ANDERSON JUNIOR COLLEGE
2018 PRELIMINARY EXAMINATION
H2 MATHEMATICS 9758/01

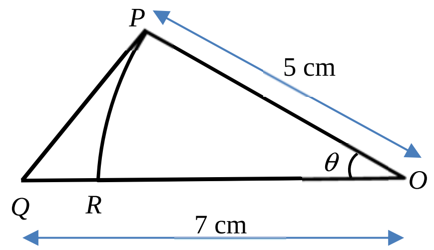
PAPER 1

Duration: 3 hours

Answer ALL questions.

1 Differentiate e^{x^2+1} with respect to x. Hence, find $\int x^3 e^{x^2+1}\, dx$. [3]

2



The diagram shows a triangle OPQ with $OP = 5$ cm, $OQ = 7$ cm and $\angle POQ = \theta$ radians. Given that OP and OR are radii of a circle with centre O and θ is a sufficiently small angle, show that the perimeter of PQR can be approximated by $a + b\theta + c\theta^2$ where a, b and c are constants to be determined. [5]

3 It is given that $y = \dfrac{ax+1}{x-b}$, where a, b are positive integers.

(i) Show that the equation of $y = \dfrac{ax+1}{x-b}$ can be written as $y = p + \dfrac{q}{x-b}$, where p and q are constants to be found in terms of a and b. [1]

(ii) Hence, describe a sequence of three transformations which transforms the graph of $y = \dfrac{ax+1}{x-b}$ on to the graph of $y = \dfrac{1}{x}$. [3]

4 **(i)** Express $\dfrac{1}{(r-1)r(r+1)}$ in partial fractions. [2]

(ii) Hence find $\displaystyle\sum_{r=2}^{n} \frac{4}{(r-1)r(r+1)}$ in terms of n. [4]

(iii) State the sum to infinity of the series in part **(ii)**. Hence, find the smallest value of n for which $\displaystyle\sum_{r=2}^{n} \frac{4}{(r-1)r(r+1)}$ is within 10^{-5} of the sum to infinity. [3]

5 It is given that $\mathrm{f}(x)=\ln\left(\sin x+\cos x\right)$ for $-\dfrac{\pi}{4}<x\leq\dfrac{\pi}{4}$.

(i) Show that $\mathrm{f}''\left(x\right)+\left[\mathrm{f}'\left(x\right)\right]^{2}+1=0$. [2]

(ii) By further differentiation of the result in part **(i)**, find the Maclaurin series for $\mathrm{f}(x)$ up to and including the term in x^3. [4]

(iii) Using the expansion obtained in part **(ii)** and the standard series from the List of Formulae (MF26), find the series expansion of $\mathrm{g}\left(x\right)=\ln\left(1+\tan x\right)$ for $-\dfrac{\pi}{4}<x\leq\dfrac{\pi}{4}$ as far as the term in x^3. [4]

6 A curve C has parametric equations

$$x=\frac{2}{1+t^{2}}, \quad y=\ln(2-t), \quad \text{where } -5<t<2 \ .$$

(i) Write down the equation of the vertical asymptote of C. [1]

(ii) Find the value of t, where $t<-0.5$, at the point on C where the tangent has gradient $-\dfrac{1}{3}$. [4]

(iii) Hence find the equation of this tangent. [2]

(iv) Find the point of intersection of C and the vertical asymptote found in part **(i)**. [2]

7 **Do not use a calculator in answering this question.**

(a) Given that $z = -2 + 3\mathrm{i}$ is a root of the equation $2z^{2}+(-1+4\mathrm{i})z+c=0$, find the complex number c and the other root. [4]

(b) The complex number u is given by $u = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \pi$.

(i) Express $u-1$ in terms of $\sin\dfrac{\theta}{2}$ and $\cos\dfrac{\theta}{2}$. Hence, find $\left|u-1\right|$ and show $\arg(u-1)=\dfrac{\pi}{2}+\dfrac{\theta}{2}$. [4]

(ii) Given that $v=-\sqrt{3}+\mathrm{i}$ and $(v^{*}u)^{8}$ is real and negative, find the smallest value of θ in terms of π. [4]

8 Referred to the origin O, a non-zero vector, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, makes an angle of α, β and γ with the positive x, y and z-axis respectively.

(i) Explain with clear workings, if $\alpha = \beta = \gamma = 45°$ is possible. [2]

It is now given that $\alpha = \beta = 45°$ and $\gamma = 90°$.

(ii) Find $\mathbf{r}$ in terms of x. [2]

(iii) $OABC$ is a tetrahedron. $\overrightarrow{OA}$ is parallel to $\mathbf{r}$ and the point B is the foot of the perpendicular from the point A to the y-axis. $\overrightarrow{AC}$ is parallel to the z-axis and $AC = AB$.

Given that the volume of tetrahedron $OABC$ is 36 unit^3, by considering the area of triangle OAB in terms of x, find the coordinates of a possible point C. [4]

[Volume of tetrahedron = $\frac{1}{3}(\text{base area})(\text{height})$]

9 The functions f and g are defined by

$$\mathrm{f} : x \mapsto (x+3)|x-2| , \quad x \in \mathbb{R} ,$$
$$\mathrm{g} : x \mapsto \frac{24}{x^2+3} - 1 , \quad x \in \mathbb{R} , \ x > 0.$$

(i) Explain why the function f^{-1} does not exist. [1]

In the rest of the question, the domain of f is restricted to $[k, 2]$ where k is the least value such that f^{-1} exists.

(ii) Write down the value of k. Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [4]

(iii) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram, showing clearly the relationship between the two graphs. Hence, find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [4]

(iv) Determine whether $\mathrm{g}^{-1}\mathrm{f}$ exists. [2]

(v) Show that g is a strictly decreasing function. Hence, without finding g^{-1}, find the range of values of x for which $\mathrm{g}^{-1}(\mathrm{f}(x)) < 1$, giving your answer in exact form. [3]

10 The human body carries out one of its main functions by consuming food and turning it into usable energy. A person's mass depends both on the rate of calories intake and on the rate of calories used. A health report shows:

> ……. the calories usage of a person per day can be estimated by……… Calories usage per day = 38.5 $\times$ (mass (in kg) of the person) …..

A man with mass m kg follows a dietary intake of u calories per day. Taking $\frac{\mathrm{d}m}{\mathrm{d}t}$ to represent the rate of change of his mass with respect to time t days, and assuming that the rate of mass change is proportional to the net excess (or deficit) in the number of calories per day, show that $\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{u}{7700} - \frac{m}{200}$. [2]

(i) Assuming that the rate of calories intake is a constant, find m in terms of u and t. [4]

(ii) A person with initial mass 80kg adopts a diet of 2500 calories per day. Write down the mass m in terms of t for this individual. [2]

A weight management company applies another model of mass function M with a differential equation given by $38.5\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{77(77M+154t)+76}{77M+154t+1}$.

Using the substitution $z = 38.5M + 77t$, find the general solution of M in terms of t. [5]

11 Mr and Mrs Lai have decided to buy an education investment fund for their daughter, Adeline, on the day she turns one-year old to provide financial security for her future education.

An insurance company offers two different types of education investment funds as shown below.

<u>SupremeEdu Fund</u>: There is a fixed annual contribution of \$1040 to the fund. The last contribution will be on the child's seventeenth birthday. This fund will earn 3.5% interest per annum.

<u>UltimateEdu Fund</u>: The first annual contribution to the fund is \$600. The subsequent contributions will increase \$80 per annum such that the second contribution will be \$680 and so on, until the last contribution on the child's seventeenth birthday. The fund will earn an annual interest of \$200.

For both funds, the contributions will be deposited into the account on the child's birthday. The interest will be credited into the account annually a day before the child's next birthday. On the child's eighteenth birthday, the total sum of money in the account will be paid out to the child.

Leave all your answers to the nearest dollar.

(i) If Mr and Mrs Lai were to choose the UltimateEdu Fund, they need to know if they are able to afford to pay for the annual contributions. Show that the last contribution to the fund is \$1880. [2]

(ii) Mr and Mrs Lai wish to invest in one of these two funds. Find the amount that Adeline would get on her eighteenth birthday from each fund. [7]

(iii) Determine, with a clear explanation, which investment fund Mr and Mrs Lai should choose. [2]

(iv) Based on the selected fund, what should be the least amount of first contribution to the fund so that Adeline will be able to receive \$50000 on her eighteenth birthday? [2]

END OF PAPER

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H2 Mathematics 9758/02

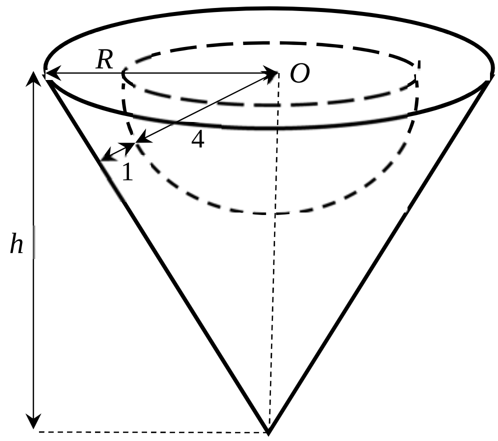
PAPER 2

Duration: 3 hours

Answer ALL questions.

