EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2018

General Certificate of Education Advanced Level

Higher 2

MATHEMATICS **9758/01**

Paper 1 [100 marks] **12 September 2018**

3 hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

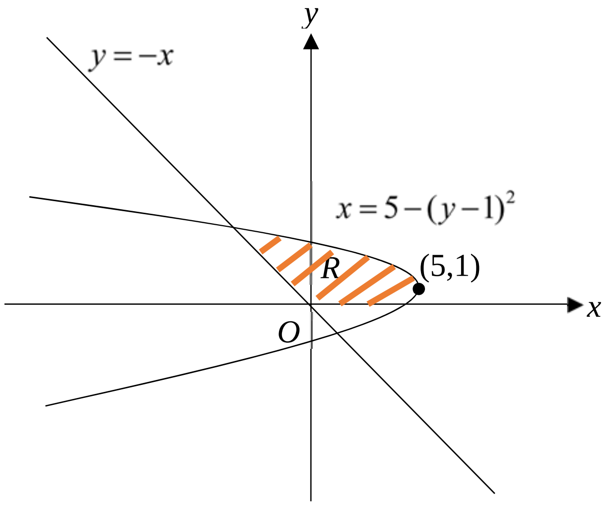
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

1



The shaded region R bounded by the curve $y = -x$, the line $y=-x$ and the x-axis is rotated about the x-axis through 360^{O}. Find the volume of the solid formed, leaving your answer to 2 decimal places. [4]

Suggested Solution
$y = -x$ $y = 1 \pm \sqrt{5-x}$ $y = 1+\sqrt{5-x}$ intersect $y=-x$ at $(-4,4)$ $y = 1-\sqrt{5-x}$ cuts the x-axis at $(4,0)$ Volume required $= \pi\int_{-4}^{5}\left(1+\sqrt{5-x}\right)^2 \mathrm{d}x - \pi\int_{4}^{5}\left(1-\sqrt{5-x}\right)^2 \mathrm{d}x - \frac{1}{3}\pi(4)^2(4)$ $= 201.06$ (to 2 d.p.)

2 **(i)** Solve the inequality $\dfrac{x^2 - ax - a}{x-a} \geq a$, where a is a positive real constant, leaving your answer in terms of a. [4]

(ii) Hence, by using a suitable value for a, solve the inequality

$$\frac{4\mathrm{e}^{2x} - \mathrm{e}^{x} - 1}{4\mathrm{e}^{x} - 1} \geq \frac{1}{4}$$

leaving your answer in exact form. [3]

Suggested Solution

(i) $\dfrac{x^2-ax-a}{x-a} \geq \quad a \quad (x \neq a)$

$$\frac{\left(x^2-ax-a\right)-a(x-a)}{x-a} \geq 0$$

$$\frac{x^2-2ax+a^2-a}{x-a} \geq 0$$

Consider $\quad x^2-2ax+\left(a^2-a\right)= \quad 0$

$$x \quad = \quad \frac{-(-2a)\pm\sqrt{(-2a)^2-4(1)\left(a^2-a\right)}}{2}$$

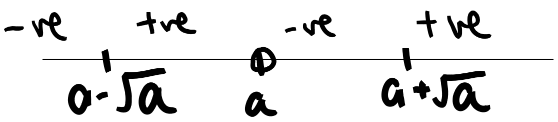
$$= \quad \frac{2a\pm\sqrt{4a}}{2} \quad = \quad a\pm\sqrt{a}$$

<u>Method 1 (test critical points)</u>

$\dfrac{x^2-2ax+a^2-a}{x-a} \geq 0$ can be rewritten as

$$\frac{\left(x-\left(a+\sqrt{a}\right)\right)\left(x-\left(a-\sqrt{a}\right)\right)}{x-a} \geq 0$$

Using sign test, we can check whether each factor is positive or negative for the different range of values of x

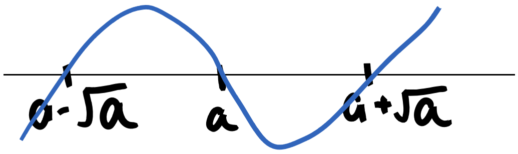


Thus, we have $\quad a-\sqrt{a} \leq x < a \quad$ or $\quad x \geq a+\sqrt{a}$

<u>Method 2</u>

$$\frac{\left(x^2-ax-a\right)-a(x-a)}{x-a} \quad \geq \quad 0$$

$$(x-a)\left[x^2-2ax+\left(a^2-a\right)\right] \quad \geq \quad 0 \quad (x \neq a)$$



$a-\sqrt{a} \le x < a \quad$ or $\quad x \ge a+\sqrt{a}$

(ii)

Given $\dfrac{4e^{2x}-e^{x}-1}{4e^{x}-1} \ge \dfrac{1}{4}$.

$$\frac{\left(e^{x}\right)^{2}-\frac{1}{4}\left(e^{x}\right)-\frac{1}{4}}{\left(e^{x}\right)-\frac{1}{4}} \ge \frac{1}{4}$$.

Replace x with e^{x} and let $a=\dfrac{1}{4}$,

$-\dfrac{1}{4} \le e^{x} < \dfrac{1}{4} \quad$ or $\quad e^{x} \ge \dfrac{3}{4}$

$0 < e^{x} < \dfrac{1}{4} \quad$ or $\quad e^{x} \ge \dfrac{3}{4}$

$x < \ln\left(\dfrac{1}{4}\right) \quad$ or $\quad x \ge \ln\left(\dfrac{3}{4}\right)$

3 The parametric equations of a curve C are $x=at$, $y=at^{3}$, where a is a positive constant.

(i) The point P on the curve has parameter p and the tangent to the curve at point P cuts the y-axis at S and the x-axis at T. The point M is the midpoint of ST. Find a Cartesian equation of the curve traced by M as p varies. [5]

(ii) Find the exact area bounded by the curve C, the line $x=0$, $x=3$ and the x-axis, giving your answer in terms of a. [3]

Suggested Solution

Solutions:

(i) $x=at, \quad y=at^{3}$

$\dfrac{dx}{dt}=a, \quad \dfrac{dy}{dt}=3at^{2}$

$$\frac{dy}{dx} = \frac{3at^2}{a}$$
$$= 3t^2$$

Therefore, gradient of the tangent at $P = 3p^2$

Equation of tangent at P:

$$y - ap^3 = 3p^2(x - ap)$$

$$y - ap^3 = 3p^2x - 3ap^3$$
$$y = 3p^2x - 2ap^3$$

For point S, $x = 0, y = -2ap^3$.

$\therefore S$ is $(0, -2ap^3)$

For point T, $y = 0, x = \frac{2ap^3}{3p^2} = \frac{2}{3}ap$

$\therefore T$ is $\left(\frac{2}{3}ap, 0\right)$

Midpoint M $= \left(\frac{1}{3}ap, -ap^3\right)$.

$$x = \frac{1}{3}ap, \quad y = -ap^3$$

$$\frac{3x}{a} = p,$$
$$y = -a\left(\frac{3x}{a}\right)^3 = -\frac{27x^3}{a^2}$$

(ii) Required area $= \int_0^3 y \, dx$

$$= \int_0^{\frac{3}{a}} at^3(a) \, dt$$

$$= \int_0^{\frac{3}{a}} at^3(a) \, dt$$

$$= a^2\left[\frac{t^4}{4}\right]_0^{\frac{3}{a}} = \frac{81}{4a^2}$$

Alternatively, find the cartesian of the given curve and use it to find the required area.

$x = at, \quad y = at^3$

$\therefore y = a\left(\frac{x}{a}\right)^3 = \frac{x^3}{a^2}$

Required area $= \int_0^3 \frac{x^3}{a^2}\,\mathrm{d}x = \frac{1}{a^2}\left[\frac{x^4}{4}\right]_0^3 = \frac{1}{a^2}\left[\frac{3^4}{4} - 0\right] = \frac{81}{4a^2}$

4 It is given that $y=\sin^{-1}x\cos^{-1}x$, where $-1\le x\le 1$.

(i) Show that $\sqrt{1-x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\cos^{-1}x-\sin^{-1}x$. [1]

(ii) Show that $\left(1-x^2\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\dfrac{\mathrm{d}y}{\mathrm{d}x}=-2$ [2]

(iii) Hence find the exact value of *A*, *B* and *C* if *y* can be expressed as $Ax+Bx^2+Cx^3$, up to (and including) the term in x^3. [4]

(iv) A student used **(iii)** to estimate that $\sin^{-1}(0.8)\cos^{-1}(0.8)\approx 0.8A+0.8^2B+0.8^3C$. Explain, with working, if his estimate is a good one. [1]

Suggested Solution

(i)

$$y=\sin^{-1}x\cos^{-1}x\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos^{-1}x}{\sqrt{1-x^2}}+\left(\frac{-\sin^{-1}x}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\cos^{-1}x-\sin^{-1}x\text{---(1) [shown]}$$

(ii)

Diff (1) wrt *x*,

$$\frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}(-2x)\frac{\mathrm{d}y}{\mathrm{d}x}+\sqrt{1-x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow(-x)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-1-1=-2\Rightarrow\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}=-2\text{ --(2)[shown]}$$

(iii) Diff (2) wrt *x*,

$$-2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(1-x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}-\left(\frac{\mathrm{d}y}{\mathrm{d}x}+x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=0$$

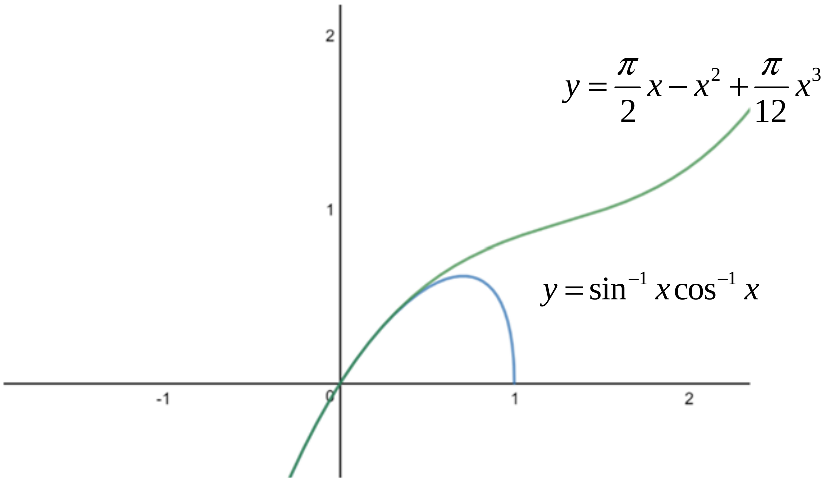
When $x=0, y=0, \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\pi}{2}, \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-2, \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=\dfrac{\pi}{2}$.

