2017 RVHS Prelim Paper 1

1. **(i)** Describe a sequence of transformations that transform the graph of $y = \ln x$ onto the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \ln(x+a)+b$ and that a and b are constants such that $a > 1$ and $b > 1$. [2]

(ii) By sketching the graph of $y = \mathrm{f}(x)$ or otherwise, sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$.

State, in terms of a and b, the coordinates of any points where $y = \dfrac{1}{\mathrm{f}(x)}$ crosses the axes and the equations of any asymptotes. [3]

2. A curve C has equation $y = \dfrac{2x^2+3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$.

(i) Sketch C, stating the equations of the asymptotes, axial intercepts and the coordinates of the turning points, if any. [3]

(ii) Using part **(i)**, solve the inequality $2x+2 \leq \mathrm{e}^x - \dfrac{5}{x-1}$. [2]

(iii) Hence, solve the inequality $2x+4 \leq \mathrm{e}^{x+1} - \dfrac{5}{x}$. [2]

3. **(i)** By using the substitution $t = 3\sec\theta$, find $\displaystyle\int \frac{\sqrt{t^2-9}}{t}\,\mathrm{d}t$. [4]

(ii) The curve C is defined by the parametric equations

$$x = \ln t, \quad y = \sqrt{t^2-9}\text{, where } t \geq 3.$$

Find the exact value of the area of the region bounded by C, the line $x = \ln 6$ and the x-axis. [4]

4. Henry and Isaac take part in a marathon race. In their first training session, they run a distance of 2.4 km each.

(a) Henry increases the distance he runs in each subsequent training session by 400 m.

(i) Find the distance he runs in the 20^{th} session. [2]

(ii) Find the minimum number of sessions he needs to attend in order to run a total distance of 99 km. [3]

(b) **(i)** Isaac increases the distance he runs in each subsequent session by x %. Find x if Isaac runs a total distance of 200 km at the end of 20 sessions. [3]

(ii) Isaac feels that the training is too tough after the first session. He decides to decrease the distance he runs in each subsequent session by 5% and increase the numbers of sessions. Will he be able to run a total distance of 200 km? Justify your answer. [2]

5. With reference to the origin O, the positon vectors of three points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Given that $|\mathbf{a}| = 4$, $|\mathbf{b}| = 3$, $\mathbf{c}$ is a unit vector and the angle AOC is $\frac{\pi}{3}$ radians.

(i) Find the value of $\mathbf{a} \cdot \mathbf{c}$ and give the geometrical interpretation of this value. [2]

(ii) Given $\mathbf{a} - \mathbf{c} = k\mathbf{b}$ where $k \in \mathbb{R}$, $k \neq 0$. By considering $(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c})$, find the exact values of k. [3]

The point M divides OC in the ratio $OM : OC = 2 : 3$.

(iii) Find the exact area of triangle AMC. [4]

6. **Do not use a calculator in answering this question.**

(a) Solve the simultaneous equations

$$z-4w=11+6\text{i} \text{ and } 3z+6\text{i}w=27$$

giving z and w in the form $x+\text{i}y$ where x and y are real. [4]

(b) **(i)** The complex numbers z and w are given as $z=4\left(\cos\frac{\pi}{3}-\text{i}\sin\frac{\pi}{3}\right)$ and $w=1+\text{i}\sqrt{3}$. w^* denotes the conjugate of w. Find the modulus r and the argument θ of $\frac{w^*}{z^2}$, where $r>0$ and $-\pi<\theta\le\pi$. [3]

(ii) Find the set of possible values of n such that $\left(\frac{w^*}{z^2}\right)^n$ is purely imaginary. [3]

7. **(a)** Show that $\int\sqrt{5-x^2}\,\text{d}x=\frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$. [4]

(b) **(i)** Let C be the curve $y^4+x^2=5$. The x-coordinate of the point P on C is 1 and the y-coordinate of the point P on C is positive. Show that the gradient of the normal to C at the point P is $4\sqrt{2}$. Hence find the equation of the normal to C at the point P in exact form. [4]

(ii) The region R is bounded by the curve C. The solid S is formed by rotating the region R through π radians about the x-axis. Using part **(a)**, find the exact volume of the solid S in terms of π. [3]

8. (a) Using differentiation, find the exact dimensions of the rectangle of largest area that can be inscribed in the ellipse, $\frac{x^2}{9}+\frac{y^2}{36}=1$. Hence, find the area of this largest rectangle. [8]

(b) In the triangle DEF, angle $EDF=\frac{\pi}{3}$ and angle $DFE=\frac{\pi}{3}+\alpha$ and $EF=6$. Given that α is sufficiently small, show that

$$DF-DE\approx d\alpha\ ,$$

where d is an exact constant to be determined. [5]

9. The line l has equation $\frac{x-2}{4}=\frac{z+3}{1}$, $y=2$ and the plane p_1 has equation $\mathbf{r}.\begin{pmatrix}1\\1\\1\end{pmatrix}=16$.

Referred to the origin O, the position vector of the point A is $2\mathbf{i}+2\mathbf{j}-3\mathbf{k}$.

(i) Find the acute angle between the line l and the plane p_1. [2]

(ii) Find the coordinates of the foot of perpendicular, N, from point A to the plane p_1. [3]

(iii) Find the coordinates of the point B which is the reflection of A in plane p_1. [2]

(iv) Hence, determine the equation of the line which is a reflection of line l in the plane p_1. [4]

(v) Another plane, p_2, contains the point B and is parallel p_1. Determine the exact distance between p_1 and p_2. [2]

10. In a farm, the growth of the population of prawns is studied.

(a) The population of prawns of size n thousand at time t months satisfies the differential equation

$$\frac{d^2n}{dt^2} = e^{-\frac{t}{5}}.$$

(i) Find the general solution of this differential equation. [2]

(ii) It is given that initially, the size of the population of prawns is 50 000. Sketch on a single diagram, two distinct solution curves for the differential equation to illustrate the following two cases for large values of t :

I. the size of the population of prawns increases indefinitely,

II. the size of the population of prawns stabilizes at a certain positive number. [3]

(b) In order for the prawns to grow faster and be more resistance to diseases, a drug is administered to the prawns. The prawn's body metabolizes (breaks down) the drug at a rate proportional to the amount of drug, x mg, present in the body at time t hours.

(i) Given that the initial dosage is 0.1 mg, show that $x = \frac{1}{10}e^{-kt}$, where $k > 0$.

[4]

(ii) The half-life of a drug is defined as the time taken for half of it to be metabolized. Given that the half-life of this drug is 4 hours, find the exact value of k. [2]

(iii) If 0.1 mg of this drug is administered to the prawn every 8 hours, show that the total amount of drug present in the prawn's body at any time t is always less than 0.15 mg. [3]

END OF PAPER

ANNEX B

RVHS H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Graphs and Transformation	(ii) y-intercept $\left(0, \dfrac{1}{(\ln a)+b}\right)$ vertical asymptote: $x=-a+e^{-b}$ horizontal asymptote: $y=0$
2	Equations and Inequalities	(ii) $x<1$ or $x \geq 2.34$ (iii) $x<0$ or $x \geq 1.34$
3	Application of Integration	(i) $3\left(\dfrac{\sqrt{t^2-9}}{3}-\cos^{-1}\left(\dfrac{3}{t}\right)\right)+c$ (ii) $3\sqrt{3}-\pi$
4	AP and GP	(ai) 10 (aii) 18 (bi) 13.2% (bii) No
5	Vectors	(i) 2; $\lvert\mathbf{a} \circ \mathbf{c}\rvert$ is the length of projection of $\mathbf{a}$ onto $\mathbf{c}$ (ii) $k=\pm\dfrac{\sqrt{13}}{3}$ (iii) $\dfrac{\sqrt{3}}{3}$
6	Complex numbers	(a) $w=-1-i$, $z=7+2i$ (bi) $\dfrac{1}{8}, \dfrac{\pi}{3}$ (bii) $\left\{n : n=\dfrac{3(2m+1)}{2},\ m \in \circ\right\}$
7	Application of Integration	(bi) $y=4\sqrt{2}x-3\sqrt{2}$ (bii) $\dfrac{5}{2}\pi^2$
8	Differentiation & Applications	(a) $x=\dfrac{3}{\sqrt{2}}$, $y=3\sqrt{2}$; 36 units^2 (b) $-4\sqrt{3}\alpha$
9	Vectors	(i) $\theta=44.4^\circ$ (ii) $(7,7,2)$