Section A: Pure Mathematics [40 marks]

1 [It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



Michelle wants to design a décor ornament for Christmas. The ornament is to be made up of a right circular cone of radius R cm and height h cm, with a hemisphere of fixed radius 4 cm being carved out from the cone as shown in the diagram above. O is the center of the circular top of both the cone and the hemisphere. The hemisphere is carved out such that the shortest distance between the curved surface of the cone and the hemisphere is 1 cm (see diagram). Show that the volume V of the ornament is given by

$$V = \frac{\pi}{3}\left(\frac{25h^3}{h^2-25}\right) - \frac{128\pi}{3} .$$

Michelle wishes to minimise the volume of the ornament so that it is light in weight. Find the exact value of h that corresponds to the minimum volume, and find the minimum volume. [8]

2 **(i)** Given that $\mathbf{a}$ and $\mathbf{b}$ are non-zero constant vectors, show that the points with position vector $\mathbf{r}$ and satisfying the equation $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=\mathbf{0}$ lie on a line. [3]

The line L_1 has equation $\left[\mathbf{r}-\begin{pmatrix}2\\1\\-1\end{pmatrix}\right]\times\begin{pmatrix}0\\1\\1\end{pmatrix}=\mathbf{0}$, and the line L_2 has equation $\mathbf{r}=\begin{pmatrix}-1\\2\\3\end{pmatrix}+\mu\begin{pmatrix}m\\0\\1\end{pmatrix}$, where m is a constant and μ is a real parameter.

(ii) Find the vector equations of the parallel planes p_1 and p_2, in scalar product form, such that p_1 contains L_1 and p_2 contains L_2, leaving your answers in terms of m. [3]

(iii) Find the acute angle between the plane p_1 and the line which contains both the points $(2,1,-1)$ and $(-1,2,3)$, leaving your answer in terms of m. [2]

(iv) It is now given that L_1 and L_2 is contained in a common plane p_3. Using your answer in part **(iii)**, or otherwise, find the cartesian equation of the plane p_3. [1]

3 The curve C has equation $y=ax+b+2a+\dfrac{2(b+2a)}{x-2}$, where a and b are real constants such that $a>0$, $b\neq -2a$ and $x\neq 2$.

(i) By using differentiation, find the range of values of b in terms of a such that C has no stationary points. [4]

(ii) Given that $b=-3a$, sketch C, stating the equations of any asymptotes and the coordinates of the points where C crosses the axes. [3]

(iii) On the same diagram as C, sketch the graph of $\left(\dfrac{y}{a}\right)^2-(x-1)^2=4$, showing clearly the equations of the asymptotes and turning points. Hence solve the inequality $x-1-\dfrac{2}{x-2}\geq\sqrt{(x-1)^2+4}$. [4]

4 It is given that $f(x)=\dfrac{x}{\sqrt{3+2x-x^2}}$ for $-1<x<3$.

(i) Find (a) $\int f(x)\,dx$. [4]

(b) $\int [f(x)]^2\,dx$ by using partial fractions. [3]

(ii) Find the exact area of the region bounded by the curve $y=f(x)$, the x-axis and the lines $x=1-\sqrt{3}$ and $x=2$. [3]

(iii) Find the exact volume of revolution when the region bounded by the curve $y=f(x)$, the x-axis and the lines $x=1$ and $x=2$ is rotated completely about the x-axis. [2]

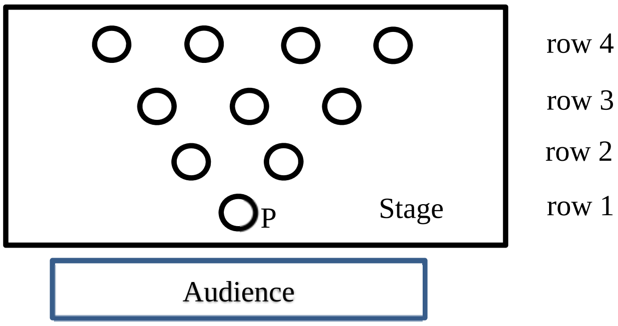
Section B: Probability and Statistics [60 marks]

5 There are 18 participants in an Idol Survival Competition. In the first round of competition, these participants are grouped into 3 groups of equal size. James and Michael are two participants in this competition.

(i) Show clearly that the number of ways the participants can be grouped is 2858856. [1]

(ii) Find the probability that James and Michael are in the same group. [2]

After many rounds of elimination, the top 10 participants (including James and Michael) remain in the competition. During their final performance, the participants stand in a triangular formation as shown in the diagram below.



Given that position P must be occupied by Michael, and that James and 2 particular participants would like to stand next to one another in the same row, find the number of ways the participants can arrange themselves. [3]

6 A farm produces a large number of eggs every day. The eggs are randomly packed into trays of 30. A small proportion p of these eggs are known to have weights which are substandard.

A check is carried out each day by taking a random sample of 50 trays and examining their weights. If more than 1 egg in a tray is substandard, the entire tray of eggs is rejected.

If 4 or more trays are rejected in the sample, the day's production is rated as 'poor'. If none of the trays is rejected in the sample, the day's production is rated as 'excellent'. Otherwise, the day's production is rated as 'fair'.

(i) State, in this context, two conditions that must be met for the number of eggs in a tray that are substandard to be well modelled by a binomial distribution. [2]

Assume now that these conditions are met.

(ii) Show that the probability G that a tray is not rejected is given by

$$G = (1-p)^{29}(1+29p).$$ [2]

Given that $G = 0.96$,

(iii) find the value of p correct to four decimal places, [1]

(iv) find the probability that two trays are rejected if the day's production is rated as 'fair'. [3]

7 Soya bean drink is sold in cups of two sizes – small and large. For each size, the amount of content, in ml, of a randomly chosen cup is normally distributed with mean and variance as given in the table. The selling prices are also given in the table.

	Mean (ml)	Variance (ml^2)	Selling Price ($)
Small	202	21	1.10
Large	405	74	2.20

The amount of content in any cup may be assumed to be independent of the amount of content in any other cup.

(i) A small cup and two large cups are selected at random. Find the probability that the total amount of content in the two large cups is less than four times the amount of content in the small cup. [2]

(ii) A boy needs at least 600 ml of soya bean drink but he only has $3.80. In what way should he make his purchase so that he has the highest probability of getting at least 600 ml? Support your answer with clear workings. [3]

(iii) A random sample of 20 small cups and *n* large cups of soya bean drink is taken. Find the least value of *n* such that there is a probability of more than 0.8 that the mean amount of content in these cups is more than 350 ml. [4]

8 To promote the sale of its products on Valentine's Day, a shop owner gives free vouchers to its customers. To get the vouchers, customers must first participate in a game. Each customer is allowed to play the game only once.

There are four rounds in the game. In each round, four cards are used. The message printed on one of the cards is 'Congratulations' and the message printed on the other three cards is 'Thank you'. These cards are placed facedown, and the customer would choose a card to flip in that round. Two points are scored if the card with the message 'Congratulations' is flipped. Otherwise, one point is deducted. Each customer is equally likely to choose any of the cards to flip.

At the end of the game, the customer's final score is X, where X is the sum of the points scored in the four rounds.
The shop will then reward the customer with a voucher worth $\$\,(8+2X)$.

(i) Show that $\mathrm{P}(X=2)=\dfrac{27}{128}$. [1]

(ii) Tabulate the probability distribution of X. [3]

(iii) Find the mean and variance of the value of the voucher a customer receives from the shop. [4]

If 100 customers visit the shop and play the game, what is the total value of the vouchers the shopkeeper is expected to give away? Estimate the probability that the total value of the vouchers given away would be more than \$650. [3]

9 **(a)** The following set of bivariate data was obtained for x and y:

$$n=8,\quad \sum(x-150)=47,\quad \sum(x-150)^2=483,$$

$$\sum(y-170)=87,\quad \sum(y-170)^2=1101,\quad \sum(x-150)(y-170)=677\,.$$

Find the

(i) equation of the least-squares regression line of x on y,

(ii) product moment correlation coefficient. [4]

(b) To understand the correlation between the level of happiness and the time spent at work, the human resource department of an organisation polls nine of its officers. The number of work hours per week (t) of each officer and the happiness index (h) are recorded, where a higher value of h indicates a higher level of happiness. The data are shown in the table below:

t	20.1	22.0	24.4	25.3	28.8	36.5	40.6	46.0	55.1
h	24.5	16.3	18.6	12.5	5.2	4.7	1.4	1.8	0.8

(i) Draw a scatter diagram of h against t for the data and comment whether a linear model would be appropriate. [3]

It is thought that the relationship between the happiness index h and the number of work hours per week t can be modelled by one of the following formulae

$$(\mathbf{A})\quad h=a+\frac{b}{t} \qquad\qquad (\mathbf{B})\quad h=c+d\ln(t)$$

where a, b, c and d are constants.