Therefore, $\sin^{-1}x\cos^{-1}x\approx\left(\dfrac{\pi}{2}\right)x+\dfrac{(-2)}{2!}x^2+\dfrac{\left(\frac{\pi}{2}\right)}{3!}x^3=\dfrac{\pi}{2}x-x^2+\dfrac{\pi}{12}x^3$.

(iv) The estimate is not good as $\sin^{-1}(0.8)\cos^{-1}(0.8)=0.597$ (to 3 sf)

But $\frac{\pi}{2}(0.8)-(0.8)^2+\frac{\pi}{12}(0.8)^3=0.751$ (to 3 sf)

Alternative Explanation



The graphs illustrated that at $x=0.8$, the two graphs are quite different from each other.

5 **(a)** Referred to the origin *O*, the points *A* and *B* have position vectors $\mathbf{a}$ and $\mathbf{b}$. Point *C* is on the line which contains *A* and is parallel to **b.** It is given that the vectors $\mathbf{a}$ and $\mathbf{b}$ are both of magnitude 2 units and are at an angle of $\sin^{-1}(1/6)$ to each other. If the area of triangle *OAC* is 3 units^2, use vector product to find the possible position vectors of *C* in terms of $\mathbf{a}$ and $\mathbf{b}$. [5]

(b) Referred to the origin *O*, the points *P* and *Q* have position vectors $\mathbf{p}$ and $\mathbf{q}$ where $\mathbf{p}$ and $\mathbf{q}$ are non-parallel, non-zero vectors. Point *R* is on *PQ* produced such that $PQ:QR=1:\lambda$. Point *M* is the mid-point of *OR*.

(i) Find the position vector of *R* in terms of λ, $\mathbf{p}$ and $\mathbf{q}$. [1]

F is a point on *OQ* such that *F*, *P* and *M* are collinear.

(ii) Find the ratio *OF*:*FQ*, in terms of λ. [4]

Suggested Solution

(a)

$\overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b}$ for some $\lambda \in \mathbb{R}$.

Area of triangle *OAC*

$= \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{OC}\right|$

$= \frac{1}{2}\left|\mathbf{a} \times (\mathbf{a} + \lambda\mathbf{b})\right|$

$= \frac{1}{2}\left|\mathbf{a} \times \mathbf{a} + \lambda(\mathbf{a} \times \mathbf{b})\right|$

$= \frac{1}{2}|\lambda|\left|(\mathbf{a} \times \mathbf{b})\right|$

$= \frac{1}{2}|\lambda||\mathbf{a}||\mathbf{b}|\sin\theta$

$= \frac{1}{2}|\lambda|(2)(2)\frac{1}{6} = \frac{1}{3}|\lambda|$

Since area of triangle OAC= 3,

$\frac{1}{3}|\lambda| = 3$

$\lambda = 9 \text{ or } -9$

$\overrightarrow{OC} = \mathbf{a} \pm 9\mathbf{b}$

(b)

By the ratio theorem,

$$\overrightarrow{OQ} = \frac{\overrightarrow{OR} + \lambda\overrightarrow{OP}}{1+\lambda}$$

$$\overrightarrow{OR} = (1+\lambda)\mathbf{q} - \lambda\mathbf{p}$$

Since the point F lies on line OQ, $\overrightarrow{OF} = t\mathbf{q}$, for some $t \in \mathbb{R}$.

$$\begin{aligned}\overrightarrow{PM} &= \overrightarrow{OM} - \overrightarrow{OP} \\ &= \frac{1}{2}\overrightarrow{OR} - \mathbf{p} \\ &= \frac{(1+\lambda)}{2}\mathbf{q} - \frac{\lambda}{2}\mathbf{p} - \mathbf{p} \\ &= \frac{(1+\lambda)}{2}\mathbf{q} - \left(\frac{\lambda}{2}+1\right)\mathbf{p}\end{aligned}$$

Since the point F also lies on line PM,

$$\begin{aligned}\overrightarrow{OF} &= \mathbf{p} + s\overrightarrow{PM}, \text{for some } s \in \mathbb{R}. \\ &= \mathbf{p} + s\left[\frac{(1+\lambda)}{2}\mathbf{q} - \left(\frac{\lambda}{2}+1\right)\mathbf{p}\right] \\ &= \left(1 - \frac{s\lambda}{2} - s\right)\mathbf{p} + \frac{s(1+\lambda)}{2}\mathbf{q}\end{aligned}$$

Since $\mathbf{p}$ and $\mathbf{q}$ are non-parallel & non-zero vectors, comparing coefficients of $\mathbf{p}$ and $\mathbf{q}$ against $\overrightarrow{OF} = t\mathbf{q}$, we have

$$1 - \frac{s\lambda}{2} - s = 0$$

$$s\left(\frac{\lambda}{2}+1\right) = 1$$

$$s = \frac{1}{\frac{\lambda}{2}+1} = \frac{2}{\lambda+2}$$

$$\begin{aligned}\overrightarrow{OF} &= \frac{s(1+\lambda)}{2}\mathbf{q} = \frac{2}{\lambda+2}\left(\frac{1+\lambda}{2}\right)\mathbf{q} \\ &= \frac{1+\lambda}{2+\lambda}\mathbf{q} = \frac{1+\lambda}{2+\lambda}\overrightarrow{OQ}\end{aligned}$$

Thus, $OF : FQ = 1+\lambda : 1$

6 Do not use a calculator in answering this question.

(a) It is given that two complex numbers z and w satisfy the following equations

$$13z = (4-7\mathrm{i})w,$$
$$z - 2w = 5 - 4\mathrm{i}.$$

Find z and w. [4]

(b) It is given that $q = -\sqrt{3} - \mathrm{i}$.

(i) Find an exact expression for q^6, giving your answer in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [3]

(ii) Find the three smallest positive whole number values of n for which $\dfrac{q^n}{q^*}$ is purely imaginary.

[4]

Suggested Solution
(a) From (2): $z = 5 - 4\mathrm{i} + 2w$ Sub into equation (1): $13(5-4\mathrm{i}+2w) = (4-7\mathrm{i})w$ $(22+7\mathrm{i})w = -13(5-4\mathrm{i})$ $w = \dfrac{-13(5-4\mathrm{i})}{22+7\mathrm{i}} \times \dfrac{22-7\mathrm{i}}{22-7\mathrm{i}}$ $= \dfrac{-13(110-35\mathrm{i}-88\mathrm{i}-28)}{22^2+7^2}$ $= \dfrac{-13}{533}(82-123\mathrm{i}) = -2+3\mathrm{i}$ Sub into $z = 5 - 4\mathrm{i} + 2w$ $z = 5 - 4\mathrm{i} + 2(-2+3\mathrm{i})$ $=1+2\mathrm{i}$
(b)(i) $q = -\sqrt{3} - \mathrm{i}$ $\arg(q) = -\dfrac{5\pi}{6}$ $\lvert q \rvert = 2$

Thus, $q = 2e^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}$

$$q^6 = \left(2e^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}\right)^6 = 2^6 e^{\mathrm{i}(-5\pi)} = 64e^{\mathrm{i}(\pi)}$$

(ii)

$$\begin{aligned}\arg\left(\frac{q^n}{q*}\right) &= \arg(q^n) - \arg\left(q*\right)\\ &= n\arg\left(q\right) + \arg\left(q\right)\\ &= (n+1)\arg\left(q\right)\\ &= (n+1)\left(-\frac{5\pi}{6}\right)\\ &= -\left(n+1\right)\frac{5\pi}{6}\end{aligned}$$

For $\frac{q^n}{q*}$ to be imaginary, $\arg\left(\frac{q^n}{q*}\right) = \pm\frac{\pi}{2}$

Thus

$$-\left(n+1\right)\frac{5\pi}{6} = \frac{\left(2k+1\right)\pi}{2}, k \in ¢$$

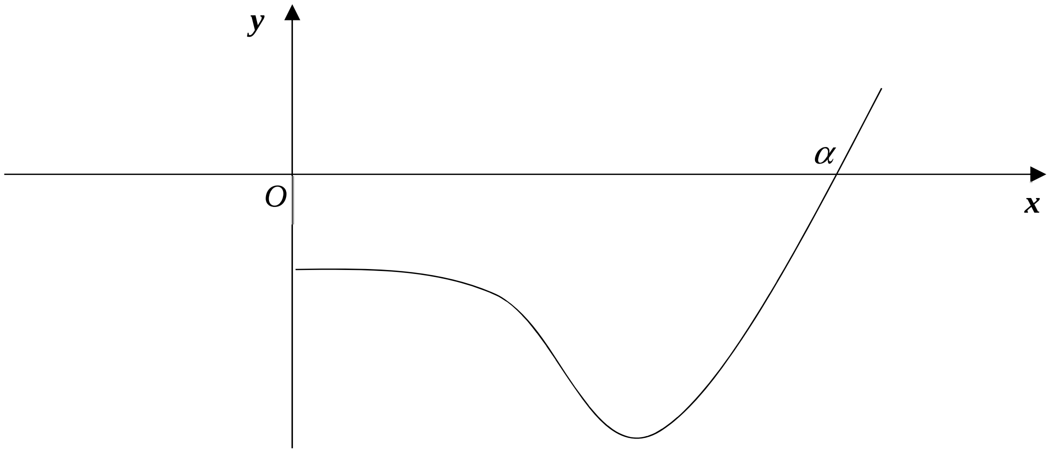
$$\left(n+1\right) = -\frac{3\left(2k+1\right)}{5}, k \in ¢$$

$$n+1 = 3, 9, 15...$$

$$n = 2, 8, 14, ..$$

The smallest positive whole number values of *n* are 2, 8, 14.