		(iii) $(12,12,7)$ (iv) $l_{BC}:\mathbf{r}=\begin{pmatrix}14\\2\\0\end{pmatrix}+s\begin{pmatrix}2\\-10\\-7\end{pmatrix},\ s\in\circ$ (v) $5\sqrt{3}$units
10	Differential Equations	(ai) $n=25\mathrm{e}^{-\frac{t}{5}}+Ct+D$ (bii) $k=-\frac{1}{4}\ln\frac{1}{2}=\frac{\ln 2}{4}$

1

(i)

Step 1: Translation of a units in the negative x-axis direction;

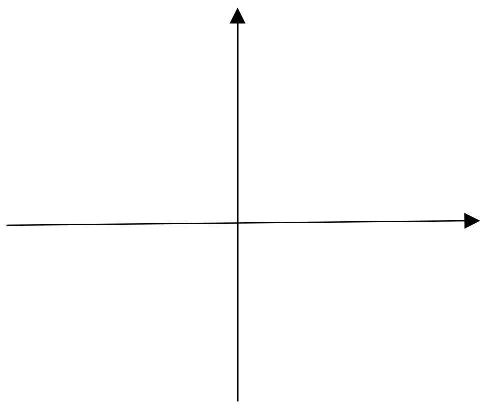
Step 2: Translation of b units in the positive y-axis direction.

OR

Step 1: Translation of b units in the positive y-axis direction;

Step 2: Translation of a units in the negative x-axis direction.

(ii)



y-intercept $\left(0, \dfrac{1}{(\ln a)+b}\right)$

vertical asymptote: $x = -a + e^{-b}$

horizontal asymptote: $y = 0$

2

(i)

By long division,

$$y = \frac{2x^2+3}{x-1}$$
$$= 2x + 2 + \frac{5}{x-1}$$

y-intercept A $(0, -3)$

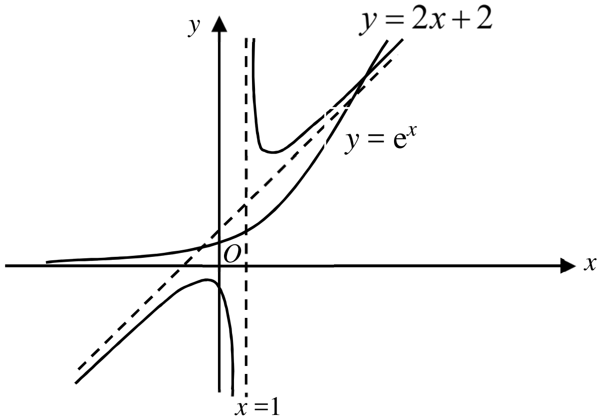
Max point B $(-0.581, -2.32)$

Min point C $(2.58, 10.3)$

(ii)

$$2x+2 \le e^x - \frac{5}{x-1}$$

$$2x+2+\frac{5}{x-1} \le e^x$$



Intersection of both curves: (2.34, 10.4)

$x<1$ or $x \ge 2.34$

(iii)

Replacing x by $x+1$

$x+1<1$ or $x+1 \ge 2.34$

$x<0$ or $x \ge 1.34$

3 (i)

Given $t = 3\sec\theta \Rightarrow \frac{dt}{d\theta} = 3\sec\theta\tan\theta$

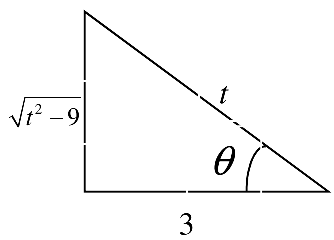
$$\int \frac{\sqrt{t^2-9}}{t}\,dt$$

$$= \int \sqrt{9\sec^2\theta - 9}\left(\frac{1}{3\sec\theta}\right)(3\sec\theta\tan\theta)\,d\theta$$

$$= 3\int \tan^2\theta\,d\theta$$

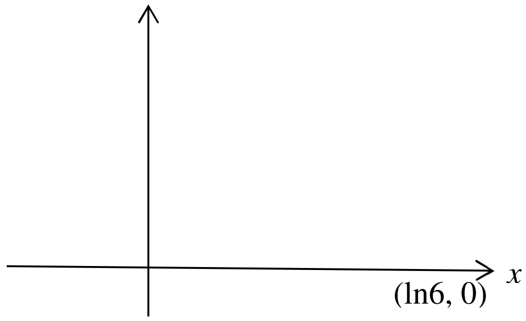
$$= 3\int \sec^2\theta - 1\,d\theta$$

$$= 3(\tan\theta - \theta) + c$$



$$= 3\left(\frac{\sqrt{t^2-9}}{3} - \cos^{-1}\left(\frac{3}{t}\right)\right) + c$$

(ii)



$$\frac{dx}{dt} = \frac{1}{t}$$

Area of $S = \int_{\ln 3}^{\ln 6} y \, dx$

$$= \int_3^6 \sqrt{t^2-9}\left(\frac{1}{t}\right) dt$$

$$= \int_3^6 \frac{\sqrt{t^2-9}}{t} \, dt$$

$$= 3\left[\frac{\sqrt{t^2-9}}{3} - \cos^{-1}\left(\frac{3}{t}\right)\right]_3^6$$

$$= 3\left(\frac{\sqrt{27}}{3} - \frac{\pi}{3}\right)$$

$$= 3\sqrt{3} - \pi$$

4 (ai)

AP: $a = 2.4$, $d = 0.4$

Distance he runs in the 20th session

$= 2.4 + (20-1)(0.4)$

$= 10$ km

(aii)

$S_n \geq 99$

$\Rightarrow \quad \frac{n}{2}\left[2(2.4)+(n-1)(0.4)\right] \geq 99$

$\Rightarrow \quad n\left[4.8+0.4n-0.4\right] \geq 198$

$\Rightarrow \quad 0.4n^2 + 4.4n - 198 \geq 0$

$\Rightarrow \quad n \leq -28.4 \quad$ or $\quad n \geq 17.4$

(rejected as $n > 0$)

Least value of $n = 18$

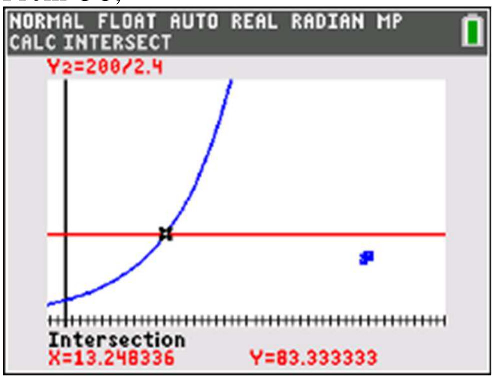
He needs a minimum of 18 sessions.

