ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS 9758/01

Paper 1 **13 August 2018**

3 hours

Additional Materials: Cover Sheet
Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **7** printed pages.

ANGLO-CHINESE JUNIOR COLLEGE
MATHEMATICS DEPARTMENT
JC2 Preliminary Examination 2018

MATHEMATICS 9758
Higher 2
Paper 1

/ 100

Index No: [ ][ ][ ][ ]   Form Class: ____________

Name: ___________________________

Calculator model: ____________________

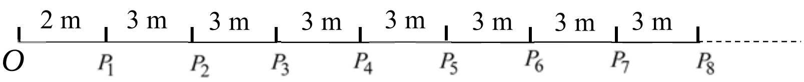
Arrange your answers in the same numerical order.

Place this cover sheet on top of them and tie them together with the string provided.

Question No.	Marks
1	/5
2	/4
3	/7
4	/7
5	/8
6	/8
7	/9
8	/9
9	/13
10	/14
11	/16

Summary of Areas for Improvement			
Knowledge (K)	Careless Mistakes (C)	Read/Interpret Qn wrongly (R)	Presentation (P)

1 Points $P_1, P_2, P_3, ..., P_n$ are marked on a straight line increasingly far away from the origin O such that distance $OP_1 = 2$ m and the distances between subsequent adjacent points are all 3 m as shown in the diagram below.



(i) Express the distances OP_n and P_1P_n in terms of n. [2]

(ii) Hence, or otherwise, show that the total distance $OP_n + P_1P_n + P_2P_n + ... + P_{n-1}P_n$ is $\frac{3}{2}n^2 + \frac{3}{2}n - 1$. [3]

2 Two series S_n and T_{n-1} are given by

$$S_n = \sum_{r=1}^{n}\left(\frac{r}{2^{r-1}}\right) \text{ and } T_{n-1} = \sum_{r=1}^{n-1}\left(\frac{r}{2^r}\right).$$

By listing the terms of S_n and T_{n-1}, write $S_n - T_{n-1}$ as a geometric series and find $S_n - T_{n-1}$, leaving your answer in terms of n. [3]

Hence find the value of $S_n - T_{n-1}$ as $n \to \infty$. [1]

3 **(a)** The quadrilateral *ABCD* is such that *P*, *Q*, *R* and *S* are the midpoints of *AB*, *BC*, *CD* and *DA* respectively. Prove that *PQRS* is a parallelogram. [3]

(b) Referred to the origin *O*, points *A*, *B* and *C* have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, where $\mathbf{a}$ is a unit vector, $|\mathbf{b}| = 3$, $|\mathbf{c}| = \sqrt{3}$ and angle *AOC* is $\frac{\pi}{6}$ radians. Given that $3\mathbf{a} + \mathbf{c} = k\mathbf{b}$ where $k \neq 0$, by considering $(3\mathbf{a} + \mathbf{c}) \cdot (3\mathbf{a} + \mathbf{c})$, find the exact values of k. [4]

4 In the triangle *ABC*, *AB* = 2, *BC* = 3 and angle *ABC* = $\frac{\pi}{3} - \theta$ radians. Given that θ is a sufficiently small angle, show that

$$AC \approx \left(7 - 6\sqrt{3}\,\theta + 3\theta^2\right)^{\frac{1}{2}} \approx a + b\theta + c\theta^2,$$

where constants a, b and c are to be determined in exact form. [7]

5 A function h is said to self-inverse if $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$ for all x in the domain of h.

Functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{5x-3}{x-5}, \; x \in \mathbb{R}, \; x \neq a, \text{ where } a \text{ is a constant,}$$

$$\mathrm{g} : x \mapsto \ln x, \; x \in \mathbb{R}, \; x \geq \mathrm{e}^{10}.$$

(i) State the value of a and explain why this value has to be excluded from the domain of f. [2]

(ii) Show that f is self-inverse. [2]

(iii) Find the exact values of b such that $\mathrm{f}^4(b) - 2 = \mathrm{f}^{-1}(b)$. [2]

(iv) Find the exact range of fg. [2]

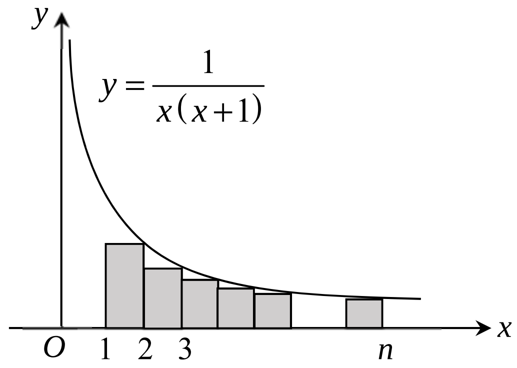
6 Find

(a) $\int (\sin^{-1} 2x) \frac{x}{\sqrt{1-4x^2}} \, \mathrm{d}x$. [4]

(b) $\int \frac{x-1}{x^2+2x+6} \, \mathrm{d}x$. [4]

7 **(i)** Express $\frac{1}{x(x+1)}$ in partial fractions. [1]

(ii)



The graph of $y = \frac{1}{x(x+1)}$, for $0 \leq x \leq n$, is shown in the diagram. Rectangles, each of width 1 unit, are drawn below the curve from $x = 1$ to $x = n$, where $n \geq 3$.

By considering $\sum_{x=a}^{b} \frac{1}{x(x+1)}$ where a and b are constants to be found, find the total area of the $n - 1$ rectangles in terms of n. [3]

Find the actual area bounded by the curve $y = \frac{1}{x(x+1)}$, the x-axis and the lines $x = 1$ and $x = n$. [2]

Hence show that $\frac{1}{2}-\ln 2<\frac{1}{n+1}+\ln\left(1-\frac{1}{n+1}\right)$ for all $n\geq 3$. [1]

Using a standard series from the List of Formulae (MF26), show that, for all $n\geq 3$,

$$\frac{1}{2}-\ln 2<\sum_{r=2}^{\infty}\frac{-1}{r(n+1)^r}.$$ [2]

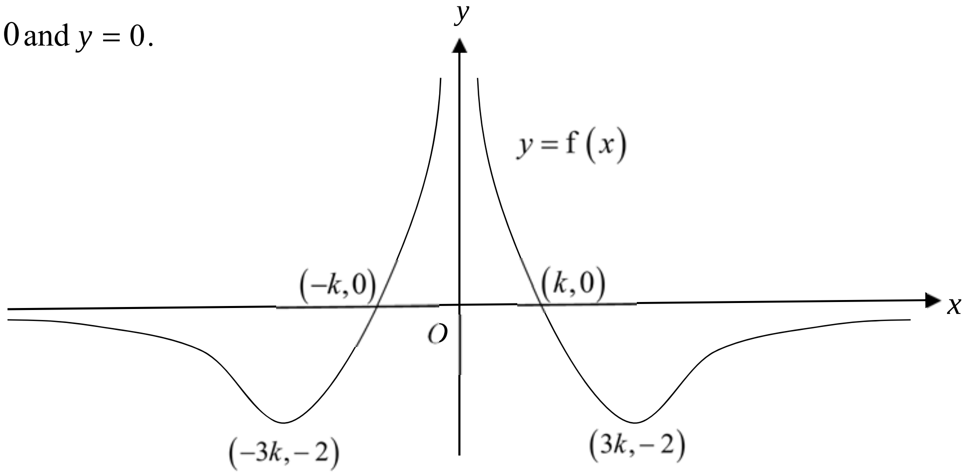
8 **(a)** Without using a calculator, solve the inequality $\frac{x+5}{x-1}>\frac{9}{x}$. [3]

Hence find the set of values of x which satisfies $\frac{e^x+5}{e^x-1}>9e^{-x}$. [2]

(b) It is given that $y=\frac{1+kx}{x^2-1}$, $x\in\mathbb{R}$, $x\neq\pm1$, and $-1\leq k\leq 1$. Find, in terms of k, the set of values that y can take. [4]

9 **(a)** The diagram shows the sketch of $y=f(x)$, where k is a positive constant.

The curve passes through the points with coordinates $(-k,0)$ and $(k,0)$, and has two minimum points with coordinates $(-3k,-2)$ and $(3k,-2)$. The asymptotes are $x=0$ and $y=0$.



Sketch on separate diagrams, the graphs of

(i) $y=f(2x-2k)$, [3]

(ii) $y=f'(x)$, [3]

(iii) $y=\frac{1}{f(x)}$, [3]

showing clearly, in terms of k, the equations of any asymptote(s), the coordinates of any turning point(s) and any points where the curve crosses the x- and y-axes, where possible.

(b) The curve C has equation $y=\dfrac{3x^2+9x+7}{x+2}$.

(i) By expressing the equation of C in the form $y=3x+a+\dfrac{b}{x+2}$, where a and b are non-zero constants, write down the equations of the asymptotes of C. [2]

(ii) Describe a pair of transformations that will transform C to the graph of $y=3x+\dfrac{1}{x}$. [2]

10 (a) Lion City Hospital is situated on the island of Jali. At the hospital, there are 3 classes of wards: A, B and C. The table below shows the number of patients admitted to each class of wards over 4 days, as well as the total amount collected by the hospital for bed charges.

	Day 1	***Day 2***	***Day 3***	***Day 4***
Class A	20	25	30	30
Class B	35	37	k	62
Class C	38	40	61	65
Total bed charges ($)	14150	16300	20600	23050

Assuming that all the patients admitted to a ward of the same class pay the same bed charge, find the value of k. [4]

(b) An epidemic occurred in the island of Jali and the number of people, P, who have caught the disease t weeks after the epidemic has started is given by the logistic growth equation proposed by Verhulst shown below.

$$\frac{dP}{dt}=0.9P\left(1-\frac{P}{300}\right)$$

It is given that when the epidemic started, 5 people had the disease.

(i) Show that $P=\dfrac{A}{Be^{-0.9t}+1}$, where A and B are constants. [5]

(ii) Sketch a graph of P against t. [2]

(iii) Given that Lion City Hospital has 350 beds, explain clearly whether this hospital can be used solely as a quarantine centre for all the people who have caught the disease during the epidemic. [1]

(iv) Once the epidemic reaches its peak, the spread of the disease will begin to slow down. Find the number of people who had the disease when the disease was spreading most rapidly. [2]

11

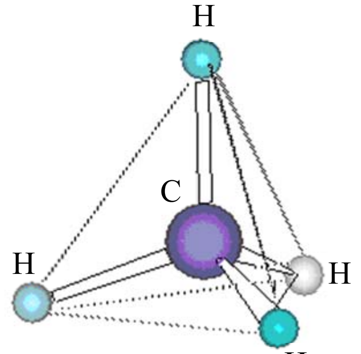


Figure 1.