7



It is given that $f(x) = 2x^6 - 4x^4 - 6x^2 - 7$. The diagram shows the curve with equation $y = f(x)$ for $x \geq 0$. The curve crosses the positive x-axis at $x = \alpha$.

(i) Find the value of α, giving your answer correct to 3 decimal places. [1]

(ii) Show that $f(x) = f(-x)$ for all real values of x. What can be said about the six roots of the equation $f(x) = 0$? [4]

It is given that $g'(x) = f(x)$, for all real values of x.

(iii) Determine the x-coordinates of all the stationary points of graph of $y = g(x)$ and determine their nature. [3]

(iv) For which values of x is the graph of $y = g(x)$ concave upwards? [3]

Suggested Solution
(i) Using GC, $\alpha = 1.804$ (3 d.p)
(ii) $f(x) = 2x^6 - 4x^4 - 6x^2 - 7$ $f(-x) = 2(-x)^6 - 4(-x)^4 - 6(-x)^2 - 7$ $= 2x^6 - 4x^4 - 6x^2 - 7$ $= f(x)$ $\Rightarrow f(x) = f(-x) \quad (\text{shown})$ Since, $f(x) = f(-x)$, two real roots of $f(x) = 0$ are 1.804 and -1.804. (From (i))

The remaining 4 complex roots are in a form of 2 complex conjugate pairs and also are negatives of each other.

The four complex roots can be written as x_1, x_2, x_3, x_4, where

$x_1 = -x_3$ while $x_2 = -x_4$.

Also $x_1* = x_2$ while $x_3* = x_4$

In other words, the complex roots are of the forms:

$x_1 = a + b\mathrm{i}$

$x_2 = a - b\mathrm{i}$

$x_3 = -a - b\mathrm{i}$

$x_4 = -a + b\mathrm{i}$

(iii)

$y = \mathrm{g}(x)$ has stationary points when $\mathrm{g}'(x) = \mathrm{f}(x) = 0$, i.e. at $x = 1.804$ or -1.804.

From the graph, we can see

	$x = 1.804^-$	x=1.804	$x = 1.804^+$
$\mathrm{g}'(x) = \mathrm{f}(x)$	-ve	0	+ve

$y = \mathrm{g}(x)$ has a minimum point at $x = 1.804$.

From the graph, we can see

	$x = -1.804^-$	$x = -1.804$	$x = -1.804^+$
$\mathrm{g}'(x) = \mathrm{f}(x)$	+ve	0	-ve

$y = \mathrm{g}(x)$ has a maximum point at $x = -1.804$.

(iv)

$y = \mathrm{g}(x)$ is concave upwards when $\mathrm{g}''(x) = \mathrm{f}'(x)$ is positive.

$\mathrm{f}'(x) = 12x^5 - 16x^3 - 12x$

$\mathrm{f}'(x) = 0$

Using GC the real roots are $x = 0, -1.37, 1.37$ (these are also the *x*-coordinates of the stationary points of $\mathrm{f}(x)$.)

From graph, we can tell that $f'(x)$ is positive for $x \in [-1.37, 0] \cup [1.37, \infty)$

8 **(a)** **(i)** Show that, for $r \in ¢$, $r \geq 2$,

$$\frac{1}{(r-1)!} - \frac{2}{r!} + \frac{1}{(r+1)!} = \frac{r^2-r-1}{(r+1)!}.$$ [1]

Let $S_n = \sum_{r=2}^{n} \frac{r^2-r-1}{(r+1)!}$.

(ii) Hence find S_n in terms of n. [3]

(iii) Show that S_n converges to a limit L, where L is to be determined. [2]

(iv) Find the least integer value of n such that S_n differs from L by less than 10^{-10}. [2]

(b) **(i)** Suppose that f is a continuous, strictly decreasing function defined on $[1, \infty)$, with $\mathrm{f}(x) > 0$, $x \geq 1$. According to the Maclaurin-Cauchy test, then the infinite series $\sum_{n=1}^{\infty} \mathrm{f}(n)$ is convergent if and only if the integral $\int_1^{\infty} \mathrm{f}(x)\,\mathrm{d}x$ is finite. By applying the Maclaurin-Cauchy test on the function f defined by $\mathrm{f}(x) = \frac{1}{x}$, $x \geq 1$, determine if the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent. [2]

(ii) Let p be a positive number. By considering the Maclaurin-Cauchy test, show that if $p > 1$, the infinite series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots \frac{1}{n^p} + \ldots$ is convergent. [2]

Suggested Solution
(i) For $r \in ¢$, $r \geq 2$, $\frac{1}{(r-1)!} - \frac{2}{r!} + \frac{1}{(r+1)!}$ $= \frac{r(r+1) - 2(r+1) + 1}{(r+1)!}$ $= \frac{r^2+r-2r-2+1}{(r+1)!}$ $= \frac{r^2-r-1}{(r+1)!}$
(ii)

$$Sn=\sum_{r=2}^{n}\frac{r^2-r-1}{(r+1)!}$$

$$=\sum_{r=2}^{n}\left[\frac{1}{(r-1)!}-\frac{2}{r!}+\frac{1}{(r+1)!}\right]$$

$$=\left\{\left[\frac{1}{1!}-\frac{2}{2!}+\frac{1}{3!}\right]\right.$$

$$+\left[\frac{1}{2!}-\frac{2}{3!}+\frac{1}{4!}\right]$$

$$+\left[\frac{1}{3!}-\frac{2}{4!}+\frac{1}{5!}\right]$$

$$+\left[\frac{1}{4!}-\frac{2}{5!}+\frac{1}{6!}\right]$$

M

$$+\left[\frac{1}{(n-3)!}-\frac{2}{(n-2)!}+\frac{1}{(n-1)!}\right]$$

$$+\left[\frac{1}{(n-2)!}-\frac{2}{(n-1)!}+\frac{1}{n!}\right]$$

$$\left.+\left[\frac{1}{(n-1)!}-\frac{2}{n!}+\frac{1}{(n+1)!}\right]\right\}$$

$$=\frac{1}{1!}-\frac{2}{2!}+\frac{1}{2!}+\frac{1}{n!}-\frac{2}{n!}+\frac{1}{(n+1)!}$$

$$=\frac{1}{2}-\frac{1}{n!}+\frac{1}{(n+1)!}$$

$$=\frac{1}{2}-\frac{n}{(n+1)!}$$

(iii)

$$Sn=\frac{1}{2}-\frac{n}{(n+1)!}$$

As $n\to\infty$, $\frac{n}{(n+1)!}\to 0$.

$\therefore$ $S_n\to\frac{1}{2}$, thus S_n converges and $L=\frac{1}{2}$

(iv)

For $\left\lvert S_n - \frac{1}{2} \right\rvert < 10^{-10}$ $\frac{n}{(n+1)!} < 10^{-10}$ From GC, $n \geq 14$ $\therefore$ the least value of n =14.
(b)(i) $\int_1^{\infty} \frac{1}{x} \mathrm{d}x = [\ln x]_1^{\infty}$ $[\ln x]_1^{\infty}$ is not finite and thus, by the Maclaurin-Cauchy test, $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.
(b)(ii) $\int_1^{\infty} \frac{1}{x^p} \mathrm{d}x = \int_1^{\infty} x^{-p} \mathrm{d}x$ $= \frac{1}{(-p+1)} \left[x^{-p+1} \right]_1^{\infty}$ $= 0 - \frac{1}{(-p+1)}$ $= \frac{1}{p-1}$ is finite, whenever $p > 1$. Thus, by the Maclaurin-Cauchy test, whenever $p > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent.

9 A drilling company plans to install a straight pipeline *AB* through a mountain. Points (x, y, z) are defined relative to a main control site at the foot of the mountain at $(0,0,0)$, where units are metres. The *x*-axis points East, the *y*-axis points North and the *z*-axis points vertically upwards. Point *A* has coordinates $(-200,150,10)$ while point *B* has coordinates $(100,10,a)$, where *a* is an integer. Point *B* is at a higher altitude than Point *A*.

(i) Given that the pipeline *AB* is of length 337 metres, find the coordinates of *B*. [3]

A thin flat layer of rock runs through the mountain and is contained in the plane with equation $20x + y + 2z = -837$.

(ii) Find the coordinates of the point where the pipeline meets the layer of rock. [4]

To stabilise the pipeline, the drilling company decides to build 2 cables to join points *A* and *B* to the layer of rock. Point *A* is joined to Point *P* while point *B* is joined to Point *Q*.