(ii) Find, correct to 5 decimal places, the value of the product moment correlation coefficient for each model. Explain which is the better model. [3]

(iii) Use the model you have chosen in part **(ii)** to predict the happiness index for an officer who works 15.0 hours per week. Comment on the reliability of your prediction. [3]

10 The public relations officer of In-star-gram stated in an education report that a student spends an average of 2.49 hours a day on his/her social media platform. A survey is done to collect the usage time, x hours, of 100 students in the age group 13 to 16 years old, and the data is summarised as follows:

$$\sum x = 250.5 \text{ and } \sum x^2 = 628.45 .$$

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) A hypothesis test is carried out at the $\alpha\%$ level of significance, and it was found that the public relations officer had underestimated the usage hours. Find the set of values of α and state any necessary assumption(s). [6]

(iii) Explain, in context, the meaning of the p - value found in part **(ii)**. [1]

In another survey done to collect the usage hours of 100 students in the age group 17 to 20 years old, it is found that the average usage is k hours. Assuming now that the population standard deviation is 1.5 hours, find the range of values of k such that there is sufficient evidence to conclude that the average usage hours of In-star-gram is not 2.49 hours at the 5% level of significance. [4]

END OF PAPER

H2 Mathematics 9758/01 (JC2) 2018
Paper 1 Solutions

Qn	Solutions
1	$\frac{\mathrm{d}}{\mathrm{d}x}e^{x^2+1}=2x\,e^{x^2+1}$ $\int x^3 e^{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int x^2\left(2x e^{x^2+1}\right)\mathrm{d}x$ $= \frac{1}{2}\left[x^2 e^{x^2+1}-\int (2x)e^{x^2+1}\,\mathrm{d}x\right]$ $= \frac{1}{2}\left[x^2 e^{x^2+1}-e^{x^2+1}\right]+c$ $= \frac{1}{2}\left[x^2 -1\right]e^{x^2+1}+c$
2	$PQ^2=5^2+7^2-2(5)(7)\cos\theta$ $\Rightarrow PQ=\sqrt{74-70\cos\theta}$ Perimeter of PQR = PQ + QR + arc length PR $=\sqrt{74-70\cos\theta}+2+5\theta \approx \left(74-70\left(1-\frac{\theta^2}{2}\right)\right)^{\frac{1}{2}}+2+5\theta$ $=\left(4+35\theta^2\right)^{\frac{1}{2}}+2+5\theta$ $=2\left(1+\frac{35}{4}\theta^2\right)^{\frac{1}{2}}+2+5\theta$ $\approx 2\left(1+\frac{35}{8}\theta^2\right)+2+5\theta = 4+5\theta+\frac{35}{4}\theta^2$ $\therefore\ a=4,\quad b=5,\quad c=\frac{35}{4}$
3	**(i)** $y=\frac{ax+1}{x-b}$ $=\frac{a(x-b)+1+ab}{x-b}=a+\frac{1+ab}{x-b}$ **(ii)** The sequence of 3 transformations are: (a) translation of b units in the negative x-direction, followed by (b) translation of a units in the negative y-direction, followed by (c) scaling parallel to the y-axis with scale factor $\frac{1}{1+ab}$.
4	**(i)** $\frac{1}{(r-1)r(r+1)}=\frac{A}{r-1}+\frac{B}{r}+\frac{C}{r+1}$

By Cover Up rule or any other methods,

$A = \dfrac{1}{2}$, $B = -1$ and $C = \dfrac{1}{2}$

$$\therefore \frac{1}{(r-1)r(r+1)} = \frac{1}{2(r-1)} - \frac{1}{r} + \frac{1}{2(r+1)}$$

(ii) $$\sum_{r=2}^{n} \frac{4}{(r-1)r(r+1)}$$

$$= 4\sum_{r=2}^{n}\left(\frac{1}{2(r-1)} - \frac{1}{r} + \frac{1}{2(r+1)}\right)$$

$$= 4\left(\frac{1}{2(1)} - \frac{1}{2} + \frac{1}{2(3)}\right.$$
$$+\frac{1}{2(2)} - \frac{1}{3} + \frac{1}{2(4)}$$
$$+\frac{1}{2(3)} - \frac{1}{4} + \frac{1}{2(5)}$$
.
.
.
$$+\frac{1}{2(n-3)} - \frac{1}{n-2} + \frac{1}{2(n-1)}$$
$$+\frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2(n)}$$
$$\left.+\frac{1}{2(n-1)} - \frac{1}{n} + \frac{1}{2(n+1)}\right)$$

$$= 4\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2n} - \frac{1}{n} + \frac{1}{2(n+1)}\right)$$

$$= 1 - \frac{2}{n(n+1)} \quad \text{or} \quad 1 - \frac{2}{n} + \frac{2}{n+1}$$

(iii) $$\sum_{r=2}^{n} \frac{4}{(r-1)r(r+1)} = 1 - \frac{2}{n(n+1)}$$

As $n \to \infty, \dfrac{2}{n(n+1)} \to 0$,   Sum to infinity = 1

$$1 - \left(1 - \frac{2}{n(n+1)}\right) < 10^{-5}$$

$$\frac{2}{n(n+1)} < 10^{-5}$$

$$n(n+1) > 2(10^{5})$$

Using G.C, $n < -447.71$ (rejected, $n > 0$) or $n > 446.71$

Least n = 447

5

(i) $\mathrm{f}\left(x\right) = \ln\left(\sin x + \cos x\right)$

$\Rightarrow e^{\mathrm{f}(x)} = \sin x + \cos x$

Differentiating wrt x :

$e^{\mathrm{f}(x)}\,\mathrm{f}'\left(x\right) = \cos x - \sin x$

$e^{\mathrm{f}(x)}\,\mathrm{f}''\left(x\right) + e^{\mathrm{f}(x)}\,\mathrm{f}'\left(x\right) \times \mathrm{f}'\left(x\right) = -\sin x - \cos x = -e^{\mathrm{f}(x)}$

$\Rightarrow \mathrm{f}''\left(x\right) + \left[\mathrm{f}'\left(x\right)\right]^2 = -1$ since $e^{\mathrm{f}(x)} \neq 0$

$\Rightarrow \mathrm{f}''\left(x\right) + \left[\mathrm{f}'\left(x\right)\right]^2 + 1 = 0$ (shown)

<u>Alternative Method 1</u>

$\mathrm{f}\left(x\right) = \ln\left(\sin x + \cos x\right)$

$$\mathrm{f}'\left(x\right) = \frac{\cos x - \sin x}{\sin x + \cos x}$$

$$f''(x)=\frac{(\sin x+\cos x)(-\sin x-\cos x)-(\cos x-\sin x)(\cos x-\sin x)}{(\sin x+\cos x)^2}$$