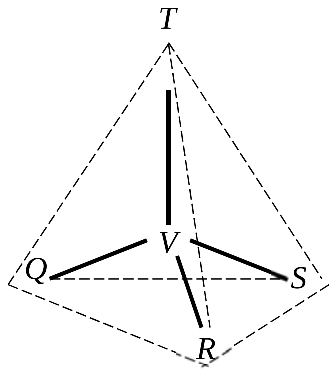


Figure 2.

Figure 1 shows a methane molecule consisting of a carbon atom with four hydrogen atoms symmetrically placed around it. Figure 2 shows the tetrahedron structure of the methane molecule with the centres of the hydrogen atoms represented by points *Q*, *R*, *S* and *T* and the centre of the carbon atom represented by point *V*.

The points *Q*, *R* and *S* has coordinates (8, 1, 8), (8, 7, 2) and (2, 1, 2) respectively and form an equilateral triangle.

(i) Find a cartesian equation of the plane *p* which passes through the points *Q*, *R* and *S*. [4]

(ii) Find a cartesian equation of the plane p_1 which passes through the midpoint of *QR* and is perpendicular to *QR*. [2]

Plane p_2 which passes through the midpoint of *RS* and is perpendicular to *RS* has equation $x+y=9$.

(iii) Find a vector equation of the line *l* where p_1 and p_2 meet. [1]

(iv) The point *T* is on the line *l* such that *QRST* is a regular tetrahedron with *QR* = *QT*. Show that the possible coordinates for point *T* is (2, 7, 8). Hence, or otherwise, find the coordinates of a point on plane *p* that is closest to point *T*. [5]

(v) Given that *TV* is $\frac{3}{4}$ of the vertical height of tetrahedron *QRST*. Find the coordinates of point *V* and hence show that the bonding angle *TVQ* of the methane molecule is $109.5°$ (correct to 1 decimal place). [4]

ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS 9758/02

Paper 2 **20 August 2018**

3 hours

Additional Materials: Cover Sheet
Answer Paper
List of Formulae (MF26)

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ANGLO-CHINESE JUNIOR COLLEGE
MATHEMATICS DEPARTMENT
JC2 Preliminary Examination 2018

MATHEMATICS 9758
Higher 2
Paper 2

/ 100

Index No: ☐☐☐☐ Form Class: ____________

Name: ____________________________

Calculator model: ____________________

Arrange your answers in the same numerical order.

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Question No.	Marks
1	/10
2	/8
3	/10
4	/12
5	/7
6	/8
7	/9
8	/10
9	/12
10	/14

Summary of Areas for Improvement			
Knowledge (K)	Careless Mistakes (C)	Read/Interpret Qn wrongly (R)	Presentation (P)

Section A: Pure Mathematics [40 marks]

1 The curve C has equation $\frac{(x-1)^2}{2^2}+y^2=1$.

(i) Sketch C, giving the exact coordinates of any points of intersection with the axes. [2]

(ii) Use the substitution $x=\sin\theta$, where $0\le\theta\le\frac{\pi}{2}$, to find the exact value of

$\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\ \mathrm{d}x.$ [4]

(iii) The region bounded by C for $x<0$ and the line $y=\frac{\sqrt{3}}{2}(x+1)$ is rotated through 2π radians about the y-axis. Using the result in **(ii)**, find the exact volume of revolution formed. [4]

2

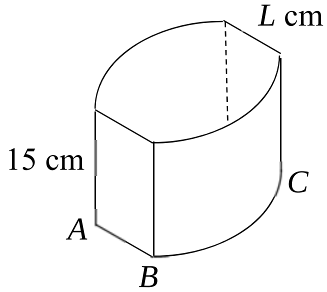


Fig. 1

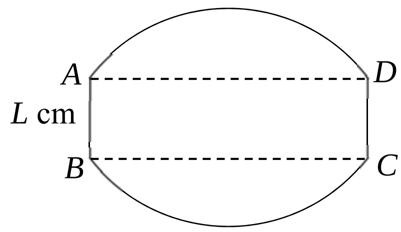


Fig. 2

Fig. 1 shows an open container with vertical sides and a height of 15 cm. Fig. 2 shows the base $ABCD$ of the container which is made up of

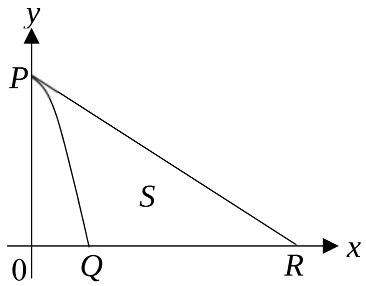
- a rectangle, where $AB = DC = L$ cm,
- and two segments of a circle of radius r, where each of the arc AD and arc BC is $\frac{1}{3}$ of the circumference of a circle with radius r.

Both r and L are variables. Assume the sides and base of the container are of negligible thickness. Show that the area of the base $ABCD$ is

$$\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)r^2+\sqrt{3}\,Lr.$$ [2]

It is given that the volume of the container is 1000 cm^3, and the external surface area of the four vertical sides and the base is to be the least possible value. Use differentiation to find the value of r that would give the least possible external surface area, and justify why it is the least possible. [6]

3



A curve C has parametric equations

$$x = k\cos 3t,\ \ y = k\sin t,\ \text{ where } 0 \le t \le \frac{\pi}{6} \text{ and } k \text{ is a positive constant.}$$

C meets the x-axis at Q and y-axis at P. The tangent to C at the point P meets the x-axis at R. The region bounded by the curve C, the line PR and the x-axis is denoted by S (see diagram).

(i) Find the exact value of $\int_0^{\frac{\pi}{6}} \sin t \sin 3t \; \mathrm{d}t$. [3]

(ii) Find the equation of the line PR, in terms of k, simplifying your answer. [3]

(iii) Hence find the exact area of S, giving your answer in terms of k. [4]

4 **(a)** The complex numbers z and w is such that $z = x + \mathrm{i}y$ and $w = u + \mathrm{i}v$, where x, y, u and v are real numbers

(i) Express $\frac{w}{z}$ in cartesian form $a + \mathrm{i}b$. [2]

(ii) If $\mathrm{Re}\left(\frac{w}{z}\right) = \frac{\mathrm{Re}(w)}{\mathrm{Re}(z)}$ where $\mathrm{Re}(z) \neq 0$. Show that either z is real or $\frac{w}{z}$ is real. [3]

(b) Given $\left(\mathrm{i}e^{\frac{\pi}{3}}z\right)^* = \frac{(1+\mathrm{i})^6}{\left(-1+\mathrm{i}\sqrt{3}\right)^8}$ where z^* is the conjugate of a complex number z.

Without using a calculator, find $|z|$ and $\arg z$ exactly, where $-\pi < \arg z \le \pi$. [5]

(c) The roots of the equation $x^4 - 2x^3 + ax^2 - x + b = 0$, where a and b are non-zero real numbers, are z_1, z_2, z_3 and z_4. It is given that $z_1^{\,2} + z_2^{\,2} + z_3^{\,2} + z_4^{\,2} < 0$. Explain why at most two of z_1, z_2, z_3 and z_4 are real. [2]

Section B: Statistics [60 marks]

5 The management of a supermarket wishes to analyse the effectiveness of an advertising campaign. Before the campaign, the daily takings $\$X$ is normally distributed with mean \$72 300 and standard deviation \$4410.

Immediately after the campaign, a sample of 30 shopping days is taken and the mean daily takings was found to be $\$\bar{x}$. A test is carried out, at the 5% significance level, to determine whether the campaign is effective, assuming that there is no change in the standard deviation of the daily takings after the campaign.

(i) State appropriate hypotheses for the test. [1]

(ii) By stating a necessary assumption, find the set of values of $\bar{x}$, to the nearest dollar, for which the result of the test would be to reject the null hypothesis at 5% level of significance. [4]

(iii) The 30 readings taken on X immediately after the campaign is summarised as follows.

$$\sum(x-70\ 000)=129\ 000$$

By finding $\bar{x}$, state the conclusion of the test, at the 5% significance level, whether the campaign is effective. [2]

6 On average, 40% of Christmas tree light bulbs manufactured by a company are red and the rest are blue. The lights are sold in boxes of 20. Assume that the number of red light bulbs in a box has a binomial distribution.

(i) Find the probability that a box of 20 light bulbs contains fewer red light bulbs than blue light bulbs. [1]

(ii) In n randomly chosen boxes, the probability that there will be at least 2 boxes with fewer red light bulbs than blue is at most 0.999. Find the greatest value of n. [3]

(iii) A random sample of 50 boxes is chosen. Using an approximate distribution, find the probability that the mean number of red light bulbs in 1 box will not exceed 7.5. [3]

(iv) A customer selects boxes of light bulbs at random from a large consignment until she finds a box with fewer red lights than blue. Give a reason why a binomial distribution is not an appropriate model for the number of boxes selected. [1]

7 Ozone, is a colourless gas that is always found in the air that we breathe. It is also the main ingredient of smog, which presents a serious air quality problem in many cities. A research on a city's air quality found that the wind speed, s, will result in a change in the ozone level, y, given below, where s and y are measured in suitable units.

s	7.0	8.0	9.2	11.1	12.7	14.4	19.6	27.7	35.0	39.3
y	49	39	36	34	31	27	24	22	41	20

(i) Draw a scatter diagram for these values. [1]

(ii) Circle the point on the scatter diagram that does not seem to follow the trend and label it as P. Suggest a possible reason for it. [2]

For the remaining parts of this question, you should **omit** point P.

(iii) Suppose that the relationship between s and y are modelled by an equation of the form $y = a - b \ln s$, where a and b are positive constants.