(iii) Assuming that the minimum length of cable is to be used, find the length *PQ*. [2]

(iv) Show that the pipeline is at an angle of 10.8^0 to the horizontal plane. [2]

(v) After the pipeline is completed, a ball bearing is released from point *B* to roll down the pipeline to check for obstacles. The ball bearing loses altitude at a rate of 0.3*t* metres per second, where *t* is the time (in seconds) after its release. Find the speed at which the ball bearing is moving along the pipeline 10 seconds after its release. [3]

Suggested Solution
(i) Given that length of BA is 337 $\overrightarrow{AB} = \begin{pmatrix} 300 \\ -140 \\ a-10 \end{pmatrix}$ $\sqrt{300^2 + (-140)^2 + (a-10)^2} = 337$ $(a-10)^2 = 3969$ $a - 10 = \pm 63$ $a = 73 \text{ or } -53$ Since B is of higher altitude, $a > 10$ and thus a= 73. Coordinates of B are $(100,10,73)$.

(ii)

$$\overrightarrow{AB} = \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix}$$

Equation of line AB:

$$\mathbf{r} = \begin{pmatrix} -200 \\ 150 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix}, \lambda \in \mathbb{R}$$

Let C be the point of intersection between the line and plane. Since C is on the plane,

$$\left[\begin{pmatrix} -200 \\ 150 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix} \right] \cdot \begin{pmatrix} 20 \\ 1 \\ 2 \end{pmatrix} = -837$$

$$-4000 + 150 + 20 + \lambda(6000 - 140 + 126) = -837$$

$$5986\lambda = 2993 \Rightarrow \lambda = \frac{1}{2}$$

$$\overrightarrow{OC} = \begin{pmatrix} -200 \\ 150 \\ 10 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix} = \begin{pmatrix} -50 \\ 80 \\ 41.5 \end{pmatrix}$$

The coordinates of C are (-50,80,41.5).

(iii)

Length of projection of AB onto the plane

$$= \frac{\left| \overrightarrow{AB} \times \begin{pmatrix} 20 \\ 1 \\ 2 \end{pmatrix} \right|}{\sqrt{405}}$$

$$= \frac{\left| \begin{pmatrix} 300 \\ -140 \\ 63 \end{pmatrix} \times \begin{pmatrix} 20 \\ 1 \\ 2 \end{pmatrix} \right|}{\sqrt{405}} = \frac{\left| \begin{pmatrix} -343 \\ 660 \\ 3100 \end{pmatrix} \right|}{\sqrt{405}}$$

$$= \frac{\sqrt{10163249}}{\sqrt{405}} = 158 \text{ m (to 3 sf)}$$

(iv)

Let acute angle between horizontal plane and AB be α.

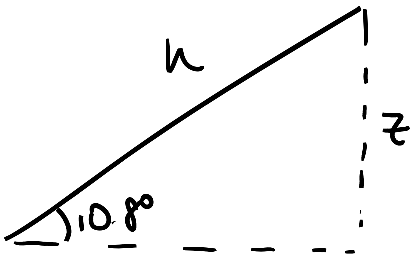
Normal to horizontal plane is **k.**

$$\sin\alpha = \frac{\left|\overrightarrow{AB}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\left|\overrightarrow{AB}\right|}$$

$$\sin\alpha = \frac{\begin{pmatrix}300\\-140\\63\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{337} = \frac{63}{337}$$

$\alpha = 10.8^0$

(v) Let h refer to the sloped distance the ball bearing has moved and z the altitude the ball bearing has moved.



Given that $\frac{\mathrm{d}z}{\mathrm{d}t} = -0.3t$, we need to find $\frac{\mathrm{d}h}{\mathrm{d}t}$.

$$\sin\left(10.8^0\right) = \frac{z}{h}$$

$$\frac{63h}{337} = z$$

Differentiate with respect to t:

$$\frac{63}{337}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{337}{63}\left(-0.3t\right)$$

When t=10

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{337}{63}\left(-0.3(10)\right)$$

$$= -16.0 ms^{-1}$$

Speed of the bearing= 16.0 m/s

10 An epidemiologist is studying the spread of a disease, dengue fever, which is spread by mosquitoes, in town *A*. *P* is defined as the number of infected people (in thousands) *t* years after the study begins. The epidemiologist predicts that the rate of increase of *P* is proportional to the product of the number of infected people and the number of uninfected people. It is known that town *A* has 10 thousand people of which a thousand were infected initially.

(i) Write down a differential equation that is satisfied by *P*. [1]

(ii) Given that the epidemiologist projects that it will take 2 years for half the town's population to be infected, solve the differential equation in **(i)** and express *P* in terms of *t*. [6]

(iii) Hence, sketch a graph of *P* against *t*. [2]

A second epidemiologist proposes an alternative model for the spread of the disease with the following differential equation:

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{2\cos t}{(2-\sin t)^2} \qquad (*).$$

(iv) Using the same initial condition, solve the differential equation (*) to find an expression of *P* in terms of *t*. [3]

(v) Find the greatest and least values of *P* predicted by the alternative model. [2]

(vi) The government of town *A* deems the alternative model as a more realistic model for the spread of the disease as it more closely follows the observed pattern of the spread of the disease. What could be a possible factor contributing to this? [1]

Suggested Solution
(i) $\frac{\mathrm{d}P}{\mathrm{d}t}=kP(10-P)$
(ii) $\frac{\mathrm{d}P}{\mathrm{d}t}=kP(10-P)$ **<u>Method 1 to integrate:</u>** $\int \frac{1}{P(10-P)}\,\mathrm{d}P=k\int \mathrm{d}t$ $\frac{1}{10}\int \frac{1}{P}+\frac{1}{10-P}\,\mathrm{d}P=k\int \mathrm{d}t$ $\frac{1}{10}\left[\ln\lvert P\rvert-\ln\lvert(10-P)\rvert\right]=kt+C$ $\frac{1}{10}\ln\left\lvert\frac{P}{10-P}\right\rvert=kt+c$

$$\frac{1}{10}\ln\left(\frac{P}{10-P}\right)=kt+C$$

$$\ln\left(\frac{P}{10-P}\right)=10kt+C_1$$

$$\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$$

<u>Method 2 to integrate</u>

$$\int\frac{1}{P(10-P)}\,\mathrm{d}P=k\int\mathrm{d}t$$

$$\int\frac{1}{25-(P-5)^2}\,dP=k\int\mathrm{d}t$$

$$\frac{1}{10}\ln\left|\frac{5+(P-5)}{5-(P-5)}\right|=kt+c$$

$$\frac{1}{10}\ln\left|\frac{P}{10-P}\right|=kt+c$$

From either Method 1 or 2,

since $P>0, 10-P\geq 0$

$$\frac{1}{10}\ln\left(\frac{P}{10-P}\right)=kt+C$$

$$\ln\left(\frac{P}{10-P}\right)=10kt+C_1$$

$$\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$$

<u>Substitute in values into solution</u>

Sub $t=0, P=1$

$$\frac{P}{10-P}=e^{10kt+C_1}=Ae^{10kt}$$

$$\frac{1}{9}=Ae^0\Rightarrow A=\frac{1}{9}$$

$$\frac{P}{10-P}=\frac{1}{9}e^{10kt}$$

Sub $t=2, P=5$

$$\frac{5}{10-5}=\frac{1}{9}e^{10(2)k}$$

$$1=\frac{1}{9}e^{20k}$$

$$e^{20k}=9 \Rightarrow k=\frac{1}{20}\ln(9)\approx 0.10986$$

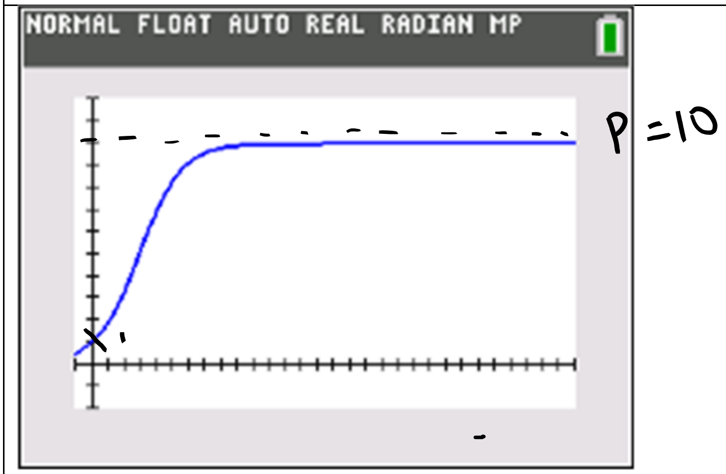
So we have

$$\frac{P}{10-P}=\frac{1}{9}e^{\frac{t}{2}\ln(9)}$$

$$9P=(10-P)e^{\frac{t}{2}\ln(9)}$$

$$P\left(9+e^{\frac{t}{2}\ln(9)}\right)=10e^{\frac{t}{2}\ln(9)}$$

$$P=\frac{10e^{\frac{t}{2}\ln(9)}}{9+e^{\frac{t}{2}\ln(9)}}$$



(iv)

$$\frac{\mathrm{d}P}{\mathrm{d}t}=\frac{2\cos t}{(2-\sin t)^2}=(-2)(-\cos t)(2-\sin t)^{-2}$$

$$P=\frac{-2(2-\sin t)^{-1}}{-1}=\frac{2}{2-\sin t}+c$$

Sub $t=0, P=1$

$$1=\frac{2}{2-\sin 0}+c$$
$$c=1-1=0$$

Thus $P=\dfrac{2}{2-\sin t}$

(v)

Since $-1\le\sin t\le1$,

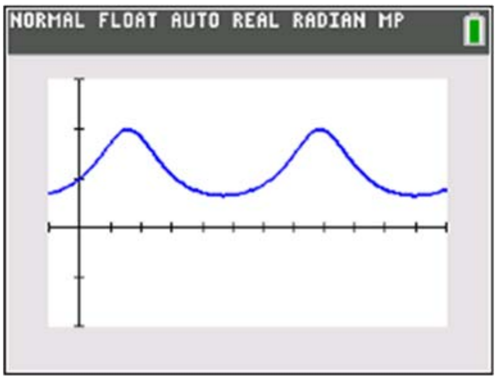
Largest value of P is when $\sin t=1$

Largest value of P= 2

Smallest value of P is when $\sin t=-1$

Smallest value of P= 2/3

(vi)



We can use the GC to plot $P=\dfrac{2}{2-\sin t}$.

