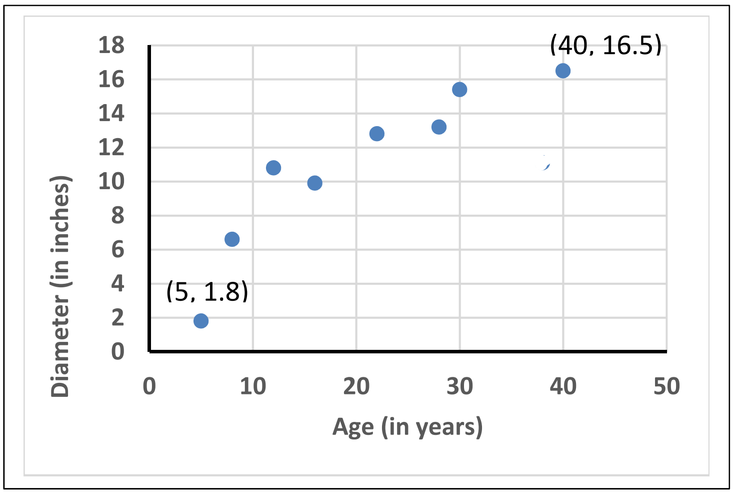
6(a) **[2]**	$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ $= 0.5 + 0.65 - \mathrm{P}(A \cap B)$ $= 1.15 - \mathrm{P}(A \cap B)$ Since $0 \le \mathrm{P}(A \cup B) \le 1$, $0.15 \le \mathrm{P}(A \cap B) \le \min\{\mathrm{P}(A), \mathrm{P}(B)\} \Rightarrow 0.15 \le \mathrm{P}(A \cap B) \le 0.5$ If A and B are independent, $\mathrm{P}(A \cap B) = (0.5)(0.65) = 0.325 \Rightarrow \mathrm{P}(A \cup B) = 0.825$ $\therefore 0.65 \le \mathrm{P}(A \cup B) \le 1$ and $\mathrm{P}(A \cup B) \ne 0.825$. Range of $\mathrm{P}(A \cup B)$ is $[0.65, 1] \setminus \{0.825\}$.

(b)(i) **[3]**	$\mathrm{P}(A' \mid B') = 1 - P(A \mid B')$ $= 1 - 0.6$ $= 0.4$ $\mathrm{P}(A' \cap B') = \mathrm{P}(B')\,\mathrm{P}(A' \mid B')$ $= (0.35)(0.4)$ $= 0.14$ $\mathrm{P}(A \cup B) = 1 - P(A' \cap B')$ $= 1 - 0.14$ $= 0.86$ Alternatively, $\mathrm{P}(A \cap B') = \mathrm{P}(B')\ \mathrm{P}(A \mid B')$ $= (0.35)(0.6)$ $= 0.21$ $\mathrm{P}(A \cup B) = \mathrm{P}(B) + \mathrm{P}(A \cap B')$ $= (1 - 0.35) + 0.21$ $= 0.86$
(b)(ii) **[1]**	$P(A' \cup B') = \mathrm{P}(A') + \mathrm{P}(B') - \mathrm{P}(A' \cap B')$ $= 0.5 + 0.35 - 0.14$ $= 0.71$ Alternatively, $P(A' \cup B') = 1 - \mathrm{P}(A \cap B) = 1 - [\mathrm{P}(A) - \mathrm{P}(A \cap B')]$ $= 1 - [0.5 - 0.21]$ $= 0.71$

7(i) **[2]**	By Central Limit Theorem, since $n = 50$ is large, $\bar{X} \sim \mathrm{N}\left(33.8, \frac{1.5^2}{50}\right)$ approximately. $\mathrm{P}(\lvert \bar{X} - 34 \rvert < 0.5) = \mathrm{P}(33.5 < \bar{X} < 34.5) = 0.921\,(3\text{s.f.})$
(ii) **[3]**	$H_o : \mu = 33.8$ $\qquad$ $H_1 : \mu > 33.8$ Under H_0, since $n = 60$ is large, $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{k^2}{n}\right)$ approximately by Central Limit Theorem, with $\mu_0 = 33.8$, $n = 60$. At 5% level of significance, to accept test, we reject H_0, p-value ≤ 0.05 $\mathrm{P}(\bar{X} \geq 34.3) \leq 0.05$ $\mathrm{P}\left(Z \geq \frac{34.3 - 33.8}{k/\sqrt{60}}\right) \leq 0.05$ $\frac{34.3 - 33.8}{k/\sqrt{60}} \geq 1.6448536$ $k \leq 2.35$ (3s.f.)  From GC, $k \leq 2.35$ (3s.f.) Hence, the set of values of k is $\{k \in \mathbb{R} : 0 < k \leq 2.35\}$.
(iii) **[3]**	$H_o : \mu = 33.8$ $\qquad$ $H_1 : \mu \neq 33.8$ Perform a 2-tail test at 5% level of significance.