(a) State the product moment correlation coefficient for this model. [1]

(b) Explain, in context, why $y = a - b \ln s$ may not be a good model. [1]

(iv) Use the model $y = c + \frac{d}{s}$, to predict the value of y when $s = 50$. Comment on the reliability of your prediction. [2]

(v) Find the intersection point of the least squares regression line in part **(iv)** with the least squares regression line of $\frac{1}{s}$ on y. [1]

(vi) Explain why the regression line of $\frac{1}{s}$ on y should not be used. [1]

8 An electronic device contains 5 components, two of which are faulty. To isolate the faults, the components are tested one by one in a random order until both the faulty components are identified. The random variable X denotes the number of tests required to locate both the faulty components. Show that $\mathrm{P}(X = 3) = \frac{1}{5}$. [2]

(i) Find the probability distribution of X. [2]

(ii) Find $\mathrm{P}(X_1 = 3 \mid X_1 + X_2 \le 5)$ if X_1 and X_2 are two independent observations of X. [3]

The cost C (in dollars) depends in part on the number of tests required and is given by the formula $C = 10 + 3X$.

(iii) Find Var(C). [3]

9 A shop sells two brands of car battery, A and B. The battery life, in months, of each brand of car battery have independent normal distributions. The means and standard deviations of these distributions, are shown in the following table.

	Mean battery life	Standard deviation
Brand A	30	8
Brand B	25	σ

(i) Explain why the battery life of a randomly chosen Brand B battery cannot be normally distributed if $\sigma = 22$. [1]

(ii) The probability that the battery life of a randomly chosen Brand B battery is within 5 months of the mean battery life of Brand B batteries is 0.8. Find the variance of the distribution of Brand B battery life. [3]

Use $\sigma = 4$ for the rest of the question.

(iii) Sketch the distributions of the battery life of the Brand A battery and the Brand B battery on a single diagram. [1]

(iv) Find the probability that the battery life of a randomly chosen Brand A car battery is exactly 26 months. [1]

(v) Find the probability that the difference between the mean battery life of 3 randomly chosen Brand B batteries and 75% of the battery life of a randomly chosen Brand A battery is less than 3 months. State the parameters of any distribution you use. [4]

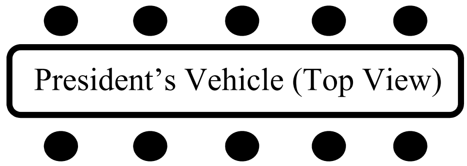
(vi) The manufacturer of Brand B battery replaces for free all batteries that fail within the warranty period of k months. If they are willing to replace for free less than 1% of all batteries sold, find the longest warranty period (to the nearest integer), that the manufacturer can offer. [2]

10 The President's bodyguard unit consists of 40 men. There are 10 experts in firearms, 10 experts in unarmed combat, 10 experts in tactical driving and 10 experts in first aid. Each man is an expert in only one area. In preparation for a historical summit, a team of 10 bodyguards is to be chosen from the 40 men.

(i) Find the number of ways in which the team can be chosen, if there are at least 2 experts chosen from each area. [3]

The selected team comprises of 3 experts in firearms, 3 experts in unarmed combat, 2 experts in tactical driving and 2 experts in first aid.

(ii) These 10 bodyguards are to walk alongside the president's vehicle in a tactical formation as shown below.



Find the number of ways the 10 bodyguards can stand, if the 2 experts in first aid must not be standing next to each other on the same side of the vehicle. [3]

(iii) The ten bodyguards are to stand at random in a circular formation surrounding the president as he walks. Find the probability that the 3 experts in firearms are separated from each other. [3]

Each of the bodyguard is wearing a surveillance earpiece whose electrical circuit is controlled by two relay switches, A and B. The probability that switch A fails is 0.1 and the probability that switch B fails is 0.23. The probability that both switches do not fail is 0.7.

(iv) Find the probability that only switch B fails. [2]

A third relay switch C is added to the electrical circuit. The event that switch C fails is independent of the event that switch A fails. The probability that switch C fails given that switch A has failed is 0.15.

(v) Find the probability that switch C fails but not A. [1]

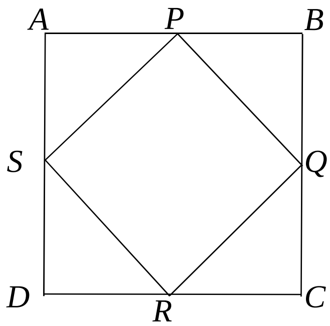
(vi) Hence find the maximum probability that, switches B and C fail but not A. [2]

Anglo-Chinese Junior College
2018 H2 Mathematics Prelim Paper 1 Solution

Qn	Solution	Remarks
1	(i) $OP_n = 2+3(n-1) = 3n-1$ $P_1P_n = 3(n-1) = 3n-3$ Total distance $= OP_n + P_1P_n + P_2P_n + P_3P_n + ... + P_{n-1}P_n$ $= (3n-1)+(3n-3)+(3n-6)$ $+(3n-9)+...+3$ $= (3n-1)+\frac{n-1}{2}((3n-3)+3)$ $= (3n-1)+\frac{(n-1)(3n)}{2}$ $= 3n-1+\frac{3}{2}n^2-\frac{3}{2}n$ $= \frac{3}{2}n^2+\frac{3}{2}n-1$ (*shown*)	
2	(i) $S_n = \sum_{r=1}^{n}\left(\frac{r}{2^{r-1}}\right) = \frac{1}{2^0}+\frac{2}{2^1}+\frac{3}{2^2}+...+\frac{n-1}{2^{n-2}}+\frac{n}{2^{n-1}}$ $T_{n-1} = \sum_{r=1}^{n-1}\left(\frac{r}{2^r}\right) = \frac{1}{2^1}+\frac{2}{2^2}+\frac{3}{2^3}+...+\frac{n-1}{2^{n-1}}$ $\therefore\ S_n - T_{n-1} = \frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-1}}$ $S_n - T_{n-1} = \frac{1}{2^0}+\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^{n-1}}$ $= \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$ $= 2\left[1-\frac{1}{2^n}\right] = 2-\left(\frac{1}{2}\right)^{n-1}$ (ii) As $n \to \infty$, $\left(\frac{1}{2}\right)^{n-1} \to 0$, $\therefore S_n - T_{n-1} \to 2$	

3 (a)

$\overrightarrow{OP} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$



$\overrightarrow{OQ} = \dfrac{\overrightarrow{OB} + \overrightarrow{OC}}{2}$

$\overrightarrow{OR} = \dfrac{\overrightarrow{OC} + \overrightarrow{OD}}{2}$

$\overrightarrow{OS} = \dfrac{\overrightarrow{OD} + \overrightarrow{OA}}{2}$

$\overrightarrow{PQ} = \dfrac{\overrightarrow{OB} + \overrightarrow{OC}}{2} - \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2} = \dfrac{\overrightarrow{OC} - \overrightarrow{OA}}{2}$

$\overrightarrow{SR} = \dfrac{\overrightarrow{OC} + \overrightarrow{OD}}{2} - \dfrac{\overrightarrow{OD} + \overrightarrow{OA}}{2} = \dfrac{\overrightarrow{OC} - \overrightarrow{OA}}{2} = \overrightarrow{PQ}$

Since $\overrightarrow{SR} = \overrightarrow{PQ}$, $\therefore PQRS$ is a parallelogram.

Alternatively,

$\overrightarrow{SP} = \dfrac{\overrightarrow{OA} + \overrightarrow{OB}}{2} - \dfrac{\overrightarrow{OD} + \overrightarrow{OA}}{2} = \dfrac{\overrightarrow{OB} - \overrightarrow{OD}}{2}$

$\overrightarrow{RQ} = \dfrac{\overrightarrow{OB} + \overrightarrow{OC}}{2} - \dfrac{\overrightarrow{OC} + \overrightarrow{OD}}{2} = \dfrac{\overrightarrow{OB} - \overrightarrow{OD}}{2} = \overrightarrow{SP}$

Since $\overrightarrow{RQ} = \overrightarrow{SP}$, $\therefore PQRS$ is a parallelogram.

(b)

$(3\mathbf{a} + \mathbf{c}) \cdot (3\mathbf{a} + \mathbf{c}) = 9\mathbf{a} \cdot \mathbf{a} + 3\mathbf{a} \cdot \mathbf{c} + 3\mathbf{c} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{c}$

$(3\mathbf{a} + \mathbf{c}) \cdot (3\mathbf{a} + \mathbf{c}) = 9\mathbf{a} \cdot \mathbf{a} + 6\mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}$

$k\mathbf{b} \cdot k\mathbf{b} = 9\mathbf{a} \cdot \mathbf{a} + 6\mathbf{a} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{c}$

$k^2 |\mathbf{b}|^2 = 9|\mathbf{a}|^2 + 6|\mathbf{a}||\mathbf{c}|\cos\dfrac{\pi}{6} + |\mathbf{c}|^2$

$k^2 |\mathbf{b}|^2 = 9(1) + 6(1)(\sqrt{3})\cos\dfrac{\pi}{6} + (\sqrt{3})^2$

$k^2 (3)^2 = 21$

$k = \pm\dfrac{\sqrt{21}}{3}$

4	$AC^2 = AB^2 + BC^2 - 2AB\times BC\cos\left(\frac{\pi}{3}-\theta\right)$ $= 2^2 + 3^2 - 2\times 2\times 3\cos\left(\frac{\pi}{3}-\theta\right)$ $= 4+9-12\left(\cos\frac{\pi}{3}\cos\theta + \sin\frac{\pi}{3}\sin\theta\right)$ $= 13-12\left(\frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$ $\approx 13-6\left(1-\frac{1}{2}\theta^2+\sqrt{3}\,\theta\right)$ $= 7-6\sqrt{3}\,\theta+3\theta^2$ $AC \approx \left(7-6\sqrt{3}\,\theta+3\theta^2\right)^{\frac{1}{2}}$ (*shown*) $\left(7-6\sqrt{3}\,\theta+3\theta^2\right)^{\frac{1}{2}}$ $= \sqrt{7}\left(1-\frac{6\sqrt{3}}{7}\theta+\frac{3}{7}\theta^2\right)^{\frac{1}{2}}$ $\approx \sqrt{7}\left(1+\frac{1}{2}\left(-\frac{6\sqrt{3}}{7}\theta+\frac{3}{7}\theta^2\right)+\frac{\frac{1}{2}.-\frac{1}{2}}{\frac{1}{2}}\left(-\frac{6\sqrt{3}}{7}\theta+\frac{3}{7}\theta^2\right)^2\right)$ $= \sqrt{7}\left(1-\frac{3\sqrt{3}}{7}\theta+\frac{3}{14}\theta^2-\frac{1}{8}\left(\frac{36\times 3}{49}\theta^2\right)\right)$ $= \sqrt{7}\left(1-\frac{3\sqrt{3}}{7}\theta+\frac{3}{14}\theta^2-\frac{27}{98}\theta^2\right)$ $= \sqrt{7}\left(1-\frac{3\sqrt{3}}{7}\theta-\frac{6}{98}\theta^2\right) = \sqrt{7}\left(1-\frac{3\sqrt{3}}{7}\theta-\frac{3}{49}\theta^2\right)$ $= \sqrt{7}-\frac{3\sqrt{21}}{7}\theta-\frac{3\sqrt{7}}{49}\theta^2$ *or* $\sqrt{7}-\frac{3\sqrt{3}\sqrt{7}}{7}\theta-\frac{3\sqrt{7}}{49}\theta^2$ $\therefore\ AC \approx \sqrt{7}-\frac{3\sqrt{21}}{7}\theta-\frac{3\sqrt{7}}{49}\theta^2$ *or* $\sqrt{7}-\frac{3\sqrt{3}\sqrt{7}}{7}\theta-\frac{3\sqrt{7}}{49}\theta^2$ $a=\sqrt{7},\ b=-\frac{3\sqrt{21}}{7},\ c=-\frac{3\sqrt{7}}{49}$	