The second model could be deemed more suitable, as it shows oscillating values of *P*, which could correspond to the population of the mosquitoes which could vary seasonally. (For example, when the season is hot and rainy, the environment is more conducive for the breeding of mosquitoes.)

EUNOIA JUNIOR COLLEGE

JC2 Preliminary Examination 2018

General Certificate of Education Advanced Level

Higher 2

MATHEMATICS 9758/02

Paper 2 [100 marks]

21 September 2018

3 hours

Additional Materials: Answer Paper

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and question number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use paper clips, highlighters, glue or correction fluid.

Answer **all** questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 **(i)** Given that f is a continuous function, explain, with the aid of a sketch, why the value of

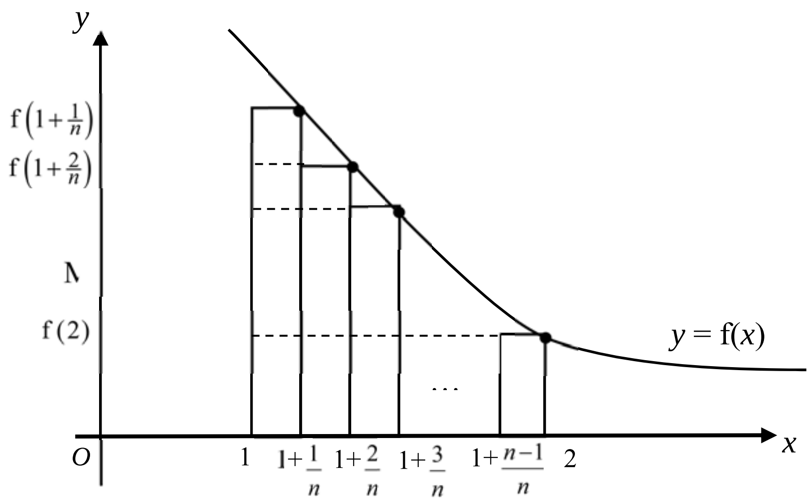
$$\lim_{n\to\infty} \frac{1}{n}\left\{ \mathrm{f}\left(\frac{n+1}{n}\right) + \mathrm{f}\left(\frac{n+2}{n}\right) + \ldots + \mathrm{f}\left(\frac{2n}{n}\right) \right\}$$

is $\int_1^2 \mathrm{f}(x)\,\mathrm{d}x$ [2]

(ii) Hence evaluate $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} \frac{n}{n+r}$ exactly. [3]

Suggested Solution

(i)



As shown in the diagram, the area under the curve $y = \mathrm{f}(x)$ from $x = 1$ to $x = 2$ can be approximated by the total areas of the n rectangles with width $\frac{1}{n}$ and heights given by

$\mathrm{f}\left(1+\frac{1}{n}\right), \mathrm{f}\left(1+\frac{2}{n}\right), \mathrm{f}\left(1+\frac{3}{n}\right), \ldots, \mathrm{f}\left(1+\frac{n}{n}\right)$.

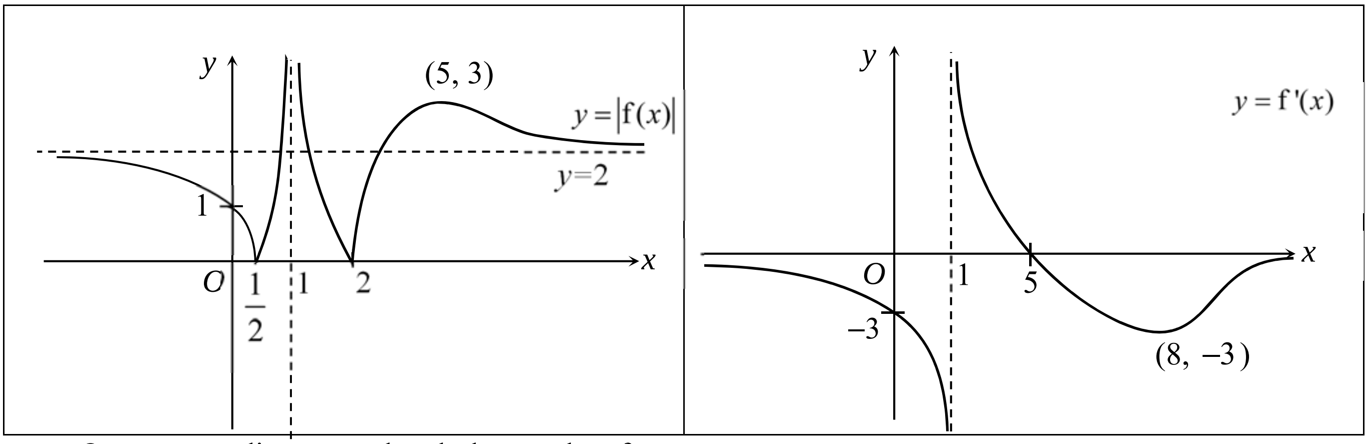
As n increases, the approximations will get better and approach the exact area as a limit,

i.e. $$\lim_{n\to\infty} \frac{1}{n}\left\{ \mathrm{f}\left(\frac{n+1}{n}\right) + \mathrm{f}\left(\frac{n+2}{n}\right) + \ldots + \mathrm{f}\left(\frac{2n}{n}\right) \right\} = \int_1^2 \mathrm{f}(x)\,\mathrm{d}x$$ [Shown]

(ii) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} \frac{n}{n+r} = \int_1^2 \mathrm{f}(x)\,\mathrm{d}x$ where $\mathrm{f}(x) = \frac{1}{x}$

$$= \int_1^2 \frac{1}{x}\,\mathrm{d}x = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln 2$$

2 The diagrams below show the graphs of $y = |f(x)|$ and $y = f'(x)$.



On separate diagrams, sketch the graphs of:

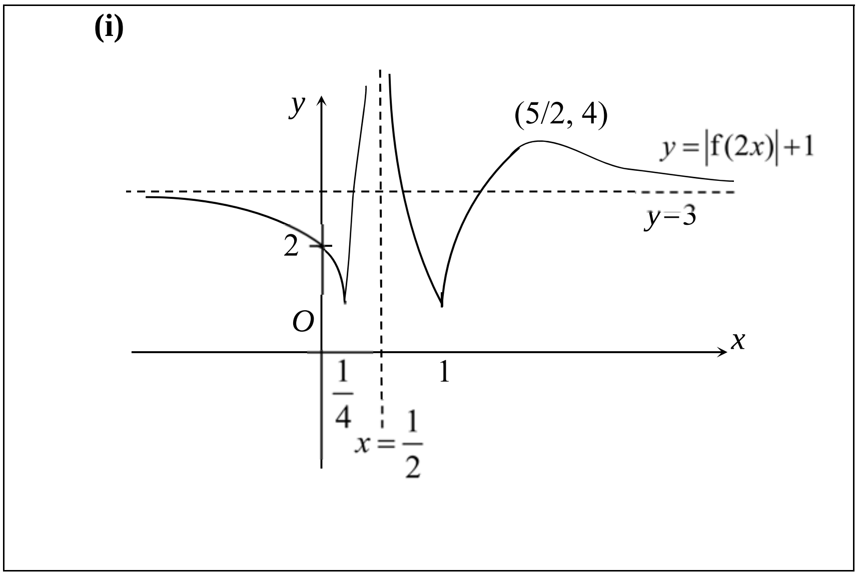
(i) $y = |f(2x)| + 1$ [2]

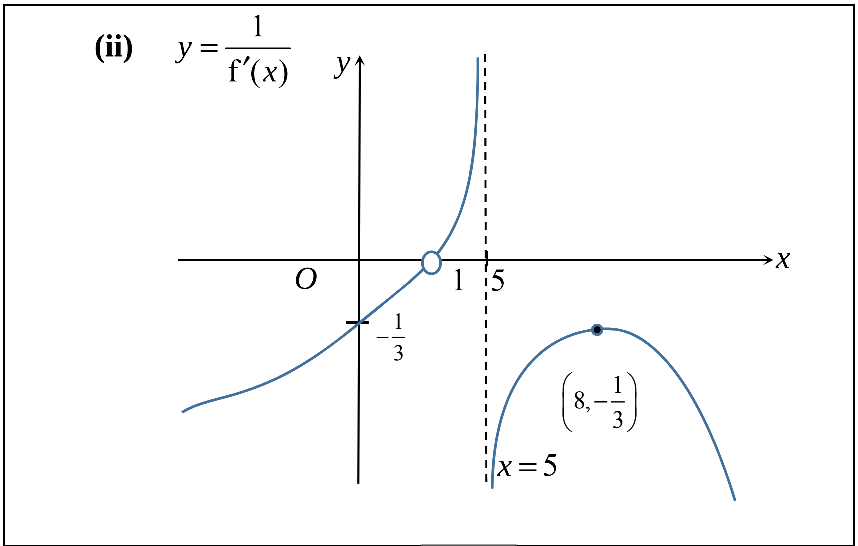
(ii) $y = \dfrac{1}{f'(x)}$ [3]

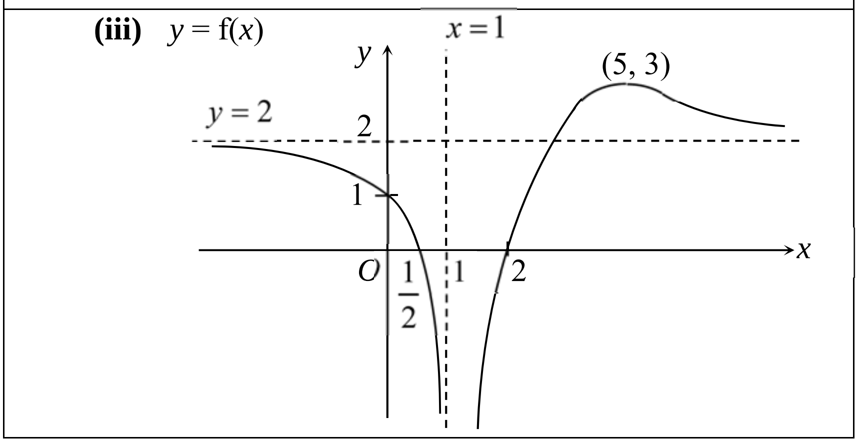
(iii) $y = f(x)$ [3]

showing clearly, in each case, the intersection(s) with the axes, the coordinates of the turning point(s) and the equation(s) of the asymptotes.