$$=-1-\frac{(\cos x-\sin x)^2}{(\sin x+\cos x)^2}\quad=-1-\left[f'(x)\right]^2$$

$$\Rightarrow\quad f''(x)+\left[f'(x)\right]^2+1=0\quad\text{(shown)}$$

Alternative Method 2

$$\text{Let } f(x)=\ln(\sin x+\cos x)$$

$$f'(x)=\frac{\cos x-\sin x}{\sin x+\cos x}$$

$$(\sin x+\cos x)f'(x)=\cos x-\sin x$$

$$(\sin x+\cos x)f''(x)+(\cos x-\sin x)f'(x)=-\sin x-\cos x$$

$$f''(x)+\frac{\cos x-\sin x}{\sin x+\cos x}\left[f'(x)\right]=-1$$

$$f''(x)+\left[f'(x)\right]^2+1=0\text{ (shown)}$$

(ii) From $f''(x)+\left[f'(x)\right]^2+1=0$

Differentiating wrt x,

$$f'''(x)+2\left[f'(x)\right]\left[f''(x)\right]=0$$

When $x=0, f(0)=0, f'(0)=1, f''(0)=-2, f'''(0)=4$

By Maclaurin's Theorem,

$$y=0+x+\frac{(-2)}{2!}x^2+\frac{4}{3!}x^3+\ldots$$

$$=x-x^2+\frac{2}{3}x^3\quad\text{(up to term in } x^3\text{)}$$

(iii) $g(x)=\ln(1+\tan x)$

$$=\ln\left(\frac{\cos x+\sin x}{\cos x}\right)$$

$$=\ln(\cos x+\sin x)-\ln(\cos x)$$

$$\approx\left(x-x^2+\frac{2}{3}x^3\right)-\ln\left(1-\frac{x^2}{2}\right)$$

$$=x-x^2+\frac{2}{3}x^3-\left[\left(-\frac{x^2}{2}\right)-\ldots\right]$$

$$=x-\frac{1}{2}x^2+\frac{2}{3}x^3\text{ (up to term in } x^3\text{)}$$

6	(i) when $t \to 2$; $y \to -\infty$ and $x \to \frac{2}{5}$, Equation of the vertical asymptote is $x = \frac{2}{5}$. (ii) $x = \frac{2}{1+t^2} \Rightarrow \frac{dx}{dt} = \frac{-2(2t)}{(1+t^2)^2}$; $y = \ln(2-t) \Rightarrow \frac{dy}{dt} = \frac{-1}{2-t}$, hence $\frac{dy}{dx} = \frac{(1+t^2)^2}{4t(2-t)}$ $\frac{(1+t^2)^2}{4t(2-t)} = -\frac{1}{3} \Rightarrow 3t^4 + 2t^2 + 8t + 3 = 0$, By GC $t = -1$ or $t = -0.436114$ (rej. as $t < -0.5$) (iii) $\frac{dy}{dx} = \frac{-1}{3}$ when $t = -1$, $x = 1$, $y = \ln(3)$ Equation of tangent is $y - \ln(3) = \frac{-1}{3}(x-1)$ i.e. $y = \frac{-x}{3} + \frac{1}{3} + \ln 3$. (iv) Solving $x = \frac{2}{5}$ and $x = \frac{2}{1+t^2}$, $y = \ln(2-t)$ $\frac{2}{5} = \frac{2}{1+t^2} \Rightarrow t = -2$ hence $y = \ln(2-(-2)) = \ln(4)$ Point of intersection is $\left(\frac{2}{5},\ \ln 4\right)$.
7	**(a)** Since $z = -2 + 3\mathrm{i}$ is a root of the equation $2z^2 + (-1+4\mathrm{i})z + c = 0$, $2(-2+3\mathrm{i})^2 + (-1+4\mathrm{i})(-2+3\mathrm{i}) + c = 0$ $2(4-9-12\mathrm{i}) + (2-8\mathrm{i}-3\mathrm{i}-12) + c = 0$ $c = 20 + 35\mathrm{i}$ $\therefore\ 2z^2 + (-1+4\mathrm{i})z + 20 + 35\mathrm{i}$ $= \left(z-(-2+3\mathrm{i})\right)\left(2z-(a+\mathrm{i}b)\right)$ $= 2z^2 + (4-6\mathrm{i}-a-b\mathrm{i})z + (-2a-3b-2b\mathrm{i}+3a\mathrm{i})$ Comparing real and imaginary parts of coefficient of z: $-1 = 4 - a$ and $4 = -6 - b$ $\Rightarrow a = 5$ and $b = -10$ The other root is $\frac{1}{2}(5-10\mathrm{i}) = \frac{5}{2} - 5\mathrm{i}$. **OR**

$\therefore \quad 2z^2+(-1+4\text{i})z+20+35\text{i}$
$=\left(z-(-2+3\text{i})\right)\left(2z-(a+\text{i}b)\right)$

Compare the coefficient of z^0,
$20+35\text{i}=(-2+3\text{i})(a+\text{i}b)$

$$(a+\text{i}b)=\frac{20+35\text{i}}{-2+3\text{i}}$$
$$=\frac{20+35\text{i}\left(-2-3\text{i}\right)}{-2+3\text{i}\left(-2-3\text{i}\right)}$$
$$=5+10\text{i}$$

The other root is $\frac{1}{2}(5-10\text{i})=\frac{5}{2}-5\text{i}$.

OR

Let c be p + iq, where p and q are real numbers.
$2z^2+(-1+4\text{i})z+p+\text{i}q$
$=(z-(-2+3\text{i}))(2z-(a+\text{i}b))$
$=2z^2+(-a-\text{i}b+4-6\text{i})z+(-2+3\text{i})(a+\text{i}b)$
$=2z^2+(-a+4-\text{i}\,(b+6))z+(-2a-3b+\text{i}\,(3a-2b))$
Compare real and imaginary parts of the coefficient of z:
$-a+4=-1$ and $-b-6=4$
$\Rightarrow$ $a=5$ and $b=-10$
$\therefore$ The other root is $\frac{1}{2}(5-10\text{i})=\frac{5}{2}-5\text{i}$
Compare real and imaginary parts of the coefficient of z^0:
$p=-2a-3b$ and $q=3a-2b$
$\Rightarrow$ $p=20$ and $q=35$
$\Rightarrow c=20+35\text{i}$

(b)(i) $u-1=\cos\theta-1+\text{i}\sin\theta$
$$=-2\sin^2\frac{\theta}{2}+2\text{i}\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$=2\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}+\text{i}\cos\frac{\theta}{2}\right)$$

$$\left|u-1\right|=2\sin\frac{\theta}{2}$$

$\arg(u-1)=\pi-\tan^{-1}\left(\cot\frac{\theta}{2}\right)$ since $u-1$ is in 2nd quadrant.
$$=\pi-\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)$$
$$=\pi-\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$
$$=\frac{\pi}{2}+\frac{\theta}{2}$$

OR

$u-1=\cos\theta-1+\mathrm{i}\sin\theta$

$=-2\sin^2\dfrac{\theta}{2}+2\mathrm{i}\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

$=2\sin\dfrac{\theta}{2}\left(-\sin\dfrac{\theta}{2}+\mathrm{i}\cos\dfrac{\theta}{2}\right)$

$=2\mathrm{i}\sin\dfrac{\theta}{2}\left(\cos\dfrac{\theta}{2}+\mathrm{i}\sin\dfrac{\theta}{2}\right)$

$|u-1|=\left|2\mathrm{i}\sin\dfrac{\theta}{2}\right|=2\sin\dfrac{\theta}{2}$

$\arg(u-1)=\arg(\mathrm{i})+\dfrac{\theta}{2}=\dfrac{\pi}{2}+\dfrac{\theta}{2}$

(b)(ii) $v^*=-\sqrt{3}-\mathrm{i}=2\mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}$

$(v^*u)^8=\left(2\mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}\mathrm{e}^{\theta}\right)^8=2^8\mathrm{e}^{\mathrm{i}\left(-\frac{40\pi}{6}+8\theta\right)}$

Given $(w^*u)^8$ is real and negative,

$-\dfrac{20}{3}\pi+8\theta=\text{K}\ ,-\pi,\ -3\pi,\ -5\pi,\ -7\pi,\ \text{K}$

$8\theta=\ \text{K}\ ,\ -3\pi+\dfrac{20}{3}\pi,\ -5\pi+\dfrac{20}{3}\pi,\ -7\pi+\dfrac{20}{3}\pi,\ \text{K}$

Smallest $\theta=\dfrac{5\pi}{24}$

8 **(i)** Using direction cosines,

$\cos 45^\circ=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$ $\dfrac{1}{\sqrt{2}}=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$ $-x^2+y^2+z^2=0$	$\cos 45^\circ=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$ $\dfrac{1}{\sqrt{2}}=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$ $x^2-y^2+z^2=0$	$\cos 45^\circ=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$ $\dfrac{1}{\sqrt{2}}=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$ $x^2+y^2-z^2=0$

From G.C, solving the 3 equations, $x=y=z=0$

Since $\mathbf{r}$ is a <u>non-zero vector</u>, $\alpha=\beta=\gamma=45^\circ$ is not possible.

Alternatively,

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\dfrac{1}{\sqrt{2}}\right)^2+\left(\dfrac{1}{\sqrt{2}}\right)^2+\left(\dfrac{1}{\sqrt{2}}\right)^2=\dfrac{3}{2}\neq 1$

$\alpha=\beta=\gamma=45^\circ$ is not possible.

(ii) When $\alpha=\beta=45^\circ$ and $\gamma=90^\circ$, $x=y$ and $z=0$ $\quad \mathbf{r}=\begin{pmatrix}x\\y\\0\end{pmatrix}=\begin{pmatrix}x\\x\\0\end{pmatrix}$

(iii) Let $\mathbf{a}=\begin{pmatrix}x\\x\\0\end{pmatrix}$. Since point B is the foot of perpendicular from point A to the y-axis,

$\Rightarrow \mathbf{b}=\begin{pmatrix}0\\x\\0\end{pmatrix}$.