5	(i) $a = 5$ 5 has to be excluded from the domain of f as it does not have an image under f, which will then mean that f is not a function. ii) Let $y = \frac{5x-3}{x-5}$. $y = \frac{5x-3}{x-5}$ $y(x-5) = 5x-3$ $yx - 5y = 5x - 3$ $yx - 5x = 5y - 3$ $x = \frac{5y-3}{y-5}$ $f^{-1}(x) = \frac{5x-3}{x-5} = f(x)$ $\therefore$ f is self-inverse. (iii) Since f is self-inverse, $f^{-1}(x) = f(x)$. $\therefore f^4(x) = ff[f^2(x)]$ $= ff[ff^{-1}(x)]$ $= ff[x]$ $= x$ $f^4(b) - 2 = f^{-1}(b)$ $b - 2 = f(b)$ $b - 2 = \frac{5b-3}{b-5}$ $b^2 - 7b + 10 = 5b - 3$ $b^2 - 12b + 13 = 0$ $b = 6 + \sqrt{23}$ or $b = 6 - \sqrt{23}$ (iv) Take the $R_g = [10, \infty)$ as the restricted domain of f and read off the corresponding range. $[e^{10}, \infty) \xrightarrow{g} [10, \infty) \xrightarrow{f} (5, \frac{47}{5}]$ $R_{fg} = (5, \frac{47}{5}]$	

6	(a) Let $u=\sin^{-1}2x$, $\dfrac{dv}{dx}=\dfrac{x}{\sqrt{1-4x^2}}=x\left(1-4x^2\right)^{-\frac{1}{2}}$ $\dfrac{du}{dx}=\dfrac{2}{\sqrt{1-4x^2}}$, $v=\int x\left(1-4x^2\right)^{-\frac{1}{2}}=-\dfrac{1}{8}\int -8x\left(1-4x^2\right)^{-\frac{1}{2}}dx$ $=-\dfrac{1}{4}\sqrt{1-4x^2}+C$ $\int \sin^{-1}2x\dfrac{x}{\sqrt{1-4x^2}}\,\mathrm{d}x$ $=\left[\left(\sin^{-1}2x\right)\left(-\dfrac{1}{4}\sqrt{1-4x^2}\right)\right]-\int\left(-\dfrac{1}{4}\sqrt{1-4x^2}\right)\left(\dfrac{2}{\sqrt{1-4x^2}}\right)\mathrm{d}x$ $=\left[-\dfrac{1}{4}\left(\sin^{-1}2x\right)\sqrt{1-4x^2}\right]+\int\dfrac{1}{2}\mathrm{d}x$ $=\left[-\dfrac{1}{4}\left(\sin^{-1}2x\right)\sqrt{1-4x^2}\right]+\dfrac{1}{2}x+C$ (b) $\int\dfrac{x-1}{x^2+2x+6}\,\mathrm{d}x$ $=\dfrac{1}{2}\int\dfrac{2x+2}{x^2+2x+6}\,\mathrm{d}x-\int\dfrac{2}{\left(x+1\right)^2+5}\,\mathrm{d}x$ $=\dfrac{1}{2}\ln\left\|x^2+2x+6\right\|-\dfrac{2}{\sqrt{5}}\tan^{-1}\left(\dfrac{x+1}{\sqrt{5}}\right)+C$	
7	(i) $\dfrac{1}{x\left(x+1\right)}=\dfrac{1}{x}-\dfrac{1}{x+1}$ (ii) Total area of rectangles $=\dfrac{1}{2\left(2+1\right)}+\dfrac{1}{3\left(3+1\right)}+...+\dfrac{1}{n\left(n+1\right)}$ $=\sum_{x=2}^{n}\dfrac{1}{x\left(x+1\right)}$ so $a=2$, $b=n$	

$$= \sum_{x=2}^{n} \left(\frac{1}{x} - \frac{1}{x+1} \right)$$

$$= \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \frac{1}{2} - \frac{1}{n+1}$$

Actual area $= \int_1^n \frac{1}{x(x+1)} \, \mathrm{d}x = \int_1^n \frac{1}{x} - \frac{1}{(x+1)} \, \mathrm{d}x$

$$= [\ln x - \ln(x+1)]_1^n$$
$$= \ln n - \ln(n+1) - \ln 1 + \ln 2$$
$$= \ln n - \ln(n+1) + \ln 2$$

Area of rectangles < actual area

$$\therefore \quad \frac{1}{2} - \frac{1}{n+1} < \ln n - \ln(n+1) + \ln 2$$

$$\frac{1}{2} - \ln 2 < \frac{1}{n+1} + \ln\left(\frac{n}{n+1} \right)$$

$$\frac{1}{2} - \ln 2 < \frac{1}{n+1} + \ln\left(1 - \frac{1}{n+1} \right) \quad (\textit{shown})$$

Using MF26,

$$\ln\left(1 - \frac{1}{n+1} \right) = -\frac{1}{n+1} - \frac{1}{2}\left(\frac{1}{n+1} \right)^2 - \ldots - \frac{1}{r}\left(\frac{1}{n+1} \right)^r - \ldots$$

$$\therefore \frac{1}{2} - \ln 2 < \frac{1}{n+1} - \frac{1}{n+1} - \frac{1}{2}\left(\frac{1}{n+1} \right)^2 - \ldots - \frac{1}{r}\left(\frac{1}{n+1} \right)^r - \ldots$$

$$\therefore \frac{1}{2} - \ln 2 < -\frac{1}{2}\frac{1}{(n+1)^2} - \ldots - \frac{1}{r}\frac{1}{(n+1)^r} - \ldots$$

$$\Rightarrow \frac{1}{2} - \ln 2 < \sum_{r=2}^{\infty} \frac{-1}{r(n+1)^r} \quad (\textit{shown})$$

8 (a)

$$\frac{x+5}{x-1} - \frac{9}{x} > 0$$

$$\frac{x^2 - 4x + 9}{(x-1)x} > 0$$

$$\frac{(x-2)^2 + 5}{(x-1)x} > 0$$

Since $(x-2)^2 + 5 > 0$ for all real values of x,

$(x-1)x > 0$

$x < 0$ or $x > 1$

Using previous result, replace x by e^x.

$e^x < 0$ (rejected) or $e^x > 1$

$\therefore x > 0$

(b)

$y = \dfrac{1+kx}{x^2-1}$

$yx^2 - y = 1 + kx$

$yx^2 - kx - (y+1) = 0$

For discriminant ≥ 0

$(-k)^2 - 4y(-y-1) \geq 0$

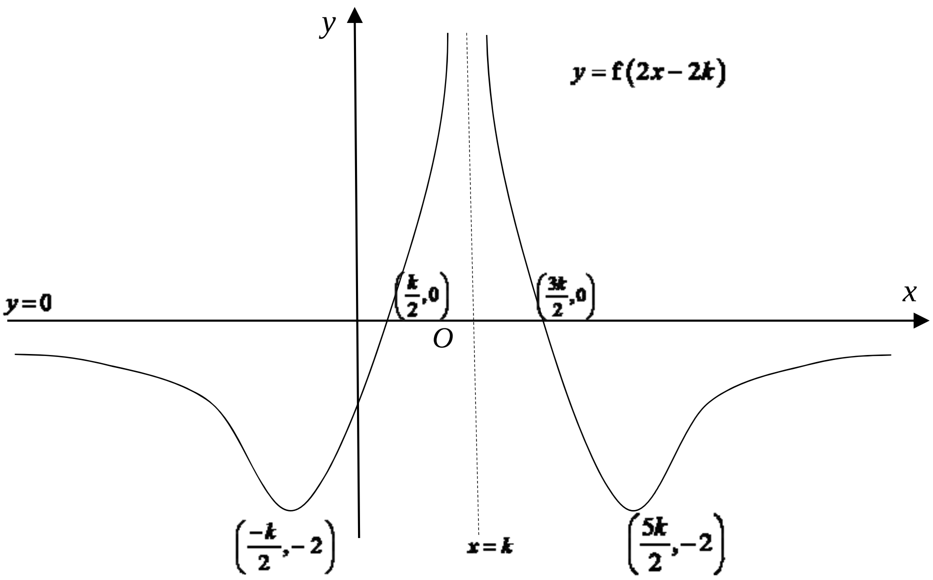
$4y^2 + 4y + k^2 \geq 0$

Consider $4y^2 + 4y + k^2 = 0$,

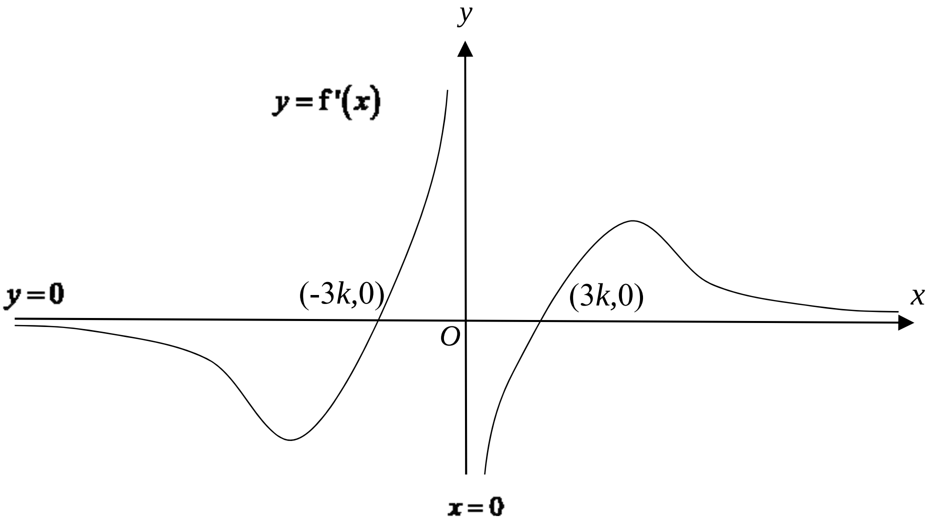
$y = \dfrac{-1 \pm \sqrt{1-k^2}}{2}$

$\therefore y \leq \dfrac{-1-\sqrt{1-k^2}}{2}$ or $y \geq \dfrac{-1+\sqrt{1-k^2}}{2}$