(bi)

$$S_{20}=\frac{2.4\left(\left(1+\frac{x}{100}\right)^{20}-1\right)}{\left(1+\frac{x}{100}\right)-1}=200$$

$$\frac{\left(1+\frac{x}{100}\right)^{20}-1}{\frac{x}{100}}=\frac{200}{2.4}$$

From GC,



x = 13.2%

(bii)

$$\text{Sum to infinity}=\frac{2.4}{1-0.95}$$
$$=48$$

Hence, total distance can never be greater than 200 km.

5 (i)

$$\mathbf{a}\cdot\mathbf{c}\ =4(1)\cos\frac{\pi}{3}=2$$

$|\mathbf{a}\cdot\mathbf{c}|$ is the length of projection of $\mathbf{a}$ onto $\mathbf{c}$

(ii)

$$(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c})=k\mathbf{b}\cdot k\mathbf{b}$$

$$\mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{c}-\mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{c}=k^2\mathbf{b}\cdot\mathbf{b}$$

$$|\mathbf{a}|^2-2\mathbf{a}\cdot\mathbf{c}+|\mathbf{c}|^2=k^2|\mathbf{b}|^2$$

$$16-2(2)+1=9k^2$$

$$k^2=\frac{13}{9}$$

$$k=\pm\frac{\sqrt{13}}{3}$$

	(iii) $\overrightarrow{MC} = \frac{1}{3}\mathbf{c}$ Area of triangle *AMC* $= \frac{1}{2}\left\|\overrightarrow{AC} \times \overrightarrow{MC}\right\|$ $= \frac{1}{2}\left\|(\mathbf{c}-\mathbf{a}) \times \frac{1}{3}\mathbf{c}\right\|$ $= \frac{1}{6}\left\|\mathbf{c}\times\mathbf{c} - \mathbf{a}\times\mathbf{c}\right\|$ $= \frac{1}{6}\left\|\mathbf{a}\times\mathbf{c}\right\|$ $= \frac{1}{6}\|\mathbf{a}\|\|\mathbf{c}\|\sin\left(\frac{\pi}{3}\right)$ $= \frac{1}{6}(4)(1)(\frac{\sqrt{3}}{2})$ $= \frac{\sqrt{3}}{3}$
6	(a) $z - 4w = 11 + 6\text{i}$ $z = 4w + 11 + 6\text{i}$ Sub above equation into $3z + 6iw = 27$, $3(4w + 11 + 6\text{i}) + 6\text{i}w = 27$ $12w + 33 + 18\text{i} + 6\text{i}w = 27$ $w(12 + 6\text{i}) = -6 - 18\text{i}$ $w = \frac{-6-18\text{i}}{12+6\text{i}}$ $= -1 - \text{i}$ $z = 4w + 11 + 6\text{i}$ $= 4(-1-\text{i}) + 11 + 6\text{i}$ $= 7 + 2\text{i}$ <u>ALT</u> $z - 4w = 11 + 6\text{i}$ $\times 3, \quad 3z - 12w = 33 + 18\text{i} \ldots(1)$ $3z + 6\text{i}w = 27 \ldots(2)$ (2) − (1), $6\text{i}w + 12w = -6 - 18\text{i}$ $w = \frac{-6-18\text{i}}{12+6\text{i}}$ $= -1 - \text{i}$

$z = 4w + 11 + 6\text{i}$

$= 4(-1-\text{i}) + 11 + 6\text{i}$

$= 7 + 2\text{i}$

(bi)

$|z| = 4$ & $\arg z = -\dfrac{\pi}{3}$

$|w| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$ & $\arg w = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

$$\left|\frac{w^*}{z^2}\right| = \frac{|w^*|}{|z^2|} = \frac{|w|}{|z|^2} = \frac{2}{16} = \frac{1}{8}$$

$$\arg\left(\frac{w^*}{z^2}\right) = \arg\left(w^*\right) - \arg\left(z^2\right)$$

$$= -\arg w - 2\arg z$$

$$= -\left(\frac{\pi}{3}\right) - 2\left(-\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}$$

(bii)

$$\frac{w^*}{z^2} = \frac{1}{8}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

$$\left(\frac{w^*}{z^2}\right)^n = \left(\frac{1}{8}\right)\left[\cos\left(\frac{n\pi}{3}\right) + i\sin\left(\frac{n\pi}{3}\right)\right]$$

For $\left(\dfrac{w^*}{z^2}\right)^n$ to be purely imaginary,

$$\cos\left(\frac{n\pi}{3}\right) = 0$$

$$\frac{n\pi}{3} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$$

$$= \frac{(2m+1)\pi}{2}, m \in \mathbb{Z}$$

$$n = \frac{3(2m+1)}{2}, m \in \mathbb{Z}$$

$$\left\{ n : n = \frac{3(2m+1)}{2},\ m \in \mathbb{Z} \right\}$$

7 (a)

$$\int \sqrt{5-x^2}\ \mathrm{d}x = x\sqrt{5-x^2} - \int \frac{-x^2}{\sqrt{5-x^2}}\ \mathrm{d}x$$

$$= x\sqrt{5-x^2} - \int \frac{\left(5-x^2\right)-5}{\sqrt{5-x^2}}\ \mathrm{d}x$$

$$= x\sqrt{5-x^2} - \int \sqrt{5-x^2}\ \mathrm{d}x + 5\int \frac{1}{\sqrt{5-x^2}}\ \mathrm{d}x$$

$$= x\sqrt{5-x^2} - \int \sqrt{5-x^2}\ \mathrm{d}x + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

$$\Rightarrow 2\int \sqrt{5-x^2}\ \mathrm{d}x = x\sqrt{5-x^2} + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c'$$

$$\Rightarrow \int \sqrt{5-x^2}\ \mathrm{d}x = \frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$$

(bi)

$y^4 + x^2 = 5$

Differentiating wrt *x*,

$$4y^3\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$$

When $x = 1$, $y^4 = 4$

$y = \pm\sqrt{2}$

At $\left(1, \sqrt{2}\right)$, $4\left(\sqrt{2}\right)^3\frac{\mathrm{d}y}{\mathrm{d}x} = -2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{4\sqrt{2}}$$

Gradient of normal at $\left(1, \sqrt{2}\right)$

$$= -\frac{1}{-\frac{1}{4\sqrt{2}}}$$

$= 4\sqrt{2}$ (shown)

Equation of normal: $y - \sqrt{2} = 4\sqrt{2}\left(x - 1\right)$

$y = 4\sqrt{2}x - 3\sqrt{2}$

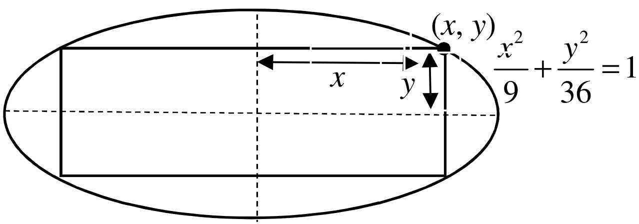
(bii)

Volume of *S*= $\pi\int_{-\sqrt{5}}^{\sqrt{5}} y^2\ \mathrm{d}x = 2\pi\int_0^{\sqrt{5}} \sqrt{5-x^2}\ \mathrm{d}x$

$$= 2\pi\left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_0^{\sqrt{5}}$$

$$= 2\pi\left[\frac{5}{2}\left(\frac{\pi}{2}\right) - 0\right] = \frac{5}{2}\pi^2$$

8 (a)



Let (*x*, *y*) be a point on the ellipse.