Suggested Solution







3 **(a)** A retirement savings account pays a compound interest of 0.2% per month on the amount of money in the account at the end of each month. A one-time principal amount of $\$P$ is deposited to open the account and a monthly pay-out of $x is withdrawn from the account at the beginning of each month, starting from the month that the account is opened.

(i) Show that the amount in the account at the end of n months after the interest has been added is given by

$$P(1.002^n) - 501x(1.002^n - 1).$$ [4]

(ii) Suppose a fixed monthly pay-out of $2,000 is to be sustained for at least 25 years, find the minimum principal amount required correct to the nearest dollar. [2]

(iii) If a principal amount of $600,000 is placed in the account, find the number of years for which a monthly pay-out of $2,000 per month can be sustained, leaving your answer correct to the nearest whole number. [2]

(b) A different retirement savings account provides an increasing amount of monthly pay-out over a period of 25 years. The pay-out in the first month is $$a$. The pay-out for each subsequent month is an increment of $$c$ from the pay-out of the previous month.

The pay-out in the final month is $4,000, and the total pay-out at the end of 25 complete years is $751,500. Find the month in which the pay-out is $2,000. [5]

Suggested Solution

(a)(i)

Month	Balance at the end of the month
1	$(P-x)(1.002) = P(1.002) - x(1.002)$
2	$(P(1.002) - x(1.002) - x)(1.002)$ $= P(1.002)^2 - x(1.002)^2 - x(1.002)$
3	$(P(1.002)^2 - x(1.002)^2 - x(1.002))(1.002)$ $= P(1.002)^3 - x(1.002)^3 - x(1.002)^2 - x(1.002)$
n	$= P(1.002)^n - x(1.002)^n - x(1.002)^{n-1} - \ldots - x(1.002)$

Balance after n months

$= P(1.002)^n - x(1.002)^n - x(1.002)^{n-1} - \ldots - x(1.002)$

$= P(1.002)^n - \dfrac{x(1.002)(1.002^n - 1)}{1.002 - 1}$

$= P(1.002)^n - 501x(1.002^n - 1)$ (shown)

(a)(ii)

$P(1.002)^{300} - 501(2000)(1.002^{300} - 1) \geq 0$

$P \geq \dfrac{501(2000)(1.002^{300} - 1)}{1.002^{300}} = 451761.1356$

Minimum principal amount required is \$451762.

(a)(iii)

$(600000)(1.002)^n - 501(2000)(1.002^n - 1) \geq 0$

$(1.002)^n(-402000) + 501(2000) \geq 0$

$(1.002)^n \leq \dfrac{1002000}{40200} = \dfrac{167}{67}$

$n \leq \dfrac{\ln\left(\dfrac{167}{67}\right)}{\ln 1.002} = 457.11$

No. of years $\leq \dfrac{457.11}{12} = 38.092$

No. of years for which the pay-out can be sustained is 38.

(b)

Month	Payout
1	a
2	a+c
3	a+2c
$\vdots$	$\vdots$
300	4000

Let \$*a* be the pay-out in the first month.

$a + 299c = 4000$ --- (1)

$\dfrac{300}{2}(2a + 299c) = 751500$

$2a + 299c = 5010$ --- (2)

Solving the simultaneous equations,

$a = 1010, \quad c = 10$

To find the month *n* with a pay-out of \$2,000:

$(1010) + (n-1)(10) = 2000$

$n = 100$

The pay-out is \$2,000 in the 100th month.

Alternative solution:

Let \$*a* be the pay-out in the first month.

$a + 299c = 4000$ --- (1)

$\dfrac{300}{2}(a + 4000) = 751500 \Rightarrow a = 1010$. Hence from (1) $c = 10$

Consider $U_n = 1010 + (n-1)(10) = 2000 \Rightarrow n = 100$

The pay-out is $2,000 in the 100th month.

4 The function f is defined by

$$f(x)=\begin{cases} 4-x & \text{for } 1\le x<3, \\ (x-4)^2 & \text{for } 3\le x<4, \end{cases}$$

and it is given that $f(x-3)=f(x)$ for all real values of *x*.

(i) State a reason why f does not have an inverse. [1]

(ii) Sketch the graph of $y=f(x)$ for $-1<x<6$. [3]

(iii) Evaluate $f(2017)$. [2]

The function g has domain $[1,4)$ and is defined by

$$g(x)=\begin{cases} 4-x & \text{for } 1\le x<3, \\ (x-4)^2 & \text{for } 3\le x<4. \end{cases}$$

(iv) By sketching $y=g(x)$ and $y=g^{-1}(x)$ on the same diagram, state the values of *x* such that $g(x)=g^{-1}(x)$. [3]

The function h, is defined by

$$h(x)=\begin{cases} \sqrt{1-x} & \text{for } 0\le x<1, \\ (x-1)^4 & \text{for } 1\le x\le 3. \end{cases}$$

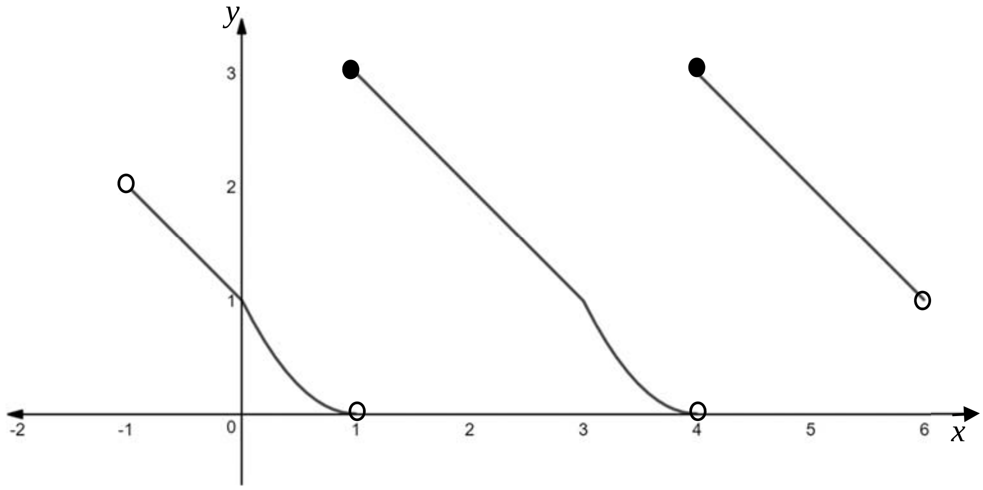
(v) Explain why hg^{-1} doesn't exist. [1]

(vi) Given that hg exist, define hg in similar form as function h. [2]

(vii) Find the range of hg. [2]

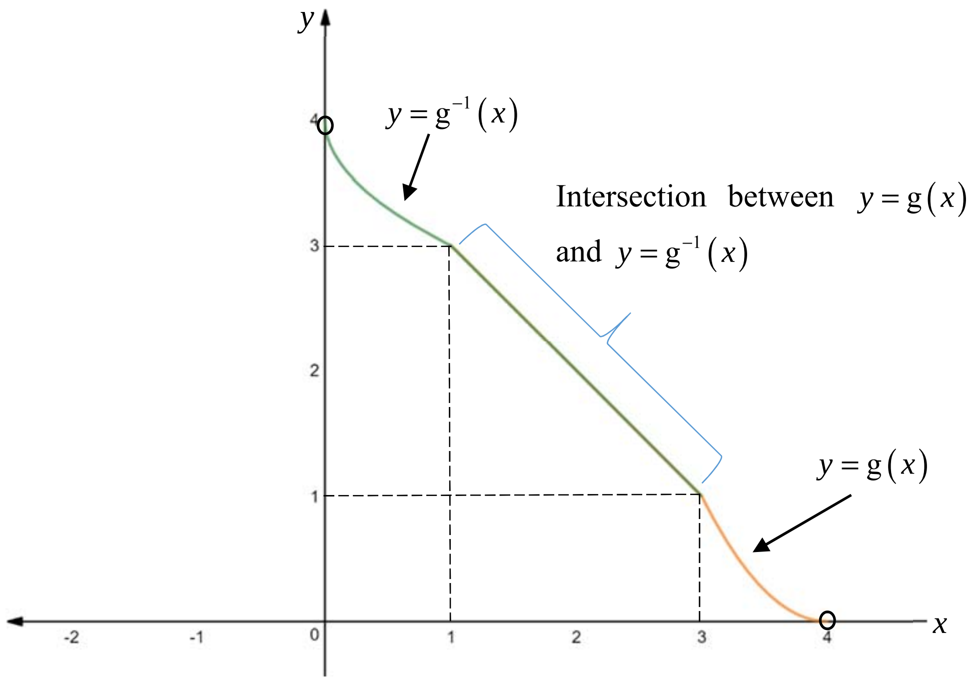
Suggested Solution
(a)(i) Since $f(x-3)=f(x)$, we can easily find 2 values of *x* (for example $f(-3)=f(0)$) to show that it's not 1-1, hence no inverse.