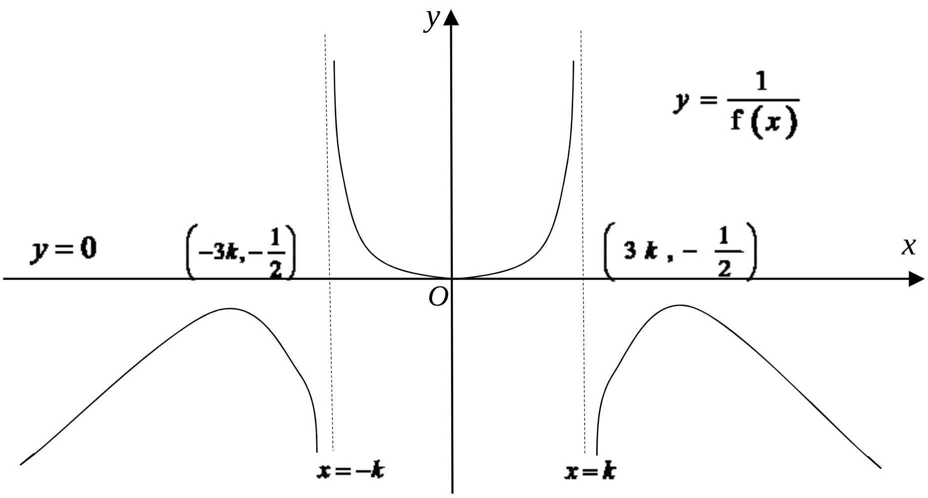
9 (a)(i)



(a)(ii)



(a)(iii)



(b)(i)

$$y = \frac{3x^2 + 9x + 7}{x + 2}$$

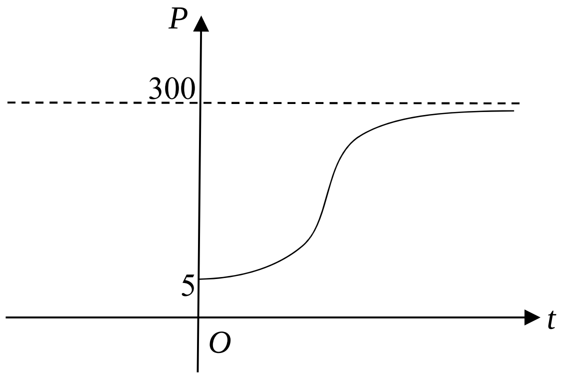
$$= 3x + 3 + \frac{1}{x + 2}$$

$$\begin{array}{r|l} & 3x+3 \\ \hline x+2 & 3x^2+9x+7 \\ & \underline{3x^2+6x} \\ & \quad\quad 3x+7 \\ & \quad\quad \underline{3x+6} \\ & \quad\quad\quad\quad 1 \end{array}$$

$\therefore a = 3,\ \ b = 1$

The equations of the asymptotes of *C* are $x = -2$ and $y = 3x + 3$.

	(b)(ii) <ins>Method 1</ins> Translation 2 units in the positive x-direction Then translation 3 units in the positive y-direction. <ins>Method 2</ins> Translation 3 units in the positive y-direction. Then translation 2 units in the positive x-direction	
10	(a) Let a, b and c be the number of patients admitted to wards A, B and C respectively. $20a+35b+38c=14150$ $25a+37b+40c=16300$ $30a+62b+65c=23050$ By GC, $a=350,\ b=150,\ c=50$ $k=\dfrac{20600-30a-61c}{b}=47$ (b)(i) $\dfrac{\mathrm{d}P}{\mathrm{d}t}=0.9P\left(1-\dfrac{P}{300}\right)$ $\dfrac{\mathrm{d}P}{\mathrm{d}t}=0.003P(300-P)$ $\int \dfrac{1}{P(300-P)}\ \mathrm{d}P=\int 0.003\ \mathrm{d}t$ $\int \dfrac{1}{P}+\dfrac{1}{(300-P)}\ \mathrm{d}P=\int 0.9\ \mathrm{d}t$ $\ln\left\|\dfrac{P}{300-P}\right\|=0.9t+C$ $\dfrac{P}{300-P}=Ae^{0.9t}$, where $A=\pm e^{C}$ $P=300Ae^{0.9t}-PAe^{0.9t}$ $P+PAe^{0.9t}=300Ae^{0.9t}$ $P\left(1+Ae^{0.9t}\right)=300Ae^{0.9t}$ $P=\dfrac{300Ae^{0.9t}}{1+Ae^{0.9t}}$ $P=\dfrac{300}{Be^{-0.9t}+1}$, where $B=\dfrac{1}{A}$	

10	when $t=0,\ P=5,$ $5=\dfrac{300}{Be^{0}+1}$ $B=\dfrac{300-5}{5}$ $B=59$ $\therefore\ P=\dfrac{300}{59e^{-0.9t}+1}$ (b)(ii)  (b)(iii) Since P does not exceed 300 in the long run, Lion City Hospital can be used solely as the quarantine centre as it has 350 beds. (b)(iv) Let $w=0.9P\left(1-\dfrac{P}{300}\right)$ Solve $\dfrac{\mathrm{d}w}{\mathrm{d}P}=0$ $0.9\left(1-\dfrac{P}{300}\right)+0.9P\left(-\dfrac{1}{300}\right)=0$ $P=150$	
11	(i) $\overrightarrow{OQ}=\begin{pmatrix}8\\1\\8\end{pmatrix},\quad \overrightarrow{OR}=\begin{pmatrix}8\\7\\2\end{pmatrix},\quad \overrightarrow{OS}=\begin{pmatrix}2\\1\\2\end{pmatrix}$ $\overrightarrow{QR}=\overrightarrow{OR}-\overrightarrow{OQ}=\begin{pmatrix}8\\7\\2\end{pmatrix}-\begin{pmatrix}8\\1\\8\end{pmatrix}=\begin{pmatrix}0\\6\\-6\end{pmatrix}=6\begin{pmatrix}0\\1\\-1\end{pmatrix}$ $\overrightarrow{QS}=\overrightarrow{OS}-\overrightarrow{OQ}=\begin{pmatrix}2\\1\\2\end{pmatrix}-\begin{pmatrix}8\\1\\8\end{pmatrix}=\begin{pmatrix}-6\\0\\-6\end{pmatrix}=6\begin{pmatrix}-1\\0\\-1\end{pmatrix}$	

11	$n=\begin{pmatrix}0\\1\\-1\end{pmatrix}\times\begin{pmatrix}-1\\0\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\\1\end{pmatrix}$ $r\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\1\\2\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=1$ The cartesian equation of the plane p is $-x+y+z=1$ (ii) Midpoint of $\overrightarrow{QR}=\dfrac{\overrightarrow{OR}+\overrightarrow{OQ}}{2}=\dfrac{\begin{pmatrix}8\\7\\2\end{pmatrix}+\begin{pmatrix}8\\1\\8\end{pmatrix}}{2}=\begin{pmatrix}8\\4\\5\end{pmatrix}$ $r\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}8\\4\\5\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=-1$ The cartesian equation of the plane p_1 is $y-z=-1$ (iii) The vector equation of the line l is $r=\begin{pmatrix}10\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-1\\1\\1\end{pmatrix}$ where $\lambda\in\mathbb{R}$	

11	(iv) Since point T is on the line l, $\overrightarrow{OT} = \begin{pmatrix} 10-\lambda \\ -1+\lambda \\ \lambda \end{pmatrix}$ $\left\|\overrightarrow{QR}\right\| = \left\|\overrightarrow{QT}\right\|$ $\left\|\begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 10-\lambda \\ -1+\lambda \\ \lambda \end{pmatrix} - \begin{pmatrix} 8 \\ 1 \\ 8 \end{pmatrix}\right\|$ $\left\|\begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix}\right\| = \left\|\begin{pmatrix} 2-\lambda \\ -2+\lambda \\ \lambda-8 \end{pmatrix}\right\|$ $\sqrt{72} = \sqrt{(2-\lambda)^2 + (-2+\lambda)^2 + (\lambda-8)^2}$ $72 = 3\lambda^2 - 24\lambda + 72$ $\lambda = 0 \quad \text{or} \quad \lambda = 8$ When $\lambda = 8$, $\overrightarrow{OT} = \begin{pmatrix} 10-8 \\ -1+8 \\ 8 \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix}$ When $\lambda = 0$, $\overrightarrow{OT} = \begin{pmatrix} 10-0 \\ -1+0 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -1 \\ 0 \end{pmatrix}$ *Method 1* $\dfrac{\begin{pmatrix} 2 \\ 7 \\ 8 \end{pmatrix} + \begin{pmatrix} 10 \\ -1 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} 6 \\ 3 \\ 4 \end{pmatrix}$ The coordinates of the point on the plane p that is closest to point T is (6, 3, 4).	