Area of rectangle, *A*

$= (2x)(2y)$

$= 4xy$

$= 4x\sqrt{36-4x^2}$

$= 8\sqrt{9x^2 - x^4}$

$= 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 8\left(\frac{1}{2}\right)\left(9x^2 - x^4\right)^{-\frac{1}{2}}\left(18x - 4x^3\right)$$

$$= \frac{4\left(18x - 4x^3\right)}{\sqrt{9x^2 - x^4}}$$

When the area is the largest,

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 0$$

$$\frac{4\left(18x-4x^3\right)}{\sqrt{9x^2 - x^4}} = 0$$

$$18x - 4x^3 = 0$$

$$2x\left(3-\sqrt{2}x\right)\left(3+\sqrt{2}x\right) = 0$$

$x = 0$ (rejected since $x \neq 0$)

or $x = \dfrac{3}{\sqrt{2}}$

or $x = -\dfrac{3}{\sqrt{2}}$ (rejected since $x > 0$)

When $x = \dfrac{3}{\sqrt{2}}$, $y = 3\sqrt{2}$

x	2.115	$\frac{3}{\sqrt{2}} \approx 2.12$	2.125
$\frac{\mathrm{d}A}{\mathrm{d}x}$	0.2013	0	-0.118
Slope	/	—	\

Area of the rectangle is a maximum

Maximum area

$= 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$

$= 8\sqrt{9\left(\frac{3}{\sqrt{2}}\right)^2 - \left(\frac{3}{\sqrt{2}}\right)^4}$

$= 36 \text{ units}^2$

<u>ALT</u>

Note: $A = 8\left(9x^2 - x^4\right)^{\frac{1}{2}}$

Since x, $y > 0$, value of x that maximises A also maximises A^2

$A^2 = 64\left(9x^2 - x^4\right)$

$\frac{\mathrm{d}A^2}{\mathrm{d}x} = 64\left(18x - 4x^3\right) = 0$

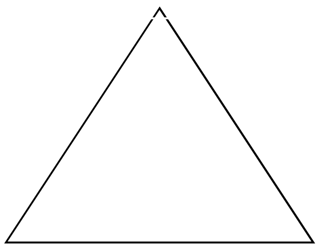
$\Rightarrow x = \frac{3}{\sqrt{2}}$

(b)

Using Sine rule,

$\frac{DF}{\sin\left(\frac{\pi}{3} - \alpha\right)} = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$

$DF = 4\sqrt{3}\sin\left(\frac{\pi}{3} - \alpha\right)$



$\frac{DE}{\sin\left(\frac{\pi}{3} + \alpha\right)} = \frac{6}{\sin\left(\frac{\pi}{3}\right)}$

$DE = 4\sqrt{3}\sin\left(\frac{\pi}{3} + \alpha\right)$

$DF - DE$

$= 4\sqrt{3}\sin\left(\frac{\pi}{3} - \alpha\right) - 4\sqrt{3}\sin\left(\frac{\pi}{3} + \alpha\right)$

$= 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right) - 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\alpha + \frac{1}{2}\sin\alpha\right)$

$\approx 4\sqrt{3}\left[\frac{\sqrt{3}}{2}\left(1 - \frac{\alpha^2}{2}\right) - \frac{1}{2}\alpha - \frac{\sqrt{3}}{2}\left(1 - \frac{\alpha^2}{2}\right) - \frac{1}{2}\alpha\right]$

$= -4\sqrt{3}\alpha$

9	(i)

$$l: \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$$

Let θ be the angle between the line l and the plane p_1.

$$\sin\theta = \left| \frac{\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\sqrt{17}\sqrt{3}} \right|$$

$$= \left| \frac{5}{\sqrt{17}\sqrt{3}} \right|$$

$$\theta = 44.4^\circ$$

(ii)

$$l_{AN}: \mathbf{r} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \mu \in \mathbb{R}$$

$$\begin{pmatrix} 2+\mu \\ 2+\mu \\ -3+\mu \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 16$$

$$2+\mu+2+\mu-3+\mu = 16$$

$$3\mu = 15$$

$$\mu = 5$$

Coordinates of $N = (7,7,2)$

(iii)
Since N is the midpoint of A and B, using ratio theorem,

$$\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$$

$$\overrightarrow{OB} = 2\overrightarrow{ON} - \overrightarrow{OA}$$

$$= 2\begin{pmatrix} 7 \\ 7 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 12 \\ 12 \\ 7 \end{pmatrix}$$

Coordinates of $B = (12,12,7)$

(iv)

Let C be the point of intersection of the line l and the plane p_1.

$$\begin{pmatrix} 2+4\lambda \\ 2 \\ -3+\lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 16$$

$$2+4\lambda+2-3+\lambda = 16$$
$$5\lambda = 15$$
$$\lambda = 3$$

$$\overrightarrow{OC} = \begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{BC} = \begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 12 \\ 12 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -10 \\ -7 \end{pmatrix}$$

$$l_{BC} : \mathbf{r} = \begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -10 \\ -7 \end{pmatrix},\ s \in \mathbb{R}$$

(v)

Since $AN = BN$,

$$BN = \sqrt{(2-7)^2+(2-7)^2+(-3-2)^2}$$
$$= \sqrt{(-5)^2+(-5)^2+(-5)^2}$$
$$= \sqrt{75}\text{units}$$
$$=5\sqrt{3}\text{units}$$

10 (ai)

Let n denote the population of prawns in thousands at time t

$$\frac{\mathrm{d}^2n}{\mathrm{d}t^2} = \mathrm{e}^{-\frac{t}{5}}$$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -5\mathrm{e}^{-\frac{t}{5}} + C$$

$$n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + D$$

(aii)

Given $n = 50,\ t = 0,$

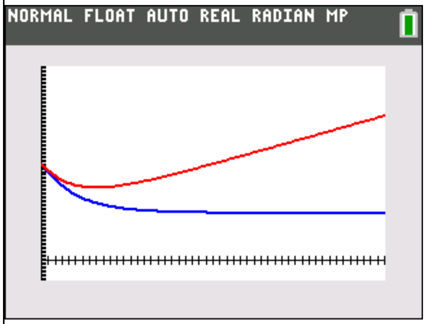
$$50 = 25 + D \Rightarrow D = 25$$

$$n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + 25$$

I Requires $C > 0$ so that $n = 25\mathrm{e}^{-\frac{t}{5}} + Ct + 25 \to \infty$ as $t \to \infty$

II Requires $C=0$ so that $n=25e^{-\frac{t}{5}}+25$

Then as $t\to\infty$, $n\to 25$



(bi)

$$\frac{dx}{dt}=-kx$$

$$\int\frac{1}{x}\,dx=-k\int 1\,dt$$

$$\ln|x|=-kt+C$$

$x=Ae^{-kt}$ where $A=\pm e^{C}$

At $t=0$, $x=0.1$,

$\therefore A=0.1$

$x=\frac{1}{10}e^{-kt}$ (shown)

(bii)

At $t=4$, $x=0.05$,

$\therefore 0.05=0.1e^{-4k}$

$\Rightarrow e^{-4k}=\frac{1}{2}$

$\Rightarrow -k=\frac{\ln\frac{1}{2}}{4}$

$\Rightarrow k=-\frac{1}{4}\ln\frac{1}{2}=\frac{\ln 2}{4}$

(biii)

Total amount of drug present in the prawn's body at any time t

$$< 0.1 + 0.1e^{-\left(\frac{\ln 2}{4}\right)8} + 0.1e^{-2\left(\frac{\ln 2}{4}\right)8} + 0.1e^{-3\left(\frac{\ln 2}{4}\right)8} + \ldots$$

$$= \frac{0.1}{1 - e^{-\left(\frac{\ln 2}{4}\right)8}}$$

$$= \frac{2}{15} < 0.15$$

$\therefore$ The total amount of drug present in the prawn's body at any time t is always less than 0.15 mg.