(ii)



(iii) $\mathrm{f}(2017)=\mathrm{f}(2014)=\mathrm{f}(2011)=\ldots=\mathrm{f}(1)=3$

(iv)



From the graph, the values of x such that $\mathrm{g}(x)=\mathrm{g}^{-1}(x)$ is $1\le x\le 3$.

(v) $R_{g^{-1}}=[1,4)\not\subset[0,3]=D_h$, hg^{-1} doesn't exist

(vi) $\mathrm{hg}(x)=\begin{cases}(3-x)^4 & \text{for } 1\le x<3,\\ \sqrt{1-(x-4)^2} & \text{for } 3\le x<4.\end{cases}$

(vii) Range of hg $=[0,16]$

Section B: Probability and Statistics [60 marks]

5 Two fair 4-sided dice each has its faces labelled '1', '2', '3' and '4'. The two dice are thrown and the absolute difference in score on their bottom faces is denoted by *X*.

(i) Find $\mathrm{P}(X = x)$ for all possible values of *x*. [2]

(ii) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [2]

Suggested Solution
(i) Consider the number of outcomes with the respective differences out of 16: $\mathrm{P}(X=0)=\frac{4}{16}=\frac{1}{4}$ $\mathrm{P}(X=1)=\frac{6}{16}=\frac{3}{8}$ $\mathrm{P}(X=2)=\frac{4}{16}=\frac{1}{4}$ $\mathrm{P}(X=3)=\frac{2}{16}=\frac{1}{8}$
(ii) $\mathrm{E}(X)=(0)\left(\frac{1}{4}\right)+(1)\left(\frac{3}{8}\right)+(2)\left(\frac{1}{4}\right)+(3)\left(\frac{1}{8}\right)=1.25$ $\mathrm{Var}(X)=\mathrm{E}(X^2)-\left(\mathrm{E}(X)\right)^2$ $=(0)^2\left(\frac{1}{4}\right)+(1)^2\left(\frac{3}{8}\right)+(2)^2\left(\frac{1}{4}\right)+(3)^2\left(\frac{1}{8}\right)-(1.25)^2$ $=0.9375$

6 **(a)** Two events A and B occur with probabilities *a* and *b* respectively.
Let $c = P\left(A \cap B\right)$.

(i) Find the minimum value of *c*, in terms of *a* and *b*, if $a+b>1$. [2]

(ii) Find the minimum value of *c* if $a=0.6$ and $b=0.2$. [1]

(b) For independent events *A* and *B*, prove that the events A' and B' are also independent. [3]

Suggested Solution

(a)

By considering $P(A \cup B) \le 1$

$P(A \cap B') + P(A \cap B) + P(B \cap A') \le 1$

Let $P(A \cap B) = c$

Then we have

$$a - c + c + b - c \le 1$$
$$c \ge a + b - 1$$

Note: Can also use $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1 \Rightarrow a + b - c \le 1 \Rightarrow c \ge a + b - 1$

Thus, minimum possible value of $P(A \cap B)$ is $a + b - 1$

when $P(A) = 0.6$ and $P(B) = 0.2$, [note that $P(A) + P(B) < 1$].

$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(A \cap B) = 0.8 - P(A \cup B)$

minimum possible value of $P(A \cap B)$ is 0

[i.e. when $P(A \cup B) = P(A) + P(B)$].

(b)

$P(A' \cap B') = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - P(A) - P(B) + P(A)P(B)$ (since A, B are indept)

$$= P(A') - P(B)[1 - P(A)]$$
$$= P(A') - P(B)[P(A')]$$
$$= P(A')[1 - P(B)]$$
$$= P(A')P(B')$$

Thus A' and B' are independent as well.

7 When Mr. Lee sends a text message to any of his students over the weekend, he gets a reply, on average, about 6 out of 10 times. On a particular weekend, Mr. Lee sends a text message to 25 students.

(i) State, in the context of this question, two assumptions needed to model the number of students that reply by a binomial distribution. [2]

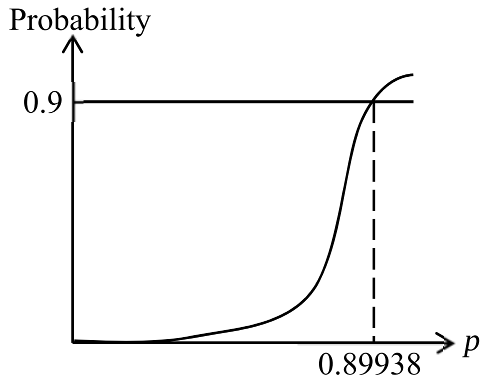
(ii) Explain why one of the assumptions stated in part (i) may not hold in this context. [1]

Assume now that these assumptions do in fact hold. Stating the distribution clearly,

(iii) find the probability that at least half of the students reply. [3]

It is given instead that the probability of a randomly chosen student replying is *p*. Find the least value of *p* such that there is at least a 90% chance of more than 20 students replying to the text message. [3]

Suggested Solution
(i) A student replying has a constant probability of 0.6. The event of a student replying is independent of another student. OR Whether a student replies is independent of another student.
(ii) Some students are more likely to reply a message. Hence the assumption of equal probability might not hold.
(iii) Let X be the number of students that reply, out of 25. $X \sim \mathrm{B}(25, 0.6)$ Required probability $= \mathrm{P}(X \geq 13)$ $= 1 - \mathrm{P}(X \leq 12)$ $= 0.84623 \quad (5\text{sf})$ $= 0.846 \quad (3\text{sf})$
(iv) Let Y be the number of students that reply, out of 25. Let p be the new probability of a student replying. $Y \sim \mathrm{B}(25, p)$ Given that $\mathrm{P}(Y > 20) \geq 0.9 \left(\text{or } \mathrm{P}(Y \leq 20) \leq 0.1\right)$



Therefore, the least probability of a student replying is 0.899 [accept 0.900].

8 A teacher, Mr. Ku, suspects that the average time a student spends on his or her mobile phone per day is μ_0 minutes. He selected a random sample of 97 students in the school who own mobile phones and recorded the amount of time each student spent on his or her phone in a randomly selected day. The results are displayed in the table below.

Time spent per day (to nearest minute)	60	65	72	90	110	180
Number of people	11	20	32	18	10	6

(i) Calculate unbiased estimates of the population mean and variance of the time a student spends on his or her mobile phone per day. [2]

The null hypothesis that the average time a student spends on his or her mobile phone per day is μ_0 minutes is tested, at 5% level of significance, against the alternative hypothesis that the average time a student spends on his or her mobile phone per day differs from μ_0 minutes.

(ii) Determine the range of values of μ_0 for which the null hypothesis is rejected. [5]

(iii) Explain, in the context of this question, the meaning of 'at 5% level of significance'. [1]

(iv) If the null hypothesis in (ii) is rejected at 5% significance level, can we reject the null hypothesis at 1% level of significance? Explain your answer. [1]

	Suggested Solution
(i)	Using a GC, The unbiased estimate of population mean is $83.134 \approx 83.1$ (3sf) The unbiased estimate of population variance is $29.011^2 \approx 841.64 \approx 842$ (3sf)
(ii)	Let X be the time a student spent on their mobile phone per day. Let μ be the population mean time a student spent on their mobile phone per day. Test: $\mathrm{H}_0 : \mu = \mu_0$ vs $\mathrm{H}_1 : \mu \neq \mu_0$ Perform a 2-tail test and 5% level of significance Under H_0, since $n = 97$ is large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(\mu_0, \frac{s^2}{n}\right)$ approximately, with unknown μ_0 and $s^2 = 841.64$ At a 5% (2-tail), the Critical Region is $\{\lvert z \rvert > 1.96\}$

Since H_0 is rejected, $|z-value| > 1.96$

z-value < –1.96 or z-value > 1.96

$$\frac{83.134-\mu_0}{\sqrt{841.64/97}} < -1.96 \quad \text{or} \quad \frac{83.134-\mu_0}{\sqrt{841.64/97}} > 1.96$$

$$\mu_0 > 88.907 \approx 88.9 \quad \text{or} \quad \mu_0 < 77.361 \approx 77.4$$

(iii) 5% level of significance means that there is a 0.05 probability of concluding that the average time a student spent on their mobile phone per day differs from μ_0 when in actual fact it did not.

(iv) Test: $\mathrm{H}_0 : \mu = \mu_0$ vs $\mathrm{H}_1 : \mu \neq \mu_0$

Since we reject Ho at 5% level of significance,

$p-value = p < 0.05$

But that doesn't imply $p < 0.01$. Hence, it's inconclusive.

9 The table below gives the world record time, in seconds, for the 100 metre race for the various years.

Year, x	1993	1994	1996	1999	2002	2005	2007	2008	2009
Time, t	9.87	9.85	9.84	9.79	9.78	9.77	9.74	9.69	9.58

(i) Draw the scatter diagram for these values, labelling the axes clearly. [2]

It is thought that the world record time t can be modelled by one of the following formulae

$$t = a + bY \quad \text{or} \quad t = c + dY^3$$

where a, b, c, d are constants and $Y = (x - 2000)$.