Method 2

The vector equation of the line l through point T perpendicular to plane p is

$$r=\begin{pmatrix}2\\7\\8\end{pmatrix}+\mu\begin{pmatrix}-1\\1\\1\end{pmatrix} \text{ where } \mu\in\mathbb{R}$$

$$\begin{pmatrix}2-\mu\\7+\mu\\8+\mu\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}=1$$

$$-2+\mu+7+\mu+8+\mu=1$$

$$3\mu=-12$$

$$\mu=-4$$

When $\mu=-4$

$$r=\begin{pmatrix}2\\7\\8\end{pmatrix}+(-4)\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}6\\3\\4\end{pmatrix}$$

The coordinates of the point on the plane p that is closest to point T is (6, 3, 4).

(v)

$$\overrightarrow{OV}=\frac{3\begin{pmatrix}6\\3\\4\end{pmatrix}+\begin{pmatrix}2\\7\\8\end{pmatrix}}{4}=\begin{pmatrix}5\\4\\5\end{pmatrix}$$

The coordinates of V is (5, 4, 5).

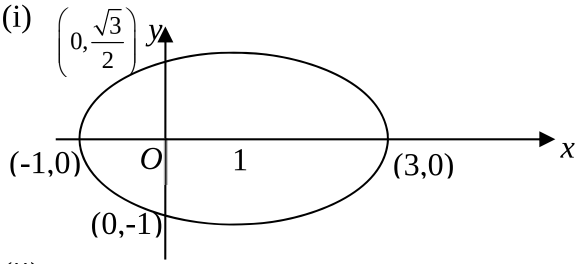
$$\overrightarrow{OQ}=\begin{pmatrix}8\\1\\8\end{pmatrix},\quad \overrightarrow{OV}=\begin{pmatrix}5\\4\\5\end{pmatrix},\quad \overrightarrow{OT}=\begin{pmatrix}2\\7\\8\end{pmatrix}$$

$$\overrightarrow{VQ}=\overrightarrow{OV}-\overrightarrow{OQ}=\begin{pmatrix}5\\4\\5\end{pmatrix}-\begin{pmatrix}8\\1\\8\end{pmatrix}=\begin{pmatrix}-3\\3\\-3\end{pmatrix}$$

$$\overrightarrow{VT}=\overrightarrow{OT}-\overrightarrow{OV}=\begin{pmatrix}5\\4\\5\end{pmatrix}-\begin{pmatrix}2\\7\\8\end{pmatrix}=\begin{pmatrix}3\\-3\\-3\end{pmatrix}$$

	$\angle TVQ=\cos^{-1}\left[\dfrac{\begin{pmatrix}-3\\3\\-3\end{pmatrix}\mathrm{g}\begin{pmatrix}3\\-3\\-3\end{pmatrix}}{\sqrt{(-3)^2+3^2+(-3)^2}\sqrt{3^2+(-3)^2+(-3)^2}}\right]$ $=\cos^{-1}\left(\dfrac{-9-9+9}{27}\right)=\cos^{-1}\left(\dfrac{-9}{27}\right)=109.471^\circ=109.5^\circ$ (1 decimal place)	

Anglo-Chinese Junior College
2018 H2 Mathematics Prelim Paper 2 Solution

Qn	Solution	Remarks
1	(i)  (ii) $\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-x^2}\,\mathrm{d}x$ $x=\sin\theta$, $\frac{dx}{d\theta}=\cos\theta$ $=\int_0^{\frac{\pi}{3}} \sqrt{1-\sin\theta^2}\cos\theta\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{3}} \cos^2\theta\,\mathrm{d}\theta$ $=\int_0^{\frac{\pi}{3}} \frac{\cos 2\theta+1}{2}\,\mathrm{d}\theta$ $=\frac{\sin\frac{2\pi}{3}}{4}+\frac{\pi}{6}$ $=\frac{\sqrt{3}}{8}+\frac{\pi}{6}$ Using $x=\sin\theta$, when $x=0$, $\sin\theta=0 \Rightarrow \theta=0$ when $x=\frac{\sqrt{3}}{2}$, $\sin\theta=\frac{\sqrt{3}}{2}\Rightarrow\theta=\frac{\pi}{3}$ (since $0\le\theta\le\frac{\pi}{3}$) (iii) Required volume $=\pi\int_0^{\frac{\sqrt{3}}{2}} x^2\,\mathrm{d}y - \left(\frac{1}{3}\right)(\pi)(1)^2\left(\frac{\sqrt{3}}{2}\right)$ $=\pi\int_0^{\frac{\sqrt{3}}{2}} x^2\,\mathrm{d}y - \frac{\pi\sqrt{3}}{6}$ $=\pi\int_0^{\frac{\sqrt{3}}{2}} \left(1-\sqrt{4(1-y^2)}\right)^2\mathrm{d}y - \frac{\pi\sqrt{3}}{6}$ $=\pi\int_0^{\frac{\sqrt{3}}{2}} 1-2\sqrt{4(1-y^2)}+4(1-y^2)\,\mathrm{d}y - \frac{\pi\sqrt{3}}{6}$ $=\pi\int_0^{\frac{\sqrt{3}}{2}} 5-4y^2-4\sqrt{(1-y^2)}\,\mathrm{d}y - \frac{\pi\sqrt{3}}{6}$ $=\pi\left[5y-\frac{4y^3}{3}\right]_0^{\frac{\sqrt{3}}{2}} -4\pi\left[\frac{\sqrt{3}}{8}+\frac{\pi}{6}\right] - \frac{\pi\sqrt{3}}{6}$ $=\pi\left[5\left(\frac{\sqrt{3}}{2}\right)-\frac{4\left(\frac{\sqrt{3}}{2}\right)^3}{3}\right]-\left[\frac{\sqrt{3}}{2}\pi+\frac{2\pi^2}{3}\right]-\frac{\pi\sqrt{3}}{6}$ = $=\frac{5\sqrt{3}}{2}\pi-\frac{\sqrt{3}}{2}\pi-\frac{\sqrt{3}}{2}\pi-\frac{2\pi^2}{3}-\frac{\pi\sqrt{3}}{6}$ $=\frac{14\sqrt{3}}{6}\pi-\pi\sqrt{3}-\frac{2\pi^2}{3}$ $=\frac{4\pi\sqrt{3}}{3}-\frac{2\pi^2}{3}$	

2	Area between arc AD and line AD $=\frac{1}{3}\pi r^2-\frac{1}{2}r^2\sin 120°=\frac{1}{3}\pi r^2-\frac{1}{2}r^2\left(\frac{\sqrt{3}}{2}\right)=\frac{1}{3}\pi r^2-\frac{\sqrt{3}}{4}r^2$ Length of line $AD=2\times r\sin 60°=\sqrt{3}\,r$ Area of base $ABCD=2\left(\frac{1}{3}\pi r^2-\frac{\sqrt{3}}{4}r^2\right)+\sqrt{3}\,rL$ $=\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)r^2+\sqrt{3}\,Lr$ (shown) Volume $=15\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)r^2+15\sqrt{3}\,Lr=1000$ $15\sqrt{3}\,Lr=1000-15\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)r^2$ $\sqrt{3}\,Lr=\frac{200}{3}-\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)r^2$ *or* $L=\frac{200}{3\sqrt{3}\,r}-\left(\frac{2\pi}{3\sqrt{3}}-\frac{1}{2}\right)r$ Perimeter of base $ABCD$ = 2 × arc AD + 2 AB $=2\left(\frac{1}{3}\times 2\pi r\right)+2L=\frac{4\pi}{3}r+2L$ Area of 4 vertical sides $=\left(\frac{4\pi}{3}r+2L\right)\times 15=20\pi r+30L$ Required area $=A=\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)r^2+\sqrt{3}\,Lr+(20\pi r+30L)$ $=\left(\frac{2\pi}{3}r^2-\frac{\sqrt{3}}{2}r^2\right)+\left(\frac{200}{3}-\frac{2\pi}{3}r^2+\frac{\sqrt{3}}{2}r^2\right)$ $+20\pi r+\frac{30}{\sqrt{3}\,r}\left(\frac{200}{3}-\frac{2\pi}{3}r^2+\frac{\sqrt{3}}{2}r^2\right)$ $=\frac{200}{3}+20\pi r+\frac{2000}{\sqrt{3}\,r}-\frac{20\pi}{\sqrt{3}}r+15r$ $\frac{\mathrm{d}A}{\mathrm{d}r}=20\pi-\frac{2000}{\sqrt{3}\,r^2}-\frac{20\pi}{\sqrt{3}}+15=0$ $\frac{2000}{\sqrt{3}\,r^2}=20\pi-\frac{20\pi}{\sqrt{3}}+15$ $r^2=\frac{2000}{\sqrt{3}\left(20\pi-\frac{20\pi}{\sqrt{3}}+15\right)}$ r = 5.271 3091 = 5.27 (3 sf)	

Note: L = 3.56 (3 sf) and A = area = 504.77 (5 sf)

Using GC, $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=15.8>0$. This area is the least possible.

Or, $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=\frac{4000}{\sqrt{3}\,r^3}=15.8>0$. This area is the least possible.

Or,

r	5.2	5.27	5.3
$\frac{\mathrm{d}A}{\mathrm{d}r}$			

Thus this area is the least possible.