2017 RVHS Prelim Paper 2

Section A: Pure Mathematics [40 Marks]

1. The curve C is defined parametrically by equations

$$x = \cos(p), \quad y = \sin^3(p), \quad 0 \leq p \leq 2\pi$$

The point P on C has parameter p. Given that p is increasing at a rate of 0.5 units per second, find the rate at which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing when $p = \frac{\pi}{3}$. [4]

2. An arithmetic sequence $u_1, u_2, u_3, \ldots$ is such that the difference between the fourteenth term and the fifth term is equal to the sum of the terms between the fifth term and the fourteenth term (both inclusive). Given further that that sum of the third, fifth and fourteenth terms is 19, find the common difference of the sequence. [4]

Hence, or otherwise, find the largest of value of n such that the sum of the first n terms is positive. [2]

3. **(i)** Express $\frac{4r+6}{(r+1)(r+2)(r+3)}$ as partial fractions. [1]

(ii) Hence find $\sum_{r=1}^{n} \frac{4r+6}{(r+1)(r+2)(r+3)}$ in terms of n. [3]

(iii) Use your answer in part **(ii)** to find the sum of the infinite series

$$\frac{3}{1\times2\times3} + \frac{5}{2\times3\times4} + \frac{7}{3\times4\times5} + \frac{9}{4\times5\times6} + \cdots.$$ [3]

4. Let $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \mathrm{e}^{\sqrt{(1-x)^3}}$ for $x \le 1$.

Show that $4\sqrt{1-x}\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6(1-x)\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$. [4]

Hence find the Maclaurin series for f(x) up to and including the term in x^2. [3]

Using the standard series of e^x and $(1+x)^n$ given in the List of Formulae (MF26), show how you could verify the correctness of the series of $\mathrm{f}(x)$ above. [4]

5. The functions f and g are defined by

$$\mathrm{f} : x \mapsto 2x^2 - x, \quad x \in \mathbb{R}, \quad x \ge 0,$$

$$\mathrm{g} : x \mapsto -3 + \frac{1}{\sqrt{2x + \frac{1}{2}}}, \quad x \in \mathbb{R}, \quad x > -\frac{1}{4}.$$

(i) Give a reason why f does not have an inverse. [1]

(ii) If the domain of f is restricted to $x \ge k$, state the least value of k for which the function f^{-1} exists, and find f^{-1} in similar form for this domain. [3]

(iii) Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same diagram if the domain of f is restricted to $x \ge k$, where k is the value found in part **(ii)**. Your diagram should show clearly the relationship between the two graphs. [3]

(iv) Solve algebraically the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for the restricted domain of f in part **(ii)**. [2]

(v) For f defined for $x \ge 0$, show that the composite function gf exists and find its range. [3]

Section B: Statistics [60 Marks]

6. A restaurant is setting up a spinning wheel for its customers to try and win vouchers. The wheel is split into 8 identical segments, comprising of \$0, \$5, \$10, \$15, \$20, \$25, \$30 and \$50.

Find the number of ways the segments can be arranged on the wheel if

(i) there are no restrictions. [1]

(ii) the \$0 segment cannot be next to the \$5 segment [2]

(iii) there must be at least two segments between the \$30 and \$50 segments. [2]

The restaurant decides to replace the \$30 and \$50 segments with another two \$0 segments.

(iv) Find the number of possible arrangements of the 8 segments. [1]

(v) Find the number of possible arrangements if the \$0 segments must be separated. [2]

7. A board game simulates players attacking each other by throwing tetrahedral (8-sided) dice. When attacking, the player throws an attack die once. An attack die has 5 of the sides printed with the number "0", 2 of the sides printed with the number "1", and 1 of the sides printed with the number "2". After the attacking player has thrown the attack die, the defending player throws a defence die once. A defence die has 2 of the sides printed with the number "0", 4 of the sides printed with the number "1" and 2 of the sides printed with the number "2". The damage dealt during a round is equal to the score shown on the attack die minus the score shown on the defence die. If the score on the defence die is more than the score on the attack die, the damage dealt will be zero.

Let A denotes the score on an attack die, and D denotes the score on a defence die.

(i) Write down the probability distributions for A and D. Hence find the expected value and variance of $A - D$. [4]

Let X denote the damage dealt during a round.

(ii) Find the probability distribution for X. Hence find the expected value and variance of X. [5]

(iii) Explain why, in the context of the question, $\mathrm{E}(X) > 0$ when $\mathrm{E}(A) < \mathrm{E}(D)$. [1]

8. A car park next to a small commercial building has a total of 12 parking lots. Land surveillance officers have been observing the usage of parking lots per day to determine if the land has been efficiently utilised. Each parking lot can be occupied by at most one vehicle per day.

(i) Denoting the number of occupied parking lots per day by X, state in context, two assumptions needed for X to be well modelled by a binomial distribution. [2]

(ii) It is further observed that for 80% of the days in the survey period, there are at least 4 occupied lots in the car park for each day. Find the probability that a parking lot is being occupied in a day. [2]

(iii) Given that at least one of the parking lots is occupied in a particular day, show that the probability that at least 2 but less than 4 lots are occupied in the particular day is given by

$$f(p)=\frac{22p^2(1-p)^9(3+7p)}{1-(1-p)^{12}}$$

where p is the probability of a parking lot being occupied in a day. What can you say about this probability if p is approximately 0.185? [5]

9. In the study of how the population of a harmful bacteria varies with temperature, scientists conducted an experiment to collect the following set of data:

Temperature (x °C)	10	12	14	16	18	20	22	24	26	28
Population (y millions)	25.4	25.1	24.4	22.9	20.8	18.3	15.4	12.2	8.8	5.3

(i) Draw a scatter diagram for the above data, labelling the axes clearly. [2]

(ii) Calculate the value of the product moment correlation coefficient. Explain why a linear model is not appropriate. [2]

It is suggested the relationship between x and y can be modelled by one of the following formulae:

$$y=a+\frac{b}{x} \quad \text{or} \quad y=a-bx^2$$

where a and b are positive constants.

(iii) Explain which of the above two models is the better model and calculate the values of a and b for the chosen model. [3]

(iv) It is required to estimate the temperature when the population of the bacteria is 10 millions. By using an appropriate regression line, find an estimate of the value of x and comment on the reliability of your answer. [2]

10. Each month the amount of electricity, X measured in kilowatt-hours (kWh), used by a household in a particular city may be assumed to follow a normal distribution with mean 950 and standard deviation σ. The charge for electricity used per month is fixed at \$0.22 per kWh.

(i) Given that 65% of the households uses less than 960 kWh of electricity in a month, find the value of σ, correct to 1 decimal place. [2]

For the rest of the question, σ is the value found in part **(i)**.

(ii) Find the probability that the difference in the amount of electricity used among 2 randomly chosen households in a particular month is not more than 30 kWh. [3]

(iii) In the month of August, the mayor of the city decides to provide 50% and 30% subsidies for the electricity bills of households in the North and South distincts of the city respectively. Find the probability that the total electricity bill of 2 randomly chosen North district households and 1 South district household is less than \$360. [4]

(iv) In December, a random sample of n households is chosen to study the mean monthly electricity usage per household in the city. Find the least value of n if the probability of the sample mean being less than 955 kWh is at least 0.9. [3]

11. Physicists are conducting an experiment involving collisions between protons and anti-protons. The mean amount of energy, $\bar{x}$ MeV, released in n collisions is found to be 1864 MeV.

One model predicts the energy released would be 1860 MeV with standard deviation 40 MeV. This is tested at a 1% level of significance against a newer model that claims a higher value.

(i) Find the least value of n such that the hypothesis that the mean amount of energy released is 1860 MeV is rejected. [5]

Given instead that $n = 600$.