Since $AC = AB$, and $\overrightarrow{AC}$ is parallel to the z-axis, $\mathbf{c}=\begin{pmatrix}x\\x\\x\end{pmatrix}$ or $\begin{pmatrix}x\\x\\-x\end{pmatrix}$.

Volume of tetrahedron = $\frac{1}{3}(\text{base area})(\text{height})$

$$36=\frac{1}{3}\left(\frac{1}{2}(x)^2\right)(x)$$

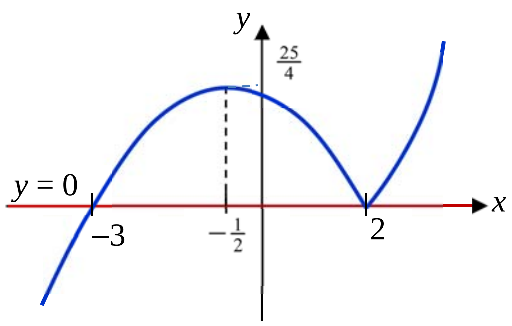
$$x=6$$

Possible coordinates of point C are $(6,6,6)$ or $(6,6,-6)$.

9 **(i)** Since f(–3) = f(2) = 0 where $-3, 2 \in D_f$, f is not one-one.

Thus f^{-1} does not exist.

OR **(i)** Since the horizontal line
y = 0, cuts the graph of
y = f(x) at more than 1 point,
f is not one to on, thus f^{-1} does not exist.



(ii) Since $k \le x \le 2$, f(x) = –(x – 2)(x + 3). Least $k = -\frac{1}{2}$.

Let $y=\mathrm{f}(x)$, where $-\frac{1}{2}\le x\le 2$.

$$y=-(x+3)(x-2)=-(x^2+x-6)=-\left(x+\frac{1}{2}\right)^2+\frac{25}{4}$$

$$\left(x+\frac{1}{2}\right)^2=\frac{25}{4}-y \quad \Rightarrow x=-\frac{1}{2}\pm\sqrt{\frac{25-4y}{4}}$$

Since $-\frac{1}{2}\le x\le 2$, $\quad x=-\frac{1}{2}+\frac{1}{2}\sqrt{25-4y}$

$\therefore \ \mathrm{f}^{-1}: \ x \mapsto -\frac{1}{2}+\frac{1}{2}\sqrt{25-4x}, \quad x\in\mathbb{R}, \ 0\le x\le\frac{25}{4}.$

9

Alternative Method

Let $y = \mathrm{f}(x)$, where $-\frac{1}{2} \le x \le 2$.

$y = -(x+3)(x-2) = -x^2 - x + 6$

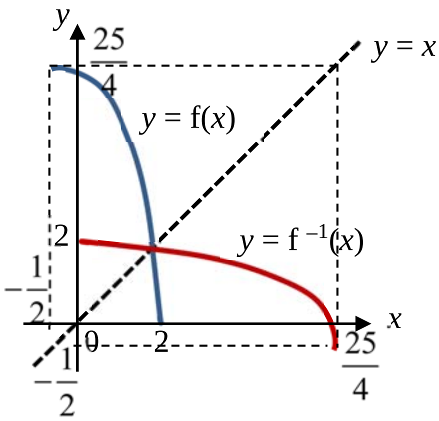
$x^2 + x + y - 6 = 0$

$x = \dfrac{-1 \pm \sqrt{1-4(y-6)}}{2} = \dfrac{-1 \pm \sqrt{25-4y}}{2}$

Since $-\frac{1}{2} \le x \le 2$, $\quad x = \dfrac{-1+\sqrt{25-4y}}{2}$

$\therefore\ \mathrm{f}^{-1}:\ x \mapsto \dfrac{-1+\sqrt{25-4x}}{2},\quad x \in \mathbb{R},\ \ 0 \le x \le \dfrac{25}{4}.$

(iii)



$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$

$\mathrm{f}(x) = x$

$-x^2 - x + 6 = x$

$x^2 + 2x - 6 = 0$

$x = \dfrac{-2 \pm \sqrt{28}}{2} = -1 \pm \sqrt{7}$

From the diagram, $x \ge 0$, $\quad \therefore\ x = -1 + \sqrt{7}$

9

(iv) Range of f $= \left[0,\ \dfrac{25}{4}\right]$, $\quad$ Domain of g^{-1} = Range of g = $(-1,\ 7)$

Since $\mathrm{R_f} \subseteq \mathrm{D}_{\mathrm{g}^{-1}}$, $\ \therefore\ \mathrm{g}^{-1}\mathrm{f}$ exists.

(v) $\mathrm{g}(x) = \dfrac{24}{x^2+3} - 1,\ \ x \in \mathbb{R},\ x > 0$

$\mathrm{g}'(x) = -\dfrac{24(2x)}{(x^2+3)^2} < 0$ for $x > 0$ since $(x^2+3)^2 > 0$

Hence, g is a strictly decreasing function.

	$g^{-1}(f(x))<1 \Rightarrow f(x)>g(1)$ (since g is a strictly decreasing fn) $\Rightarrow f(x)>5$ Since $f^{-1}(5)=-\frac{1}{2}+\frac{1}{2}\sqrt{25-4(5)}=\frac{\sqrt{5}-1}{2}$, from the graph of y = f(x), $-\frac{1}{2}\le x<\frac{\sqrt{5}-1}{2}$
9	OR $g^{-1}(f(x))<1 \Rightarrow f(x)>g(1)$ (since g is a decreasing function) $\Rightarrow f(x)>5$ $\Rightarrow -x^2-x+6>5$ $\Rightarrow x^2+x-1<0$ $\Rightarrow \frac{-1-\sqrt{5}}{2}<x<\frac{-1+\sqrt{5}}{2}$ Since $-\frac{1}{2}\le x\le 2$, $\therefore -\frac{1}{2}\le x<\frac{-1+\sqrt{5}}{2}$
10	**(i)** $\frac{dm}{dt}\propto(u-38.5m)$. Let $\frac{dm}{dt}=k(u-38.5m)$ When $(u-38.5m)=-7700$, $\frac{dm}{dt}=-1\Rightarrow k=\frac{1}{7700}$ $\Rightarrow \frac{dm}{dt}=\frac{1}{7700}(u-38.5m)=\frac{u}{7700}-\frac{m}{200}$ **(ii)** Solve DE: $\frac{dm}{dt}=\frac{u}{7700}-\frac{m}{200}=\frac{1}{200}\left(\frac{2u}{77}-m\right)$ $\int \frac{1}{\frac{2u}{77}-m}\,dm=\int \frac{1}{200}\,dt$ $-\ln\left\|\frac{2u}{77}-m\right\|=\frac{t}{200}+c$ $\frac{2u}{77}-m=\pm e^{-c}.e^{-\frac{t}{200}}=Be^{-\frac{t}{200}}$ $m=\frac{2u}{77}-Be^{-\frac{t}{200}}$ **(iii)** Given: $u=2500$ and $t=0, m=80$, $B=\frac{5000}{77}-80=-\frac{1160}{77}$

	$m=\frac{5000}{77}+\frac{1160}{77}e^{-\frac{t}{200}}$
10	**(iv)** New DE: $38.5\frac{\mathrm{d}M}{\mathrm{d}t}=-\frac{77(77M+154t)+76}{77M+154t+1}$ By substitution: $z=38.5M+77t \Rightarrow \frac{\mathrm{dz}}{\mathrm{d}t}=38.5\frac{\mathrm{d}M}{\mathrm{d}t}+77$ $\frac{\mathrm{dz}}{\mathrm{d}t}-77=-\frac{77(2z)+76}{2z+1}=-77+\frac{1}{2z+1}$ $\frac{\mathrm{dz}}{\mathrm{d}t}=\frac{1}{2z+1}$ Integrate w.r.t. t, $\int (2z+1)\,\mathrm{dz}=\int 1\,\mathrm{d}t$ $z^2+z=t+C$ $\left(z+\frac{1}{2}\right)^2-\frac{1}{4}=t+C \Rightarrow \left(z+\frac{1}{2}\right)^2=t+D$ $\Rightarrow \left(38.5M+77t+\frac{1}{2}\right)^2=t+D$ $\Rightarrow 38.5M+77t+\frac{1}{2}=\sqrt{t+D}$ (-ve square root rejected) $\Rightarrow M=\frac{1}{38.5}\left[\sqrt{t+D}-77t-\frac{1}{2}\right]$ where D is an arbitrary constant

11 **(i)** A.P with a = 600, d = 80

Last contribution is when n = 17

$T_{17} = 600 + (17-1)(80) = 1880$

The last contribution is \$1880.