3 (i)

$$\int_0^{\frac{\pi}{6}}(\sin t)(\sin 3t)\ \mathrm{d}t$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{6}}\cos 4t-\cos 2t\ \mathrm{d}t=-\frac{1}{2}\left[\frac{\sin 4t}{4}-\frac{\sin 2t}{2}\right]_0^{\frac{\pi}{6}}$$

$$=-\frac{1}{2}\left[\frac{\sin\left(\frac{2\pi}{3}\right)}{4}-\frac{\sin\left(\frac{\pi}{3}\right)}{2}-0\right]=-\frac{1}{2}\left[\frac{\left(\frac{\sqrt{3}}{2}\right)}{4}-\frac{\left(\frac{\sqrt{3}}{2}\right)}{2}\right]=\frac{\sqrt{3}}{16}$$

(ii)

$x=k\cos 3t$

$y=k\sin t$

At P, $x=0\Rightarrow t=\frac{\pi}{6}\Rightarrow y=\frac{k}{2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{k\cos t}{-3k\sin 3t}=\frac{\cos t}{-3\sin 3t}$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\frac{\pi}{6}}=\frac{\sqrt{3}}{-6}$$

$\Rightarrow$ equation of tangent at P :

$$y-\frac{k}{2}=\frac{\sqrt{3}}{-6}(x-0)$$

$$y=-\frac{\sqrt{3}}{6}x+\frac{k}{2}$$

	(iii) Area of S $=\left(\frac{1}{2}\right)\left(k\sqrt{3}\right)\left(\frac{k}{2}\right)-\int_0^k y\ \mathrm{d}x$ $=\frac{k^2\sqrt{3}}{4}-\int_{\frac{\pi}{6}}^{0}\left(k\sin t\right)\left(-3k\sin 3t\right)\ \mathrm{d}t$ $= =\frac{k^2\sqrt{3}}{4}-3k^2\int_0^{\frac{\pi}{6}}\left(\sin t\right)\left(\sin 3t\right)\ \mathrm{d}t$ $=\frac{k^2\sqrt{3}}{4}-3k^2\left(\frac{\sqrt{3}}{16}\right)$ $=\frac{k^2\sqrt{3}}{16}$	
4	(a) (i) $\frac{w}{z}=\frac{u+iv}{x+iy}=\frac{u+iv}{x+iy}\times\frac{x-iy}{x-iy}$ $=\frac{(ux+vy)+i(vx-uy)}{x^2+y^2}$ $=\frac{(ux+vy)}{x^2+y^2}+i\frac{(vx-uy)}{x^2+y^2}$ (a) (ii) $\mathrm{Re}\left(\frac{w}{z}\right)=\frac{\mathrm{Re}(w)}{\mathrm{Re}(z)}$ Hence $\frac{ux+vy}{x^2+y^2}=\frac{u}{x}$ $ux^2+vxy=ux^2+uy^2$ $\Rightarrow y(vx-uy)=0$ $\Rightarrow y=0$ or $vx-uy=0$ $\Rightarrow \mathrm{Im}(z)=0$ or $\mathrm{Im}\left(\frac{w}{z}\right)=0$ $\Rightarrow z$ is real or $\frac{w}{z}$ is real	

(b)

Method 1

$|1+i|=\sqrt{2}$, arg (1 + i) = $\frac{\pi}{4}$

$\left|-1+i\sqrt{3}\right|=2$, $\arg(-1+i\sqrt{3}) = \pi-\tan^{-1}\sqrt{3}=\frac{2\pi}{3}$

$$\left|\frac{(1+i)^6}{(-1+i\sqrt{3})^8}\right| = \frac{|1+i|^6}{\left|-1+i\sqrt{3}\right|^8} = \frac{\left(\sqrt{2}\right)^6}{2^8} = \frac{1}{2^5}$$

$$\text{Arg}\left(\frac{(1+i)^6}{\left(-1+i\sqrt{3}\right)^8}\right) = 6\arg(1+i) - 8\arg(-1+i\sqrt{3})$$

$$= 6\left(\frac{\pi}{4}\right) - 8\left(\frac{2\pi}{3}\right)$$

$$= -\frac{23\pi}{6} = \frac{\pi}{6} \text{ (adjusted)}$$

$$\left(ie^{\frac{\pi}{3}}z\right)^* = -ie^{\frac{\pi}{3}}z^*$$

$$\left|-ie^{\frac{\pi}{3}}z^*\right| = e^{\frac{\pi}{3}}|z|$$

Since $\left|-ie^{\frac{\pi}{3}}z^*\right| = \left|\frac{(1+i)^6}{(-1+i\sqrt{3})^8}\right|$,

$$e^{\frac{\pi}{3}}|z| = \frac{1}{2^5}$$

$$\therefore |z| = \frac{1}{e^{\frac{\pi}{3}}2^5}$$

$$\arg(-ie^{\frac{\pi}{3}}z^*) = -\frac{\pi}{2} + \arg(z^*) = -\frac{\pi}{2} - \arg(z)$$

Since $\arg(-ie^{\frac{\pi}{3}}z^*) = \text{Arg}\left(\frac{(1+i)^6}{\left(-1+i\sqrt{3}\right)^8}\right)$,

$$\therefore -\frac{\pi}{2} - \arg(z) = \frac{\pi}{6}$$

Hence arg z = $-\frac{2\pi}{3}$

	(b) **Method 2** $\|1+i\|=\sqrt{2}$, arg $(1+i)=\frac{\pi}{4}$ $\|-1+i\sqrt{3}\|=\sqrt{1+3}=2$, $\arg(-1+i\sqrt{3})=\pi-\tan^{-1}\sqrt{3}=\frac{2\pi}{3}$ $1+i=\sqrt{2}e^{i\frac{\pi}{4}}$ $-1+i\sqrt{3}=2e^{i\frac{2\pi}{3}}$ $\frac{(1+i)^6}{(-1+i\sqrt{3})^8}=\frac{\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^6}{\left(2e^{i\frac{2\pi}{3}}\right)^8}=\frac{2^3e^{i\frac{3\pi}{2}}}{2^8e^{i\frac{16\pi}{3}}}=\frac{1}{2^5}e^{-i\frac{23\pi}{6}}=\frac{1}{2^5}e^{i\frac{\pi}{6}}$ $\left(ie^{\frac{\pi}{3}}z\right)^*=-ie^{\frac{\pi}{3}}z^*$ Since $\left(\mathrm{i}e^{\frac{\pi}{3}}z\right)^*=\frac{(1+\mathrm{i})^6}{(-1+\mathrm{i}\sqrt{3})^8}$, $-ie^{\frac{\pi}{3}}z^*=\frac{1}{2^5}e^{i\frac{\pi}{6}}$ $\Rightarrow z^*=\frac{1}{-ie^{\frac{\pi}{3}}2^5}e^{i\frac{\pi}{6}}=\frac{i}{e^{\frac{\pi}{3}}2^5}e^{i\frac{\pi}{6}}$ $=\frac{e^{i\frac{\pi}{2}}}{e^{\frac{\pi}{3}}2^5}e^{i\frac{\pi}{6}}=\frac{e^{i\frac{\pi}{2}}e^{i\frac{\pi}{6}}}{e^{\frac{\pi}{3}}2^5}=\frac{e^{i\frac{2\pi}{3}}}{e^{\frac{\pi}{3}}2^5}$ Hence $z=\frac{e^{-i\frac{2\pi}{3}}}{e^{\frac{\pi}{3}}2^5}$ $\|z\|=\frac{1}{e^{\frac{\pi}{3}}2^5}$ and arg $z=-\frac{2\pi}{3}$ (c) Since $z_1^2+z_2^2+z_3^2+z_4^2<0$, hence equation has complex roots. Also as coefficients of x of equation are all real, hence the complex roots occur in conjugate pairs. Thus at most two of z_1, z_2, z_3 and z_4 are real.	
5	(i) To test $H_0: \mu=72300$ against $H_1: \mu>72300$ at 5 % level of significance (ii) Assume that the sample of 30 daily takings after the advertising campaign is a random sample Under H_o, $\overline{X}\sim N(72300,\frac{4410^2}{30})$	

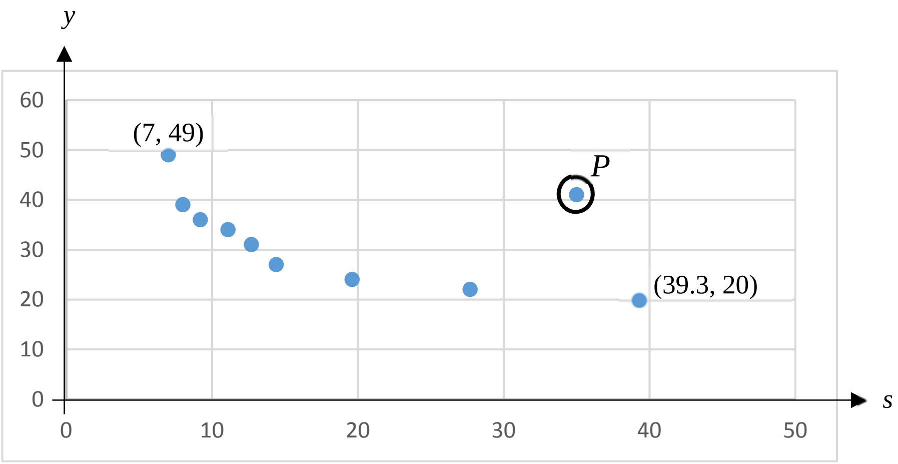
	Method 1 Using GC $P(\overline{X} > 73624) = 0.05$ Reject H_0 at 5% level of significance if $\bar{x} > 73624$ **Method 2** $Z = \dfrac{\overline{X} - 72300}{\frac{4410}{\sqrt{30}}} \sim N(0,1)$ And $P(Z > 1.64485) = 0.05$ Reject H_0 at 5% level of significance if $\dfrac{\bar{x} - 72300}{\frac{4410}{\sqrt{30}}} > 1.66485$ i.e $\bar{x} > 73624$ (iii) $\bar{x} = \dfrac{\sum(x - 70000)}{30} + 70000 = \dfrac{129000}{30} + 70000 = 74300$ Using previous result in (ii), since 74300 > 73624, we reject H_0 . There is sufficient evidence at 5% level of significance that the advertising campaign is effective	
6	(i) Let random variable X be the number of red light bulbs $X \sim B(20, 0.4)$ $P(X<20-X) = P(X<10) = P(X\leq 9) = 0.755337 = 0.755$ (to 3 s.f) (ii) Let random variable A be the number of boxes with fewer red light bulbs than blue $A \sim B(n, 0.755337)$ $P(A \geq 2) \leq 0.999$ $1 - P(A \leq 1) \leq 0.999$ $P(A \leq 1) \geq 0.001$ From GC, <table><tr><td>n</td><td>$P(A \leq 1)$</td></tr><tr><td>6</td><td>0.00419</td></tr><tr><td>7</td><td>0.00119</td></tr><tr><td>8</td><td>0.00033</td></tr></table> Greatest possible value of n = 7	

(iii)

$E(X) = 20(0.4) = 8$
$Var(X) = 20(0.4)(0.6) = 4.8$
$n = 40$ is large. From central limit theorem
$\overline{X} \sim N(8, \frac{4.8}{50})$ approximately
$P(\overline{X} \leq 7.5) = 0.05329158 = 0.0533$ (3 s.f)

(iv) In the context of the question, the number of boxes that the customer can select is not fixed (sample size is not fixed). Hence a binomial distribution is not an appropriate model as it requires a fixed number of trials.

7 (i) & (ii)



Point P could be the result of a recording error or an instrument error.