(ii) Calculate the p-value and state its meaning in context of the question. [3]

(iii) State, with a reason, whether it is necessary to assume the amount of energy released in collisions to be normally distributed for this test to be valid. [1]

Two-sigma is an indicative of how confident researchers feel their results are. For researchers to feel confident, they must be able to produce a "two-sigma" result – that is the experimental result must be at least two standard deviations away from the predicted mean under the null hypothesis.

(iv) Calculate the level of significance that corresponds to a "two-sigma" test. Hence, using your answer from part **(ii)** determine whether the experiment has met the "two-sigma" threshold. [3]

END OF PAPER

ANNEX B

RVHS H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Differentiation & Applications	0.75 units per second
2	AP and GP	-5 or $-\frac{95}{46}$; 16 or 19
3	Sigma Notation and Method of Difference	(i) $\frac{4r+6}{(r+1)(r+2)(r+3)}=\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}$ (ii) $\frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)$ (iii) $\frac{5}{4}$
4	Maclaurin series	$\mathrm{e}-\frac{3\mathrm{e}}{2}x+\frac{3\mathrm{e}}{2}x^2$
5	Functions	(ii) $k=\frac{1}{4}$; $\mathrm{f}^{-1}:x\mapsto\frac{1+\sqrt{8x+1}}{4}, x\geq-\frac{1}{8}$ (iv) $x=1$. (v) $R_{\mathrm{gf}}=(-3,-1]$
6	P&C, Probability	(i) 5040 (ii) 3600 (iii) 2160 (iv) 840 (v) 240
7	DRV	(i) $\frac{-1}{2}$, 1 (ii) $\frac{3}{16}$, $\frac{55}{256}$
8	Binomial Distribution	(ii) 0.412 (iii) $p\approx 0.185$
9	Correlation & Linear Regression	(ii) $r=-0.973$ (iii) a = 30.0 and b = 0.0308 (iv) x = 25.5 °C
10	Normal Distribution	(i) 26.0 (ii) 0.585 (iii)0.796 (iv)45

11	Hypothesis Testing	(i) 542 (ii) p-value $= 0.00715$ (iv) 2.28%

1	At point P, $x=\cos(p)$, $y=\sin^3(p)$ $\frac{\mathrm{d}y}{\mathrm{d}p}=3\cos(p)\sin^2(p)$ $\frac{\mathrm{d}x}{\mathrm{d}p}=-\sin(p)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-3\sin(p)\cos(p)=\frac{-3}{2}\sin(2p)$ Let $z=\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\mathrm{d}z}{\mathrm{d}p}\cdot\frac{\mathrm{d}p}{\mathrm{d}t}$ $=-3\cos(2p)\cdot(0.5)$ $=\frac{-3}{2}\cos(2p)$ $\left.\frac{\mathrm{d}z}{\mathrm{d}t}\right\|_{p=\frac{\pi}{3}}=\frac{-3}{2}\cos\left(\frac{2\pi}{3}\right)=0.75$ Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing at 0.75 units per second when $p=\frac{\pi}{3}$.
2	Let the first term be a and the common difference be d. $\sum_{k=5}^{14}u_k=\|u_{14}-u_5\|$ $S_{14}-S_4=\|(a+13d)-(a+4d)\|$ $\frac{14}{2}(2a+13d)-\frac{4}{2}(2a+3d)=\|9d\|$ $14a+91d-4a-6d=\|9d\|$ $10a+85d=\|9d\|$ $10a+85d=9d$ or $10a+85d=-9d$ $5a+38d=0$---- (1) or $5a+47d=0$---- (1) $u_3+u_5+u_{14}=19$ $(a+2d)+(a+4d)+(a+13d)=19$ $3a+19d=19$ ----- (2) Solving simultaneously, from GC, $a=38,\ d=-5$ or $a=\frac{893}{46},\ d=-\frac{95}{46}$ Hence the common difference is -5 or $-\frac{95}{46}$

$$S_n > 0 \qquad\qquad\qquad S_n > 0$$

$$\frac{n}{2}\left(2a+(n-1)d\right)>0 \quad \text{or} \quad \frac{n}{2}\left(2a+(n-1)d\right)>0$$

$$n(81-5n)>0 \qquad\qquad n\left(\frac{1881}{92}-\frac{95}{92}n\right)>0$$

$$0<n<\frac{81}{5}=16.2 \quad \text{or} \quad 0<n<\frac{99}{5}=19.8$$

Hence, the largest value of n is 16 or 19.

3 (i)

Let $\dfrac{4r+6}{(r+1)(r+2)(r+3)}=\dfrac{A}{r+1}+\dfrac{B}{r+2}+\dfrac{C}{r+3}$

Then by cover up rule, $A=1, B=2, C=-3$

Hence, $\dfrac{4r+6}{(r+1)(r+2)(r+3)}=\dfrac{1}{r+1}+\dfrac{2}{r+2}-\dfrac{3}{r+3}$

(ii)

$$\sum_{r=1}^{n}\frac{4r+6}{(r+1)(r+2)(r+3)}=\sum_{r=1}^{n}\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}$$

$$\begin{aligned}
&= \frac{1}{2} + \frac{2}{3} - \frac{3}{4} \\
&+ \frac{1}{3} + \frac{2}{4} - \frac{3}{5} \\
&+ \frac{1}{4} + \frac{2}{5} - \frac{3}{6} \\
&+ \frac{1}{5} + \ldots\ldots \quad \ldots\ldots \\
&+ \ldots\ldots \quad \ldots\ldots - \frac{3}{n} \\
&+ \frac{1}{n-1} + \frac{2}{n} - \frac{3}{n+2} \\
&+ \frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2} \\
&+ \frac{1}{n+1} + \frac{2}{n+2} - \frac{3}{n+3} \\
&= \frac{1}{2}+\frac{2}{3}+\frac{1}{3}-\frac{3}{n+2}+\frac{2}{n+2}-\frac{3}{n+3} \\
&= \frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)
\end{aligned}$$

(iii)

$$\frac{3}{1\times2\times3}+\frac{5}{2\times3\times4}+\frac{7}{3\times4\times5}+\frac{9}{4\times5\times6}+\ldots$$

$$=\frac{3}{1\times2\times3}+\frac{1}{2}\left(\frac{10}{2\times3\times4}+\frac{14}{3\times4\times5}+\frac{18}{4\times5\times6}+\ldots\right)$$

$$=\frac{1}{2}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{4r+6}{(r+1)(r+2)(r+3)}$$

$$=\frac{1}{2}+\frac{1}{2}\lim_{n\to\infty}\left(\frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)\right)$$

$$=\frac{1}{2}+\frac{1}{2}\times\frac{3}{2}$$

$$=\frac{5}{4}$$

4

$$y=\mathrm{e}^{\sqrt{(1-x)^3}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{\sqrt{(1-x)^3}}\left(\frac{3}{2}\right)(1-x)^{\frac{1}{2}}(-1)=\frac{-3}{2}y\sqrt{1-x}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{-3}{2}\frac{\mathrm{d}y}{\mathrm{d}x}\sqrt{1-x}+\frac{-3}{2}y\frac{-1}{2\sqrt{1-x}}=\frac{3y}{4\sqrt{1-x}}-\frac{3\sqrt{1-x}}{2}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$4\sqrt{1-x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3y-6(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}$$

Thus,

$$4\sqrt{1-x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}-3y=0 \text{ (shown)}$$