(ii)

nth contribution	Amount of money in SupremeEdu Fund
1 (1 year old)	1040
2	1040(1.035)+1040
3	(1040(1.035)+1040)(1.035) + 1040 $= 1040(1.035)^2 + 1040(1.035) + 1040$
…	…
17	$1040\left[(1.035)^{16} + (1.035)^{15} + K + (1.035) + 1\right]$
18 *no contribution	$1.035 \times 1040\left[(1.035)^{16} + (1.035)^{15} + K + (1.035) + 1\right]$

G.P with $a = 1040(1.035)$, $r = 1.035$

$$\text{Total amount} = \$\frac{1040(1.035)\left[(1.035)^{17} - 1\right]}{1.035 - 1} = \$24439.67$$

= \$24440 (nearest dollar)

nth contribution	Amount of money in UltimateEdu Fund
1 (1 year old)	600
2 (2 year old)	600+(600+80)+(200)
3	600+(600+80)+(600+80+80)+(200+200)
…	…

Total contribution is an A.P with a = 600, d = 80, n = 17

$S_{17} = \frac{17}{2}\left(2(600) + 16(80)\right) = \$21080$ OR $S_{17} = \frac{17}{2}\left(600 + 1880\right) = \$21080$

Total interest earn = 17 x \$200 = \$3400

Total amount on eighteenth birthday = \$21080 + \$3400 = \$24480

OR $\frac{17}{2}\left(2(800) + (17-1)(80)\right) = \$24480$

11	**(iii)**

	Invested amount	Amt received on 18th Birthday	Interest earned
SupremeEdu	$1040 x 17 = $17680	$24440	$24440 - $1040 x 17 = $6760
UltimateEdu	$21080	$24480	17 x $200 = $3400

Since SupremeEdu earns a higher interest with a lower invested amount, Mr and Mrs Lai should get SupremeEdu instead.

(iv) Chosen SupremeEdu

Let the first investment amount be $*a*.

$$\text{Total amount} = \frac{a(1.035)\left[(1.035)^{17}-1\right]}{1.035-1} \geq 50000$$

$$a \geq 2127.69$$

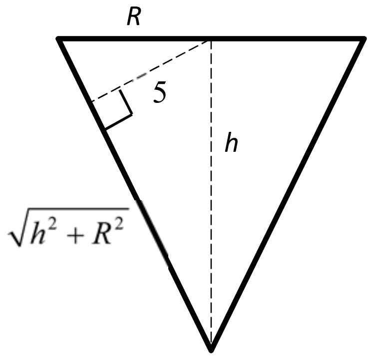
$$a = \$2128 \text{ (nearest dollar)}$$

1

Volume of the ornament (V)

$=\frac{1}{3}\pi R^2 h-\frac{2}{3}\pi(4)^3$

$=\frac{1}{3}\pi R^2 h-\frac{128}{3}\pi$



By similar triangle,

$\frac{R}{\sqrt{h^2+R^2}}=\frac{5}{h}$

$h^2R^2=25\left(h^2+R^2\right)$

$R^2=\frac{25h^2}{h^2-25}$

$\therefore V=\frac{1}{3}\pi\left(\frac{25h^2}{h^2-25}\right)h-\frac{128}{3}\pi=\frac{\pi}{3}\left(\frac{25h^3}{h^2-25}\right)-\frac{128\pi}{3}$

$\frac{dV}{dh}=\frac{25\pi}{3}\left(\frac{\left(h^2-25\right)\left(3h^2\right)-h^3(2h)}{\left(h^2-25\right)^2}\right)=\frac{25\pi}{3}\left(\frac{h^2\left(h^2-75\right)}{\left(h^2-25\right)^2}\right)$

When $\frac{dV}{dh}=0$, $h^2\left(h^2-75\right)=0$

Since $h\neq 0$, $h^2-75=0$

$\Rightarrow h=\sqrt{75}=5\sqrt{3}$ (Since $h>0$)

<u>1st derivative test</u>

	$h=\left(\sqrt{75}\right)^-$	$h=\sqrt{75}$	$h=\left(\sqrt{75}\right)^+$
$\frac{dV}{dh}=\frac{25\pi}{3}\left(\frac{h^2\left(h^2-75\right)}{\left(h^2-25\right)^2}\right)$	$h^2<75$ $h^2-75<0$ $\frac{dV}{dh}<0$	$h^2=75$ $\frac{dV}{dh}=0$	$h^2>75$ $h^2-75>0$ $\frac{dV}{dh}>0$
gradient	\	—	/

<u>2nd derivative test</u>

$\frac{d^2V}{dh^2}=\frac{\left(h^2-25\right)\left(4h^3-150h\right)-2\left(h^2-25\right)(2h)\left(h^2\right)\left(h^2-75\right)}{\left(h^2-25\right)^4}$

When $h=\sqrt{75}$, $\frac{d^2V}{dh^2}=0.51962>0$, $V=\frac{\pi}{3}\left(\frac{25\left(5\sqrt{3}\right)^3}{\left(5\sqrt{3}\right)^2-25}\right)-\frac{128\pi}{3}=206.046$ or

$\frac{\pi}{6}\left(375\sqrt{3}-256\right)$

$\therefore$When $h=5\sqrt{3}$, V is minimum. & $V_{min}=\approx 206\text{ cm}^3$ (3s.f.)

2

(i) $\left(\underset{\sim}{r}-\underset{\sim}{a}\right)\times\underset{\sim}{b}=\underset{\sim}{0}$

Either $\underset{\sim}{b}=\underset{\sim}{0}$ (reject since $\underset{\sim}{b}$ is a non-zero vector) or $\left(\underset{\sim}{r}-\underset{\sim}{a}\right)=\underset{\sim}{0}$ or $\left(\underset{\sim}{r}-\underset{\sim}{a}\right)//\underset{\sim}{b}$

$\Rightarrow \underset{\sim}{r}=\underset{\sim}{a}$ OR $\left(\underset{\sim}{r}-\underset{\sim}{a}\right)=k\underset{\sim}{b}$ for $k\in\mathbb{R}\setminus\{0\}$

$\Rightarrow \underset{\sim}{r}=\underset{\sim}{a}+k\underset{\sim}{b}$ for $k\in\mathbb{R}$ (Note that $\underset{\sim}{r}=\underset{\sim}{a}$ is included in these solutions)

This is the equation of a line passing through point with position vector $\underset{\sim}{a}$ and parallel to the vector $\underset{\sim}{b}$.

(ii) $\underset{\sim}{n}=\begin{pmatrix}0\\1\\1\end{pmatrix}\times\begin{pmatrix}m\\0\\1\end{pmatrix}=\begin{pmatrix}1\\m\\-m\end{pmatrix}$

p_1: $\underset{\sim}{r}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}=\begin{pmatrix}2\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}=2+2m$

p_2: $\underset{\sim}{r}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}=\begin{pmatrix}-1\\2\\3\end{pmatrix}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}=-1-m$

(iii) Let θ be the acute angle between the plane and the line.

$$\sin\theta=\left|\frac{\underset{\sim}{d}\bullet\underset{\sim}{n}}{\left|\underset{\sim}{d}\right|\left|\underset{\sim}{n}\right|}\right|=\frac{\left|\begin{pmatrix}-3\\1\\4\end{pmatrix}\bullet\begin{pmatrix}1\\m\\-m\end{pmatrix}\right|}{\sqrt{26}\sqrt{1+2m^2}}=\frac{|-3-3m|}{\sqrt{26+52m^2}}$$

$$\theta=\sin^{-1}\left|\frac{3+3m}{\sqrt{26+52m^2}}\right|$$

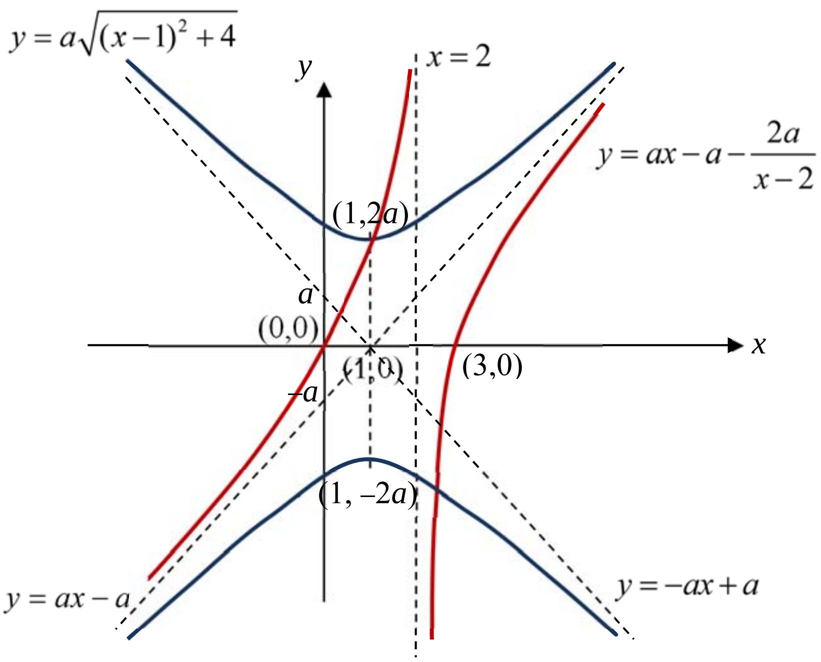
(iv) For p_3 to contain both L_1 and L_2, $\theta=0$.