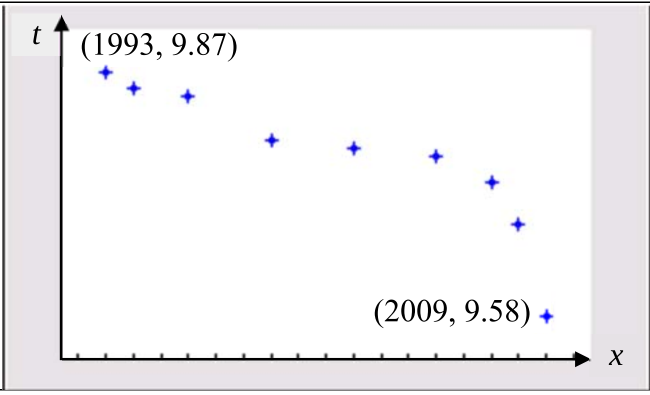
(ii) Using the value of the product moment correlation coefficient, explain which of $t = a + bY$ and $t = c + dY^3$ is the better model. [3]

(iii) Using the better model found in part **(ii)**, find the equation of a suitable regression line. Use it to estimate the world record time in the year 2018. Comment on the reliability of this estimate. [3]

(iv) Deduce, with justification, the product moment correlation coefficient between t and x. [1]

Suggested Solution

(i)



(ii) For $t = a + bY$, $r = -0.89351 \approx -0.894$ (3sf)

For $t = c + dY^3$, $r = -0.97163 \approx -0.972$ (3sf)

Since -0.972 is closer to -1 than -0.894, $t = c + dY^3$ is the better model.

(iii) Using GC, $t = 9.7985 - (2.5279 \times 10^{-4})(x - 2000)^3$

When $x = 2018$, $t = 9.7985 - (2.5279 \times 10^{-4})(2018 - 2000)^3$

$= 8.3242 \approx 8.32$ (3sf)

The estimate obtained by extrapolation is not reliable since $x = 2018$ lies outside the range of the sample data. $(1993 \le x \le 2009)$

(iv)Since the PMCC is not affected by linear transformation(mx+C with m > 0), $r = -0.89351 \approx -0.894 \quad (3\text{sf})$

10 In this question, you should state clearly the distribution of any random variables that you define.

The volume, *S*, in ml, of perfume in a randomly chosen small bottle has mean 20 and variance σ^2.

(i) If $\sigma = 15$, explain why S may not be appropriately modelled by a normal distribution. [2]

It is now assumed that *S* follows a normal distribution

(ii) Given that 6.68% of the small bottles contains more than 23 ml of perfume, find the value of σ. [2]

For the rest of the question, the volume of perfume, *S*, in ml, in a randomly chosen small bottle follows the distribution $\mathrm{N}(20, 4)$ and the volume of perfume, *L*, in ml, in a randomly chosen large bottle follows the distribution $\mathrm{N}(100, 25)$.

(iii) Calculate the probability that 6 randomly chosen small bottles and 9 randomly chosen large bottles contain a total volume of at least 1 litre of perfume. [3]

(iv) Calculate the probability that the volume of perfume in a randomly chosen large bottle differs from 6 times the volume of perfume in a randomly chosen small bottle by more than 25ml. [3]

(v) State, in this context, an assumption needed for your calculations in parts **(iii)** and **(iv)**. [1]

	Suggested Solution
(i)	Let *S* be the volume of perfume in a randomly chosen small bottle. $S \sim \mathrm{N}(20, \sigma^2)$ When $\sigma = 15$, $\mathrm{P}(S < 0) = 0.0912 \quad (3\text{sf})$ This means that there is a significant chance (9.12%) that a small bottle has a negative volume of perfume. Hence, a normal distribution with $\mu = 20, \sigma = 15$ is not appropriate. [Accept 99.7% including between (-25 to 65) which includes –ve volume.]

(ii) $S \sim \mathrm{N}\left(20, \sigma^2\right)$

Given $\mathrm{P}(S > 23) = 0.0668$

$$\mathrm{P}\left(Z > \frac{23-20}{\sigma}\right) = 0.0668$$

Using GC, $\dfrac{23-20}{\sigma} = 1.5001$

$$\begin{aligned}\sigma &= \frac{3}{1.5001} \\ &= 1.9999 \quad (5\text{sf}) \\ &= 2.00 \quad (3\text{sf})\end{aligned}$$

(iii) $S \sim \mathrm{N}\left(20, 2^2\right)$

Let L be the volume of perfume in a randomly chosen large bottle.

$L \sim \mathrm{N}\left(100, 5^2\right)$

Let $T = S_1 + \mathrm{L} + S_6 + L_1 + \mathrm{L} + L_9$

$$\begin{aligned}\mathrm{E}(T) &= \mathrm{E}(S_1) + \mathrm{L} + \mathrm{E}(S_6) + \mathrm{E}(L_1)\mathrm{L} + \mathrm{E}(L_9) \\ &= 6\mathrm{E}(S) + 9\mathrm{E}(L_9) \\ &= 6(20) + 9(100) \\ &= 1020\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(T) &= \mathrm{Var}(S_1) + \mathrm{L} + \mathrm{Var}(S_6) \\ &\quad + \mathrm{Var}(L_1)\mathrm{L} + \mathrm{Var}(L_9) \\ &= 6\mathrm{Var}(S) + 9\mathrm{Var}(L) \\ &= 6(4) + 9(25) \\ &= 249\end{aligned}$$

$T \sim \mathrm{N}\left(1020, \left(\sqrt{249}\right)^2\right)$

$$\begin{aligned}\text{Required probability} &= \mathrm{P}(T > 1000) \\ &= 0.89750 \quad (5\text{sf}) \\ &= 0.898 \quad (3\text{sf})\end{aligned}$$

(iv) $\mathrm{E}(6S-L)=6\mathrm{E}(S)-\mathrm{E}(L)$ $=6(20)-100$ $=20$ $\mathrm{Var}(6S-L)=6^2\,\mathrm{Var}(S)+\mathrm{Var}(L)$ $=36(4)+25$ $=169$ $6S-L\sim \mathrm{N}(20,13^2)$ Required probability $=\mathrm{P}(\lvert 6S-L\rvert>25)$ $=1-\mathrm{P}(-25<6S-L<25)$ $=0.35053\quad(5\mathrm{sf})$ $=0.351\quad(3\mathrm{sf})$	
(v) An assumption needed is that the distribution of volumes of perfume in **ALL** (between and within *S* and *L*) are independent of one another.	

11 A bag contains 5 cards with the letter 'A', 3 cards with the letter 'B' and 2 cards with the letter 'C'.

A A A A A B B B C C

The 10 cards are arranged at random in a row to form a letter sequence. For example, AABBCAAACB is a possible letter sequence.

(i) Find the number of possible letter sequences. [1]

(ii) Find the number of possible letter sequences if no two 'B's are next to each other **and** no two 'C's are next to each other. [4]

(iii) Find the probability that the first two letters are identical given that the second letter is **not** an 'A'. [4]

The 10 cards are now arranged at random in a circle.

(iv) Find the probability that no two 'A's are next to each other. [3]

Suggested Solution	**Comments**
(i) Number of different letter sequences $=\dfrac{10!}{5!3!2!}=2520$	
(ii) Number of different letter sequences = (Number with no adjacent 'B's) – (Number with no adjacent 'B's, but with adjacent 'C's) = (No. of ways to permutate 7 units – 5As and 2Cs, and slot in 3Bs in between the 7 units) – (No. of ways to permutate 6 units – 5As and CC, and slot in 3Bs in between the 6 units) $=\left(\dfrac{7!}{5!2!}\cdot\binom{8}{3}\right)-\left(\dfrac{6!}{5!}\cdot\binom{7}{3}\right)$ $=966$	
(iii) P(1st and 2nd letter identical | 2nd letter is not 'A') $=\dfrac{P(BB\text{ or }CC)}{P(AA'\text{ or }A'A')}$ $=\dfrac{P(BB)+\mathrm{P}(CC)}{P(AA')+P(A'A')}$	

$$=\frac{\left(\frac{3}{10}\right)\left(\frac{2}{9}\right)+\left(\frac{2}{10}\right)\left(\frac{1}{9}\right)}{\left(\frac{5}{10}\right)\left(\frac{5}{9}\right)+\left(\frac{5}{10}\right)\left(\frac{4}{9}\right)}$$

$$=\frac{8}{45}$$

Alternative Solution:

P(1st and 2nd letter identical | 2nd letter is not 'A')

$$=\frac{P(BB\text{ or }CC)}{P(\text{2nd letter is }B\text{ or }C)}$$

$$=\frac{\frac{\text{No. of permutations starting with }BB\text{ or }CC}{\text{Total no. of permutations}}}{\frac{\text{No. of permutations with 2nd letter }B\text{ or }C}{\text{Total no. of permutations}}}$$

$$=\frac{\left(\frac{8!}{5!2!}+\frac{8!}{5!3!}\right)}{\left(\frac{9!}{5!2!2!}+\frac{9!}{5!3!}\right)}$$

$$=\frac{8}{45}$$

(iv) Treat all the As as distinct (A_1, A_2, A_3, A_4, A_5), Bs as distinct (B_1, B_2, B_3) and Cs as distinct (C_1, C_2).

Probability

$$=\frac{\text{No. of ways to permutate 10 cards in a circle with all As separate}}{\text{No. of ways to permutate 10 cards in a circle}}$$

$$=\frac{\text{No. of ways to permutate }B_1B_2B_3C_1C_2\text{ in a circle, and slot in the As}}{\text{No. of ways to permutate 10 cards in a circle}}$$

$$=\frac{\left(\frac{5!}{5}\right)(5!)}{\left(\frac{10!}{10}\right)}=\frac{1}{126}$$

Alternative Solution:

Treat 5 As as identical, 3 Bs as identical and 2 Cs as identical. Since all cards are used, each outcome is equally probable.

Probability $= \dfrac{\text{No. of ways of arranging 3Bs and 2Cs in a circle, and slot in the 5As}}{\text{No. of ways of arranging 5As, 3Bs and 2Cs in a circle}}$ $= \dfrac{\left(\dfrac{5!}{5\cdot 3!2!}\right)}{\left(\dfrac{10!}{10\cdot 5!3!2!}\right)} = \dfrac{1}{126}$	