When $x = 0$,

$$y=\mathrm{e}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-3\mathrm{e}}{2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=3\mathrm{e}$$

$$\mathrm{f}(x)\approx\mathrm{f}(0)+\mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2!}x^2$$

$$\approx\mathrm{e}-\frac{3\mathrm{e}}{2}x+\frac{3\mathrm{e}}{2}x^2$$

$$(1-x)^{\frac{3}{2}}\approx1-\frac{3}{2}x+\frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(-x)^2$$

$$=1-\frac{3}{2}x+\frac{3}{8}x^2$$

$$e^{\sqrt{(1-x)^3}} \approx e^{1-\frac{3}{2}x+\frac{3}{8}x^2}$$

$$= e\left(e^{-\frac{3}{2}x+\frac{3}{8}x^2}\right)$$

$$\approx e\left(1+\left(-\frac{3}{2}x+\frac{3}{8}x^2\right)+\frac{\left(-\frac{3}{2}x+\frac{3}{8}x^2\right)^2}{2!}\right)$$

$$\approx e\left(1-\frac{3}{2}x+\frac{3}{8}x^2+\frac{9}{8}x^2\right)$$

$$= e\left(1-\frac{3}{2}x+\frac{3}{2}x^2\right)$$

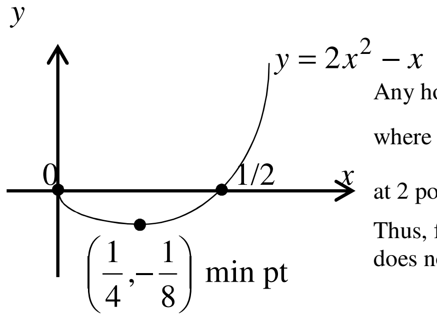
$$= e-\frac{3e}{2}x+\frac{3e}{2}x^2$$

which is the same as the above series expansion of $f(x)$

---

5 (i)

As shown in the following sketch:



Any horizontal line of the form $y = k$ where $-\frac{1}{4} < k \le 0$ will intersect the curve at 2 points.

Thus, f is not one-one and hence f^{-1} does not exist.

(ii)

From the sketch of the curve, we deduce that the least value of $k = \frac{1}{4}$ for f^{-1} to exist.

Next let $y = 2x^2 - x$. Then we have

$$y = 2\left(x^2 - \frac{1}{2}x\right)$$

$$= 2\left(x^2 - \frac{1}{2}x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right)$$

$$= 2\left(x - \frac{1}{4}\right)^2 - \frac{1}{8}$$

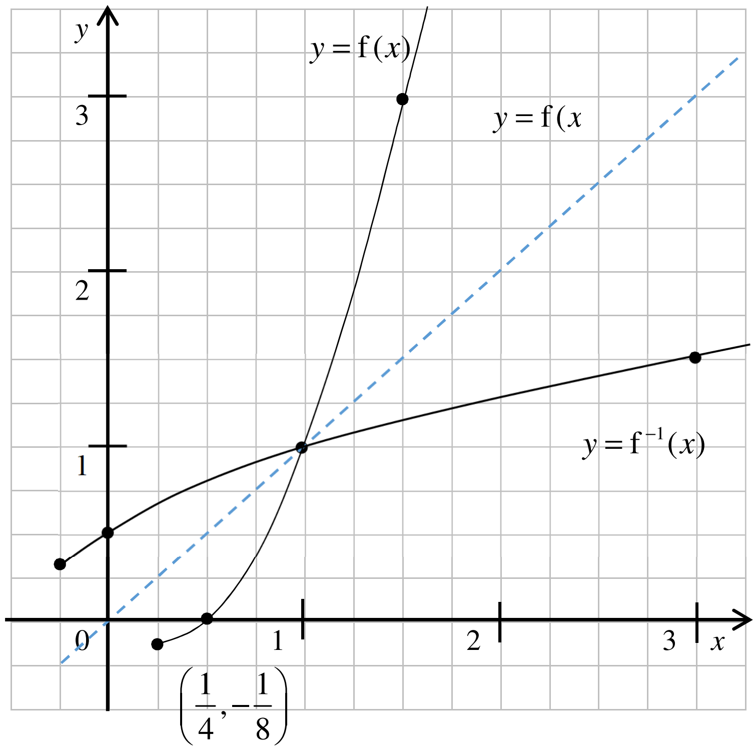
$$\left(x-\frac{1}{4}\right)^2 = \frac{1}{2}\left(y+\frac{1}{8}\right)$$

$$\Rightarrow x = \frac{1}{4} \pm \sqrt{\frac{8y+1}{16}} = \frac{1}{4} + \frac{\sqrt{8y+1}}{4} \qquad \text{since } x \ge \frac{1}{4}$$

Hence, $f^{-1} : x \mapsto \dfrac{1+\sqrt{8x+1}}{4}, x \geq -\dfrac{1}{8}$. $D_{f^{-1}} = R_f = \left[-\dfrac{1}{8}, \infty\right)$

(iii)

Sketch of $y = f(x)$ and $y = f^{-1}(x)$:



(iv)

From the sketch in part (iii) we note that to solve the equation $f(x) = f^{-1}(x)$, we can also solve $f(x) = x$

Thus, $2x^2 - x = x \Rightarrow 2x(x-1) = 0$

Therefore, in the restricted domain of $x \geq \dfrac{1}{4}$,

the solution is $x = 1$.

(v)

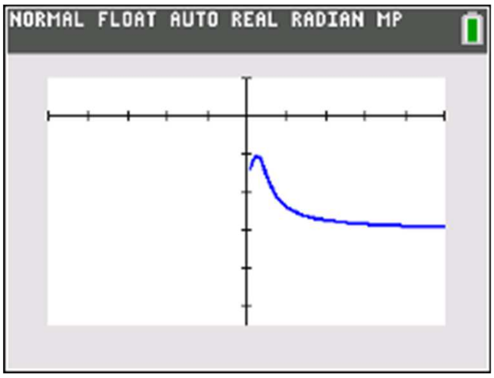
For $f : x \mapsto 2x^2 - x,\ x \in \mapsto\ x \geq 0, R_f = \left[-\dfrac{1}{8}, \infty\right)$

Also, for $g : x \mapsto -3 + \dfrac{1}{\sqrt{2x + \frac{1}{2}}},\ x \in \mapsto\ x > -\dfrac{1}{4},\ D_g = \left(-\dfrac{1}{4}, \infty\right)$

Since $R_f \subseteq D_g$, the composite function gf exists.

Then,

$$gf(x) = -3 + \frac{1}{\sqrt{2(2x^2 - x) + \frac{1}{2}}} = -3 + \frac{1}{\sqrt{4\left(x - \frac{1}{4}\right)^2 + \frac{1}{4}}}$$



Since $D_{gf} = [0, \infty)$ and $gf\left(\frac{1}{4}\right) = -3 + \frac{1}{\sqrt{\frac{1}{4}}} = -3 + 2 = -1$,

we have $R_{gf} = (-3, -1]$

<u>ALT</u>

$[0, +\infty) \to [-\frac{1}{8}, +\infty) \to (-3, -1]$

$R_{gf} = (-3, -1]$

6 (i)

$(8-1)! = 5040$

(ii)

No. of ways with \$0 and \$5 segments adjacent

$= (7-1)!2!$

$= 1440$

No. of ways without identical segments adjacent

$=$ total no. of ways $-$ no. of ways with identical segments adjacent

$= 5040 - 1440$

$= 3600$

(iii)