	Under H_0, $\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$ with $\mu_0 = 33.8$, $\sigma = 0.8$, $n = 30$. At 5% level of significance, for the test to be not accepted, do not reject H_0. From GC p-value > 0.05 $\Rightarrow$ $P(\bar{X} \geq \bar{x}) > 0.025$ and $P(\bar{X} \leq \bar{x}) > 0.025$ $\Rightarrow$ $\bar{x} < 34.1$ (3.s.f.) and $\bar{x} > 33.5$ (3.s.f.) The set of values of $\bar{x}$ is $\{\bar{x} \in ¡ : 33.5 < \bar{x} < 34.1\}$. Alternatively, we can use symmetry about $\mu = 33.8$ to get the other value.

8(i) **[2]**	Let X be the mass of an orange in grams. $X \sim N(100, 5^2)$ $X_1 + X_2 \sim N(2\times100, 2\times5^2)$ $X_1 + X_2 \sim N(200, 50)$ $P(X_1 + X_2 < 205) = 0.760$ (3 s.f.)
(ii) **[3]**	Let Y be the mass of an apple in grams. $Y \sim N(80, 3^2)$ and $X \sim N(100, 5^2)$ $Y_1 + Y_2 + Y_3 + Y_4 - 3X \sim N(4\times80 - 3\times100, 4\times3^2 + 3^2\times5^2)$ $Y_1 + Y_2 + Y_3 + Y_4 - 3X \sim N(20, 261)$ $P(Y_1 + Y_2 + Y_3 + Y_4 > 3X) = P(Y_1 + Y_2 + Y_3 + Y_4 - 3X > 0)$ $= 0.892$
(iii) **[5]**	Let B be the mass of an empty gift box. $B \sim N(50, 2^2)$ Let M be the total mass of a gift box. $M = 1.07(Y_1 + Y_2 + Y_3 + X_1 + X_2) + 0.9B$ $M \sim N(3\times1.07\times80 + 2\times1.07\times100 + 0.9\times50,$ $3\times1.07^2\times3^2 + 2\times1.07^2\times5^2 + 0.9^2\times2^2)$ $M \sim N(515.8, 91.3973)$ $P(M < k) = 0.8$ Using GC, $\therefore k = 523.8$(1d.p.) $= 524$ (3 s.f.) Required probability = (Probability that of the first 24 gift boxes given, 18 of them has total mass less than k grams)$\times$ (Probability that the 25th gift box has total mass less than k grams)= $\left[\binom{24}{18}(0.8)^{18}(0.2)^6\right](0.8) = 0.124$(3s.f.)

9(i) **[2]**	 	
(ii) **[2]**	From GC, the product moment correlation coefficient between D and t is $r =$ 0.9146 (4 d.p.) Since r is close to 1 and the revised scatter diagram indicates a positive linear correlation between D and t, a linear model is appropriate. Alternatively Although r is close to 1, scatter diagram suggests that as t increases, D is increasing at a slower rate, so a linear model would not be appropriate.	
(iii) **[2]**	From GC, the product moment correlation coefficient between D^2 and t is r = 0.9689 (4 d.p.) From GC, the product moment correlation coefficient between D and ln$t$ is r = 0.9747 (4 d.p.) The $t = ae^{bD}$ provides a better model as its product moment correlation coefficient is closer to 1.	
(v) **[4]**	From GC, the regression line of Diameter, D, on ln(age), ln$t$, is $D = -7.642653677 + 6.595868978 \ln t$. When t = 50, $t = 50 \Rightarrow \ln t = 3.9120$ $D = 18.161\ (5sf)$ $= 18.2$ inches (1 d.p.) The estimated diameter of a 50 year-old tree is 18.2 inches (1 d.p.) As the value of t = 50 is outside the data range, the linear model may no longer be suitable. Hence the estimate is not reliable.	