$\Rightarrow 3+3m=0$

$\Rightarrow m=-1$

Substituting $m=-1$ into either plane equation above, $\underset{\sim}{r}\bullet\begin{pmatrix}1\\-1\\1\end{pmatrix}=0$

$x-y+z=0$ is the Cartesian equation of p_3.

3	$y = ax + b + 2a + \dfrac{2(b+2a)}{x-2}$ $\dfrac{dy}{dx} = a - \dfrac{2(b+2a)}{(x-2)^2}$ When $\dfrac{dy}{dx} = 0$, $\quad \dfrac{2(b+2a)}{(x-2)^2} = a$ $a(x-2)^2 = 2(b+2a)$ For C to have no stationary points, $2(b+2a) < 0$ (since $a > 0$) $b < -2a$ **Alternatively,** When $\dfrac{dy}{dx} = 0$, $\quad \dfrac{2(b+2a)}{(x-2)^2} = a$ $a(x-2)^2 = 2(b+2a)$ $ax^2 - 4ax - 2b = 0$ For C to have no stationary points, Discriminant $= 16a^2 - 4a(-2b) = 8a(2a+b) < 0$ Since $a > 0$, $\quad 2a + b < 0 \Rightarrow b < -2a$
	For $b = -3a$, $\quad C$: $y = ax - a + \dfrac{-2a}{x-2}$ Equations of asymptotes: $\quad y = ax - a$, $x = 2$. $\left(\dfrac{y}{a}\right)^2 - (x-1)^2 = 4 \Rightarrow \left(\dfrac{y}{2a}\right)^2 - \left(\dfrac{x-1}{2}\right)^2 = 1$ Asymptotes: $\dfrac{y}{2a} = \pm\left(\dfrac{x-1}{2}\right) \Rightarrow y = ax - a$ or $y = -ax + a$ 

	$x-1-\dfrac{2}{x-2} \geq \sqrt{(x-1)^2+4}$ From the graphs of $y = ax - a - \dfrac{2a}{x-2}$ and $y = a\sqrt{(x-1)^2+4}$ (*i.e.* $y > 0$), $1 \leq x < 2$.
4	(i) (a) $\displaystyle\int \frac{x}{\sqrt{3+2x-x^2}}\,dx = \int \frac{\frac{-1}{2}(2-2x)+1}{\sqrt{3+2x-x^2}}\,dx$ $\displaystyle = \frac{-1}{2}\int \frac{(2-2x)}{\sqrt{3+2x-x^2}}\,dx + \int \frac{1}{\sqrt{3+2x-x^2}}\,dx$ $\displaystyle = \frac{-1}{2}.2\sqrt{3+2x-x^2} + \int \frac{1}{\sqrt{4-(x-1)^2}}\,dx$ $\displaystyle = \sin^{-1}\left(\frac{x-1}{2}\right) - \sqrt{3+2x-x^2} + c$ (i) (b) $\displaystyle\int \frac{x^2}{3+2x-x^2}\,dx = \int \left(-1+\frac{9}{4(3-x)}+\frac{1}{4(1+x)}\right)dx$ $\displaystyle = -x - \frac{9}{4}\ln(3-x) + \frac{1}{4}\ln(1+x) + c$ (ii) Area = $\displaystyle\int_0^2 \frac{x}{\sqrt{3+2x-x^2}}\,dx - \int_{1-\sqrt{3}}^{0} \frac{x}{\sqrt{3+2x-x^2}}\,dx$ $\displaystyle = \sin^{-1}\left(\frac{1}{2}\right) - \sqrt{3} - \left[\sin^{-1}\left(\frac{-1}{2}\right) - \sqrt{3}\right]$ $\displaystyle - \left\{\sin^{-1}\left(\frac{-1}{2}\right) - \sqrt{3} - \left[\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) - \sqrt{1}\right]\right\}$ where $\sqrt{3+2(1-\sqrt{3})-(1-\sqrt{3})^2} = 1$ $\displaystyle = \frac{\pi}{6} + \frac{\pi}{6} - \left[-\frac{\pi}{6} - \sqrt{3} - \left(-\frac{\pi}{3}\right) + 1\right] = \frac{\pi}{6} + \sqrt{3} - 1$ (iii) Volume = $\displaystyle\pi\int_1^2 \frac{x^2}{3+2x-x^2}\,dx$ $\displaystyle = \pi\left\{-2 - \frac{9}{4}\ln(1) + \frac{1}{4}\ln(3) - \left[-1 - \frac{9}{4}\ln(2) + \frac{1}{4}\ln(2)\right]\right\}$ $\displaystyle = \pi\left\{\frac{1}{4}\ln(3) + 2\ln(2) - 1\right\}$

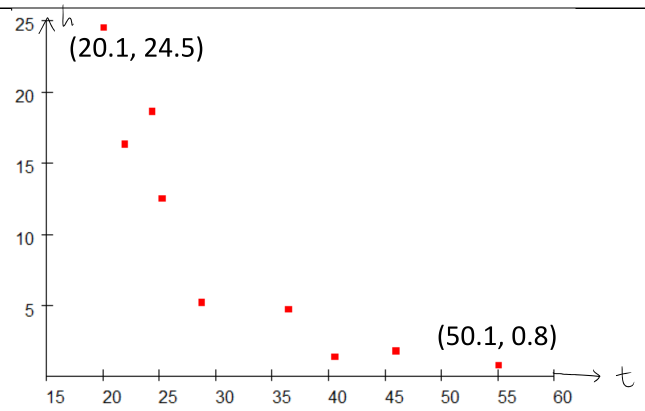
5	(i) $\dfrac{{}^{18}C_6 \times {}^{12}C_6 \times {}^{6}C_6}{3!} = 2858856$ (Shown) (ii) Number of ways M and J in the same group $= \dfrac{{}^{16}C_4 \times {}^{12}C_6 \times {}^{6}C_6}{2!} = 840840$ Required probability = $\dfrac{840840}{2858856} = 0.294$ (3 s.f.) Case 1 : 3 particular participants stand in row 3 Number of ways = $3! \times 6! = 4320$ Case 2 : 3 participants stand in row 4 Number of ways = $2 \times 3! \times 6! = 8640$ Total number of ways = 12960
6	(i) Assumption: The probability that the eggs are lighter than standard weight is constant. The weight of an (randomly chosen) egg is independent of the weight of another (randomly chosen) egg. (ii) Let X be the number of eggs that are lighter than standard weight in a tray of 30 eggs. $X \sim \mathrm{B}(30, p)$ $\mathrm{P}(X = 0, 1) = \mathrm{P}(X = 0) + \mathrm{P}(X = 1)$ $= \binom{30}{0} p^0 (1-p)^{30} + \binom{30}{1} p^1 (1-p)^{29}$ $G = (1-p)^{30} + 30p(1-p)^{29} = (1-p)^{29}(1+29p)$ (iii) $(1-p)^{29}(1+29p) = 0.96$ Using GC, $p = 0.1057495\ldots$ (iv) Let Y be the number of trays that are rejected in a day. $Y \sim \mathrm{B}(50, 1-0.96)$ $Y \sim \mathrm{B}(50, 0.04)$ $\mathrm{P}(1 \le Y \le 3) = \mathrm{P}(Y \le 3) - \mathrm{P}(Y = 0) = 0.731$ Probability that two trays are rejected if the day's production is rated as 'fair' $= \mathrm{P}(\text{2 trays are rejected} \mid \text{the day's production is 'fair'})$ $= \dfrac{\mathrm{P}(Y = 2 \text{ and } 1 \le Y \le 3)}{\mathrm{P}(1 \le Y \le 3)} = \dfrac{\mathrm{P}(Y=2)}{\mathrm{P}(1 \le Y \le 3)}$ $= \dfrac{0.2762328}{0.7309834} = 0.378$
7	(i) Let L be the amount of drink in a large cup and S be the amount of drink in a small cup. $L \sim \mathrm{N}(405, 74)$ $S \sim \mathrm{N}(202, 21)$ $L_1 + L_2 - 4S \sim \mathrm{N}(2, 484)$ $\mathrm{P}(L_1 + L_2 - 4S < 0) = 0.464$