(iii) (a) $r = -0.9258429505 = -0.926$ (3s.f)

(b) $y = a - b\ln s$ may not be a good model as it seems to suggest that as wind speed (x) increases, ozone (y) will decrease and eventually be negative in value. This is inappropriate as ozone level will never be negative in value. There is always ozone present in the air.

(iv) Least square regression line of y on $\frac{1}{s}$:

$y = 12.93341953 + \frac{227.6608861}{s}$

When $s = 50$, $y = 17.486637 = 17.5$ (3s.f)

The prediction is not reliable as $s = 50$ does not lies within the data range of the wind speed (s). Hence, we are extrapolating.

(v) Using G.C, mean of $\frac{1}{s}$ = 0.0808 (3s.f)

mean of y = 31.3 (3s.f).

The least squares regression line of y on $\frac{1}{s}$ intersects the least squares regression line of $\frac{1}{s}$ on y at (0.0808, 31.3).

(vi) The regression line of $\frac{1}{s}$ on y should not be used as y is not the independent variable. It is the dependent variable.

8

F is the event that the component is faulty
N is the event that the component is not faulty
Outcomes: NFF or FNF

P(X = 3) = $\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}+\frac{2}{5}\times\frac{3}{4}\times\frac{1}{3}=\frac{1}{10}+\frac{1}{10}=\frac{1}{5}$

(i)

x	2	3	4	5
outcomes	FF		NNFF, NFNF, FNNF	NNNFF NNFNF NFNNF FNNNF
P(X = x)	$\frac{2}{5}\times\frac{1}{4}$ $=\frac{1}{10}$	$\frac{1}{5}$	$3\times\frac{3}{5}\times\frac{2}{4}\times\frac{2}{3}\times\frac{1}{2}=\frac{3}{10}$	$4\times\frac{3}{5}\times\frac{2}{4}\times\frac{1}{3}\times\frac{2}{2}\times\frac{1}{1}=\frac{2}{5}$ or $1-\frac{1}{10}-\frac{1}{5}-\frac{3}{10}=\frac{2}{5}$

(ii) $P(X_1=3 \mid X_1+X_2\le 5) = \frac{P([X_1=3]\cap[X_1+X_2\le 5])}{P(X_1+X_2\le 5)} =$

$\frac{P(X_1=3,\, X_2=2)}{2P(X_1=3, X_2=2)+P(X_1=2, X_2=2)} =$

$\frac{P(X_1=3)P(X_2=2)}{2P(X_1=3)P(X_2=2)+P(X_1=2)P(X_2=2)}$

$=\frac{\left(\frac{1}{5}\right)\left(\frac{1}{10}\right)}{\left(\frac{1}{10}\right)^2+2\left(\frac{1}{5}\right)\left(\frac{1}{10}\right)}=\frac{2}{5}=0.4$.

(iii) $E(X) = \sum_{all\,x} xP(X=x) = 2\left(\frac{1}{10}\right)+3\left(\frac{1}{5}\right)+4\left(\frac{3}{10}\right)+5\left(\frac{2}{5}\right)=4$

$E(X^2) = \sum_{all\,x} x^2P(X=x) = 2^2\left(\frac{1}{10}\right)+3^2\left(\frac{1}{5}\right)+4^2\left(\frac{3}{10}\right)+5^2\left(\frac{2}{5}\right)=17$

Var(X) = $17-\left(4^2\right)=1$

Var(C) = 9Var(X) = $9(1)=9$

9

(i) Let random variable X be the battery life of a randomly selected Brand B car battery

$X \sim N(25, 22^2)$

$P(X < 0) = 0.128$

A randomly chosen Brand B battery cannot be normally distributed as this will result in 12.8% of Brand B batteries having a negative battery life which is not a negligible value and does not make sense.

(ii)

$$P(|X-25|<5)=0.8$$

$$P(-5<X-25<5)=0.8$$

$$P(20<X<30)=0.8$$

$$P\left(\frac{20-25}{\sigma}<Z<\frac{30-25}{\sigma}\right)=0.8$$

$$P\left(\frac{-5}{\sigma}<Z<\frac{5}{\sigma}\right)=0.8$$

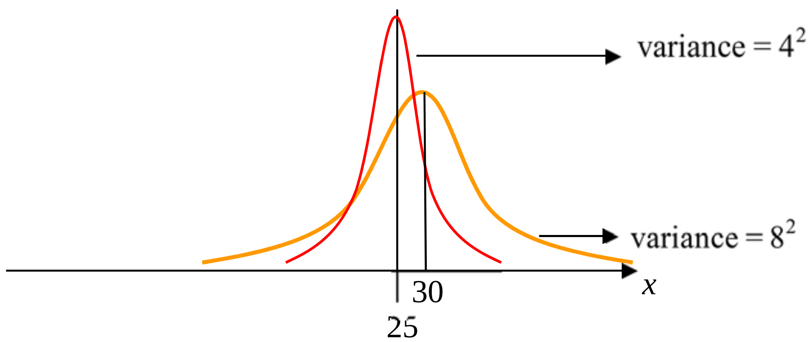
$$P\left(Z<\frac{-5}{\sigma}\right)=0.1$$

Using G.C,

$$\frac{-5}{\sigma}=-1.281551567$$

$$\sigma^2=15.221864=15.2\text{ (3s.f)}$$

(iii)



(iv) Let random variable A be the battery life of a randomly selected Brand A car battery

$P(A = 26) = 0$

(v)

$A \sim N(30, 8^2)$

$\bar{X} \sim N(25, \frac{4^2}{3})$

$0.75A \sim N(30(0.75), 8^2(0.75)^2)$ i.e $0.75A \sim N(22.5, 36)$

$$\bar{X} - 0.75A \sim N(25 - 22.5, \frac{4^2}{3} + 36)$$

$$\bar{X} - 0.75A \sim N(2.5, \frac{124}{3})$$

$$P(\left|\bar{X} - 0.75A\right| < 3) = P(-3 < \bar{X} - 0.75A < 3)$$

$= 0.33485 = 0.335$ (3 s.f)

(vi) $P(X < k) < 0.01$

From GC,
$k < 15.695$
Largest integer $k = 15$

10 (i)

Case 1:
Team comprises of 2+2+3+3 (selected from each of the expert area).

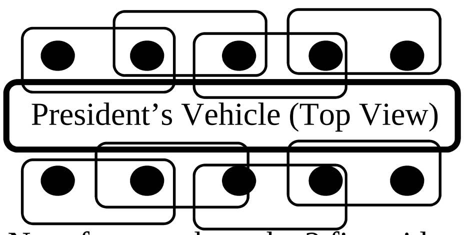
$$^{10}C_2 \times {}^{10}C_2 \times {}^{10}C_3 \times {}^{10}C_3 \times \frac{4!}{2!2!} = 174960000$$

Case 2:
Team comprises of 2+2+2+4 (selected from each of the expert area).

$$^{10}C_2 \times {}^{10}C_2 \times {}^{10}C_2 \times {}^{10}C_4 \times \frac{4!}{3!} = 76545000$$

Total ways = 174960000 + 76545000 = 251505000.

(ii)
Method 1 (complement Method):
No. of ways of standing without restrictions = 10!



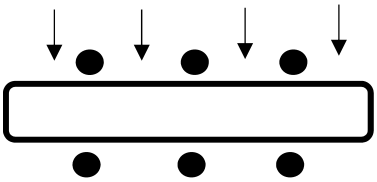
No. of ways where the 2 first aid experts are together = (8)(2!)(8!)
= 645120

No of required ways = 10! − 645120 = 2983680

Method 2:

Case 1: The two first aiders are on the same side of car and separated.

No. of ways= $\left({}^4C_2 \times 2! \times 8!\right) \times 2 = 967680$

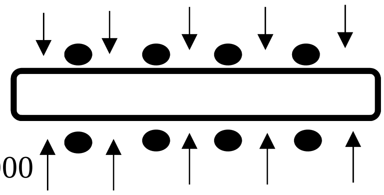


Case 2: The two first aiders are on the opposite sides of car.

No. of ways= $\left({}^5C_1 \times {}^5C_1 \times 2! \times 8!\right) = 2016000$

OR

No. of ways= $\left({}^8C_4 \times {}^4C_4 \times 5! \times 5!\right) \times 2 = 2016000$



Total no. of ways = $967680 + 2016000 = 2983680$.

(iii)

Required probability $= \dfrac{(7-1)! \times {}^7C_3 \times 3!}{(10-1)!} = \dfrac{5}{12}$

(iv)

Let event A be switch A fails and event B be switch B fails.

$\mathrm{P}(A) = 0.1 \quad \mathrm{P}(B) = 0.23 \quad \mathrm{P}(A' \cap B') = 0.7$

$$\begin{aligned}\mathrm{P}(B \cap A') &= 1 - \mathrm{P}(A' \cap B') - \mathrm{P}(A) \\ &= 1 - 0.7 - 0.1 \\ &= 0.2\end{aligned}$$

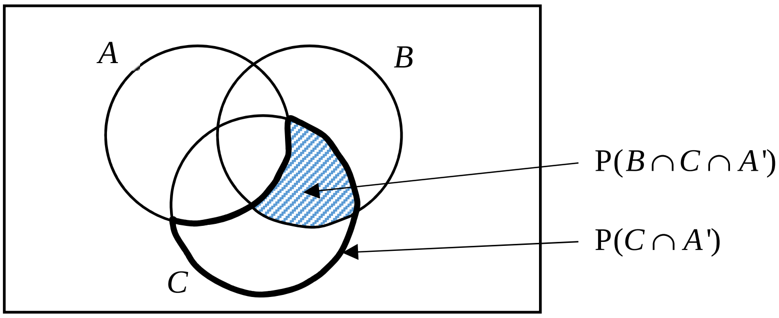
(v)

Let event C be switch C fails.

Since A and C are independent events, $\mathrm{P}(C \mid A) = \mathrm{P}(C) = 0.15$.

$$\begin{aligned}\mathrm{P}(C \cap A') &= \mathrm{P}(C) \times \mathrm{P}(A') \\ &= 0.15 \times [1 - 0.1] \\ &= 0.135\end{aligned}$$

(vi)



$0 \le \mathrm{P}(B \cap C \cap A') \le 0.135$

$\therefore$ Maximum $\mathrm{P}(B \cap C \cap A') = 0.135$