	To estimate D given t, we would need t, the age of the tree, to be the independent variable.	

10(a) **[1]**	Number of ways = $\frac{10!}{4!6!}$ or ${}^{10}C_4 = 210$
(b) **[3]**	In order for exactly 5 hearts to be together, the remaining single heart must be separated from the group of 5 hearts. No. of ways to choose and arrange the 5 hearts in the box $=\binom{6}{5}\times 5!$ No. of ways to arrange the 4 stars $= 4!$ No. of ways to slot the group of 5 hearts and the single heart into separate slots $=\binom{5}{2}\times 2!$ Required number of ways = $4!\times\binom{6}{5}\times 5!\times\binom{5}{2}\times 2! = 345600$ Or $4!\times {}^6P_5 \times {}^5P_2 = 345600$ Alternative Solutions : Arrange group of 5 hearts with 4 stars, then slot in last heart : $\binom{6}{5}\times 5!\times 5!\times\binom{4}{1}$ Total (for group of 5 hearts + 4 stars + last heart) – 6 hearts together : $\left(\binom{6}{5}5!\right)6! - 6!5!2!$
(c) **[2]**	2 Red, 2 Green, 2 Yellow, 2 Purple, 1 Black and 1 White Number of ways all the ornaments of the same colours are next to each other, arranged in a circular manner $=(6-1)!\times 2^4$ = 1920
(d)(i) **[3]**	Number of ways to select the three groups = $\frac{\binom{10}{4}\binom{6}{3}\binom{3}{3}}{2!} = 2100$ Number of ways to select when black and white are in the same group = Number of ways when black and white are in group of 4 + number of ways when black and white are in a group of 3 $=\frac{\binom{8}{2}\binom{6}{3}\binom{3}{3}}{2!}+\binom{8}{4}\binom{4}{1}\binom{3}{3}=560$ Required probability = $\frac{560}{2100}=\frac{4}{15}$ Alternative solutions : $\frac{4}{10}\times\frac{3}{9}+2\left(\frac{3}{10}\times\frac{2}{9}\right)$ or $\frac{\binom{4}{2}+2\binom{3}{2}}{\binom{10}{2}}$ or $\frac{\binom{8}{2}}{\binom{10}{4}}+\frac{2\binom{8}{1}}{\binom{10}{3}}$

(d)(ii)
[3]

Let A be the event that at least 2 heart shaped ornaments are in the group of 4.
Let B be the event that the black and white ornaments are in the same group.

$$P(A\,|\,B) = 1 - P(A'\,|\,B) = \frac{n(B) - n(A' \cap B)}{n(B)} \text{ or } 1 - \frac{P(A' \cap B)}{P(B)}$$

The cases for $A' \cap B$ are as follows:

Group of 4	Group of 3	Group of 3	Number of Ways
3 stars with a Red or Green or Yellow or Purple heart	Black and white and 1 other	The remaining	$\binom{4}{3}\binom{4}{1}\binom{4}{1}\binom{3}{3}$ $= 64$
4 stars	Black and white and 1 other	The remaining	$\binom{4}{4}\binom{4}{1}\binom{3}{3} = 4$
			Total = 68

$$P(A\,|\,B) = 1 - \frac{\frac{68}{2100}}{\frac{4}{15}} \quad \text{or} \quad \frac{560-68}{560} \quad \left(\frac{492}{560}\right)$$

$$= \frac{123}{140} \text{ or } 0.879 \text{ (3s.f, 0.87857(5s.f.))}$$

Alternatively, have to consider 4 cases.