	(ii) He can buy 3 small cups of drink OR 1 large cup and 1 small cup of drink. 3 small cups of drink : $S_1+S_2+S_2 \sim \mathrm{N}(606,63)$ $\mathrm{P}(S_1+S_2+S_2>600)=0.775$ 1 large cup and 1 small cup of drink : $L+S \sim \mathrm{N}(607,95)$ $\mathrm{P}(L+S>600)=0.764$ He should buy 3 small cups. (Note: the amount he needs to spend is the same for both cases) (iii) Let $T=\dfrac{S_1+S_2+...+S_{20}+L_1+L_2+...+L_n}{20+n}$ $T \sim N\left(\dfrac{20(202)+n(405)}{20+n},\dfrac{20(21)+n(74)}{(20+n)^2}\right)$ $T \sim N\left(\dfrac{4040+405n}{20+n},\dfrac{420+74n}{(20+n)^2}\right)$ $\mathrm{P}(T>350)>0.8$ Using GC, when n = 54, $\mathrm{P}(T>350)=0.5598<0.8$ when n = 55, $\mathrm{P}(T>350)=0.834>0.8$ Least value of n = 55 **OR** $\mathrm{P}\left(Z>\dfrac{350-\dfrac{4040+405n}{20+n}}{\sqrt{\dfrac{420+74n}{(20+n)^2}}}\right)>0.8$ $\dfrac{350-\dfrac{4040+405n}{20+n}}{\sqrt{\dfrac{420+74n}{(20+n)^2}}}<-0.8416212$ Using GC, Least value of n = 55
8	(i) $\mathrm{P}(X=2)=\left(\dfrac{1}{4}\right)^2\left(\dfrac{3}{4}\right)^2\,{}^4C_2=\dfrac{27}{128}$ (ii) C = ‘Congratulations’ (+2 points), T = ‘Thank You’ (-1 points)

X	cards	Prob	
-4	0C4T	$\left(\dfrac{3}{4}\right)^4=\dfrac{81}{256}$	$\dfrac{81}{256}$
-1	1C3T	$\left(\dfrac{1}{4}\right)^1\left(\dfrac{3}{4}\right)^3\,{}^4C_1=\dfrac{27}{64}$	$\dfrac{108}{256}$
2	2C2T	$\dfrac{27}{128}$	$\dfrac{54}{256}$
5	3C1T	$\left(\dfrac{1}{4}\right)^3\left(\dfrac{3}{4}\right)^1\,{}^4C_3=\dfrac{3}{64}$	$\dfrac{12}{256}$
8	4C0T	$\left(\dfrac{1}{4}\right)^4=\dfrac{1}{256}$	$\dfrac{1}{256}$

	(iii) E (X) $=-4\left(\frac{81}{256}\right)-1\left(\frac{27}{64}\right)+2\left(\frac{27}{64}\right)+5\left(\frac{3}{64}\right)+8\left(\frac{1}{256}\right)=-1$ Mean amount = E (8+ 2X) = (8+2 (-1)) = 6 $\text{Var}(X)=E(X^2)-[E(X)]^2$ $=\left[(-4)^2\left(\frac{81}{256}\right)+(-1)^2\left(\frac{27}{64}\right)+2^2\left(\frac{27}{64}\right)+5^2\left(\frac{3}{64}\right)+8^2\left(\frac{1}{256}\right)\right]-(-1)^2$ $=\frac{31}{4}-1=\frac{27}{4}$ Variance amount = Var (8+ 2X) = 4 Var (X) = $4\left(\frac{27}{4}\right)=27$ The total value of the vouchers the shop keeper is expected to give away = 100 (6) = \$600 Let T be the total value of the vouchers given away to 100 customers By CLT, as sample size 100 is large, $T\sim \text{N}(6(100),27(100))$ $T\sim \text{N}(600,2700)$ approximately Probability that the total value of the vouchers given away is more than \$650. $=\text{P}(T>650)\approx 0.168$
9	a(i) Let $v=x-150$, $w=y-170$ $d=\dfrac{\sum vw-\frac{\sum v\sum w}{n}}{\sum w^2-\frac{(\sum w)^2}{n}}=\dfrac{677-\frac{47\times 87}{8}}{1101-\frac{87^2}{8}}=1.0710$ $\overline{x}=\overline{v}+150=\frac{47}{8}+150=155.875$; $\overline{y}=\overline{w}+170=\frac{87}{8}+170=180.875$ Regression line of x on y : $x-\overline{x}=d(y-\overline{y})$ $\Rightarrow x-155.875=1.0710(y-180.875)$ $\Rightarrow x=1.07y-37.8$ (3sf)
(ii)	a(ii) $r_{xy}=r_{vw}=\dfrac{\sum vw-\frac{\sum v\sum w}{n}}{\sqrt{\left[\sum v^2-\frac{(\sum v)^2}{n}\right]\left[\sum w^2-\frac{(\sum w)^2}{n}\right]}}$ $=\dfrac{677-\frac{47\times 87}{8}}{\sqrt{\left(483-\frac{47^2}{8}\right)\left(1101-\frac{87^2}{8}\right)}}=0.92668\approx 0.927$

b) (i)



A linear model is not suitable, from the scatter diagram as t increases, h decrease at a decreasing rate, whereas a linear model predicts a constant rate of decrease.

(ii)

Model (A) between h and $\frac{1}{t}$, $r = 0.94438$

Model (B) between h and $\ln(t)$, $r = -0.90997$

Model A has a better fit since $|r| = 0.94438 > |r| = 0.90997$
(or) Model A has a better fit since $|r| = 0.94438$ is closer to 1.

(iii)

From GC, regression line $h = -15.264 + \dfrac{738.12}{t}$

$$h = -15.264 + \frac{738.12}{15} = 33.944 \approx 33.9 \text{ (3 s.f.)}$$

$t = 15$ lies outside the data range of t (extrapolation), thus the linear relationship between h and $\frac{1}{t}$ may not be valid beyond the data range, hence estimate is not likely to be reliable.

10 (i)

From sample,

Unbiased estimate of population mean = $\bar{x} = \dfrac{250.5}{100} = 2505$

Unbiased estimate of population variance = $s^2 = \dfrac{1}{99}\left(628.45 - \dfrac{250.5^2}{100}\right) = 0.0095707$

(ii)

Let X be the number of hours a student spend on In-star-gram and μ be the mean usage hours.

$H_0: \mu = 2.49$

$H_1: \mu > 2.49$

Since the distribution of X is unknown, population variance is unknown and $n = 100$ is large, by Central Limit Theorem, $\bar{X} : N\left(\mu, \dfrac{s^2}{n}\right)$ approximately.

Under H_0, test statistics is $Z = \dfrac{\bar{X} - 2.49}{s/\sqrt{n}} \sim N(0,1)$.

H_0 is rejected in a one tailed test under α% significance level

⇨ p-value < α%

	Sample readings : $\mu_o = 2.49$, $\overline{x} = 2.505$, $n = 100$, $s = 0.097830$ $p = 0.0626$ Since we reject H0, $\alpha > 6.26$. Assumption: The sample is randomly obtained.
(iii)	The p – value is the probability of getting a mean usage time of 2.505 or more from any sample of size 100 if the population mean is indeed 2.49. (i.e. There is 6.26% chance of getting a sample mean of 2.505 or more if the population mean is 2.49.)
	H_0: $\mu = 2.49$ H_1: $\mu \neq 2.49$ It is given that the standard deviation $\sigma = 1.5$. Since the distribution of X is unknown and $n = 100$ is large, by Central Limit Theorem, $\overline{X} : \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$ approximately. Under H_0, test statistics is $Z = \frac{\overline{X} - 2.49}{\sigma/\sqrt{n}} \sim N(0,1)$. Since there is sufficient evidence at 5% level of significance to conclude that average usage hours of Instar-gram is not 2.49 hours, H_0 is rejected. Reject H0 if $z < -1.95996$ or $z > 1.95996$. Sample readings : $\mu_o = 2.49$, $\overline{x} = k$, $n = 100$, $\sigma = 1.5$ $\frac{k - 2.49}{1.5/\sqrt{100}} < -1.95996$ or $\frac{k - 2.49}{1.5/\sqrt{100}} > 1.95996$ $\Rightarrow k < 2.20$ or $k > 2.78$ (3s.f.)