Group of 4	Group of 3	Group of 3	Number of Ways
Black and white and 2 others	Any remaining 3	The remaining	$\binom{8}{2}\binom{6}{3}\binom{3}{3}/2$ $= 280$
2 hearts (not black and white) + 2 stars			$\binom{4}{2}\binom{4}{2}\binom{4}{1}\binom{3}{3} = 144$
3 hearts (not black and white) + 1 stars	Black and white and 1 other	The remaining	$\binom{4}{3}\binom{4}{1}\binom{4}{1}\binom{3}{3} = 64$
4 hearts (not black and white)			$\binom{4}{4}\binom{4}{1}\binom{3}{3} = 4$
			Total = 492

$$P(A\,|\,B) = \frac{492}{560}$$

$$= \frac{123}{140} \text{ or } 0.879 \text{ (3s.f, 0.87857(5s.f.))}$$

<table>
<tr><td>11(i)
[2]</td><td colspan="2">The probability that Jon takes the ingredients from each bowl is constant at 0.4. However, the probability may not be constant as Jon may prefer certain ingredients and be more likely to take the ingredients from some bowls than others.
OR
Jon takes the ingredients from each bowl independently of other bowls. However, Jon may not take the ingredients independently as he may consider that some ingredients go well together while others do not.</td></tr>
<tr><td>(ii)
[2]</td><td>$X \sim \mathrm{B}(n, 0.4)$
$P = {}^{n}C_7 (0.4)^7 (0.6)^{n-7}$
Find n such that P has the greatest value.
<table><tr><th>n</th><th>P</th></tr><tr><td>16</td><td>0.18889</td></tr><tr><td>17</td><td>0.19267</td></tr><tr><td>18</td><td>0.18916</td></tr></table>From GC, $n=17$.

P has greatest value when n = 17</td><td>Alternatively, $P(n) \ge P(n-1)$
${}^{n}C_7 (0.4)^7 (0.6)^{n-7} \ge {}^{n-1}C_7 (0.4)^7 (0.6)^{n-1-7}$
$\frac{n!}{7!(n-7)!} 0.6 \ge \frac{(n-1)!}{7!(n-1-7)!}$
$\left(\frac{n}{n-7}\right) 0.6 \ge 1$
$0.6n \ge n-7$
$n \le \frac{35}{2} = 17.5$
P has greatest value when n = 17</td></tr>
<tr><td>(iii)
[2]</td><td colspan="2">When $n=6$, $X \sim \mathrm{B}(6, 0.4)$.
P(at least 3 different ingredients each on Mon and Thur)
$= [\mathrm{P}(X \ge 3)]^2 = [1 - \mathrm{P}(X \le 2)]^2 = 0.208$ (3s.f.)</td></tr>
<tr><td>(iv)
[1]</td><td colspan="2">When $n=12$, $X \sim \mathrm{B}(12, 0.4)$.
P(at least 6 different ingredients on Sat) $= \mathrm{P}(X \ge 6) = 1 - \mathrm{P}(X \le 5) = 0.335$ (3s.f.)</td></tr>
<tr><td>(v)
[1]</td><td colspan="2">Part (iv) includes the cases in part (iii) as well as other cases. For example, the case of Jon taking exactly 2 different ingredients from A to F and exactly 4 different ingredients from G to L is included in part (iv) but not in part (iii).</td></tr>
<tr><td>(vi)
[3]</td><td colspan="2">$\mathrm{P}(K > 4.5) = \mathrm{P}(K \ge 5) = 1 - \mathrm{P}(K \le 4) = 1 - \left(\frac{4}{9}\right)^2 = 1 - \frac{16}{81} = \frac{65}{81}$

Expected amount Kai pays $= \$\left[3 + 2\left(\frac{65}{81}\right)\right] = \$4.60$ (to nearest cent)

OR $\$\left[3\left(\frac{16}{81}\right) + 5\left(\frac{65}{81}\right)\right] = \$4.60$ (to nearest cent)</td></tr>
<tr><td>(vii)
[1]</td><td colspan="2">$\mathrm{P}(K = k) = \mathrm{P}(K \le k) - \mathrm{P}(K \le k-1), \quad k = 1, 2, \ldots, n$
$= \left(\frac{k}{n}\right)^2 - \left(\frac{k-1}{n}\right)^2 = \frac{k^2 - (k^2 - 2k + 1)}{n^2}$
$= \frac{2k-1}{n^2}$ (shown)</td></tr>
<tr><td>(viii)
[2]</td><td colspan="2">Expected number of different ingredients Kai takes
$= \sum_{k=0}^{n} k\ \mathrm{P}(K = k) = \frac{1}{n^2} \sum_{k=1}^{n} \left(2k^2 - k\right)$
$= \frac{1}{n^2}\left[\frac{2n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]$
$= \frac{(n+1)(4n-1)}{6n}$</td></tr>
</table>