<u>Case 1: no segment separating them</u>

$(7-1)!2! = 1440$

<u>Case 2: exactly 1 segment separating them</u>

$\binom{6}{1} 2!(6-1)! = 1440$

Total number of ways $= 5040 - 1440 - 1440$

$= 2160$

	<u>ALT</u> <u>Case 1</u>: exactly 2 segments separating them $\binom{6}{2}2!2!(5-1)!=1440$ <u>Case 2</u>: exactly 3 segments separating them $\frac{\binom{6}{3}3!2!(4-1)!}{2}=720$ Therefore, total number of ways = 2160 (iv) The segments are \$0, \$0, \$0, \$5, \$10, \$15, \$20, \$25 $\frac{(8-1)!}{3!}=840$ (v) Arrange the other 5 objects in (5 – 1)! = 24 ways Choose 3 spaces for the \$0 in $^5C_3$ = 10 ways Total = 240 ways
7	(i) Probability distribution for *A*:

a	0	1	2
P(*A* = *a*)	5/8	2/8	1/8

Probability distribution for *D*:

d	0	1	2
P(*D* = *d*)	2/8	4/8	2/8

$$E(A)=\left(\frac{5}{8}\right)(0)+\left(\frac{2}{8}\right)(1)+\left(\frac{1}{8}\right)(2)=\frac{1}{2}$$

$$E(D)=\left(\frac{2}{8}\right)(0)+\left(\frac{4}{8}\right)(1)+\left(\frac{2}{8}\right)(2)=1$$

$$E(A-D)=E(A)-E(D)=\frac{-1}{2}$$

$$E(A^2)=\left(\frac{5}{8}\right)(0)^2+\left(\frac{2}{8}\right)(1)^2+\left(\frac{1}{8}\right)(2)^2=\frac{3}{4}$$

$$E(D^2)=\left(\frac{2}{8}\right)(0)^2+\left(\frac{4}{8}\right)(1)^2+\left(\frac{2}{8}\right)(2)^2=\frac{3}{2}$$

	$\text{Var}(A)=\text{E}(A^2)-\text{E}(A)^2=\frac{3}{4}-\left(\frac{1}{2}\right)^2=\frac{1}{2}$ $\text{Var}(D)=\text{E}(D^2)-\text{E}(D)^2=\frac{3}{2}-1^2=\frac{1}{2}$ $\text{Var}(A-D)=\text{Var}(A)+\text{Var}(D)=\frac{1}{2}+\frac{1}{2}=1$ (ii) Probability distribution for *X*:

x	0	1	2
P(*X* = *x*)	$\text{P}(A=0)$ $+\text{P}(A=1)\text{P}(D\geq 1)$ $+\text{P}(A=2)\text{P}(D=2)$ $=\frac{27}{32}$	$\text{P}(A=1)\text{P}(D=0)$ $+\text{P}(A=2)\text{P}(D=1)$ $=\frac{1}{8}$	$\text{P}(A=2)\text{P}(D=0)$ $=\frac{1}{32}$

$$\text{E}(X)=\left(\frac{27}{32}\right)(0)+\left(\frac{1}{8}\right)(1)+\left(\frac{1}{32}\right)(2)=\frac{3}{16}$$

$$\text{E}(X^2)=\left(\frac{27}{32}\right)(0)^2+\left(\frac{1}{8}\right)(1)^2+\left(\frac{1}{32}\right)(2)^2=\frac{1}{4}$$

$$\text{Var}(X)=\frac{1}{4}-\left(\frac{3}{16}\right)^2=\frac{55}{256}$$

(iii)
If the score on the defence die is more than the score on the attack die, the damage dealt will be zero. So even though sometimes *A* – *D* will be less than zero, that is never considered when dealing damage. Hence, the expected damage must be greater than zero.

---

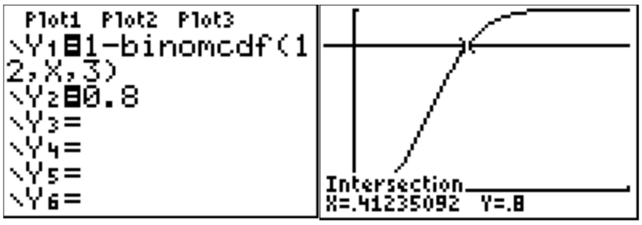
8 (i)
The 2 assumptions needed for *X* to be well modelled by a binomial distribution are as follow:

1. The occupancy of any particular parking lot in the car park is *independent* of that of another lot.
2. The probability of a parking lot being occupied in a day is *constant* for all the car park lots in the car park.

(ii)

Since for 80% of the days in the survey period, there are at least 4 occupied lots for each day, we can infer that
$P(X\geq 4)=1-P(X\leq 3)=0.8$ for $X\sim \text{B}(12, p)$.

We then use GC to plot the graph involving binomial cdf and determine the x coordinate of the intersection of the curve and the line $y = 0.8$ as shown below:



Hence, the value of p is 0.412 (3 s.f.)

(iii)

Let $X \sim \mathrm{B}(12, p)$

The required conditional probability, $\mathrm{f}(p)$

$$= \mathrm{P}(2 \le X < 4 \mid X \ge 1)$$

$$= \frac{\mathrm{P}(X = 2 \text{ or } X = 3)}{\mathrm{P}(X \ge 1)}$$

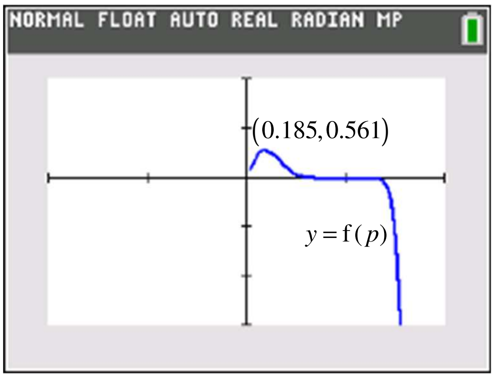
$$= \frac{\mathrm{P}(X = 2 \text{ or } X = 3)}{1 - \mathrm{P}(X = 0)}$$

$$= \frac{\binom{12}{2} p^2 (1-p)^{10} + \binom{12}{3} p^3 (1-p)^9}{1 - \left[\binom{12}{0} p^0 (1-p)^{12}\right]}$$

$$= \frac{66 p^2 (1-p)^{10} + 220 p^3 (1-p)^9}{1-(1-p)^{12}}$$

$$= \frac{22 p^2 (1-p)^9 \left[3(1-p) + 10p\right]}{1-(1-p)^{12}}$$

$$= \frac{22 p^2 (1-p)^9 (3 + 7p)}{1-(1-p)^{12}}. \quad \text{(Shown)}$$

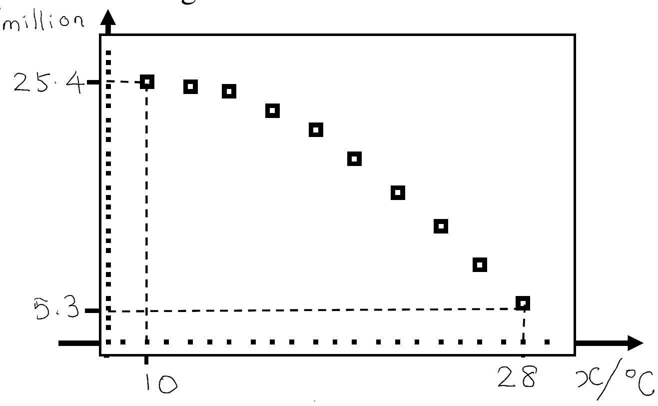


$p \approx 0.185$ give the maximum probability.

9

(i)

The required scatter diagram is as shown below:



(ii)

From GC, the correlation coefficient $r = -0.973$.
Although the value of r is close to -1 and suggests a strong negative linear relationship between x and y, the scatter diagram shows a curvilinear relationship between x and y. Thus, the a linear relationship between x and y is not appropriate.

(iii)

The scatter diagram shows that when x increases, y decreases at increasing rate. Thus, the model with $y = a - bx^2$ where a, b are positive constants is more appropriate.

Using GC, we found that $a = 29.98560169 = 30.0$ (3 s.f.)

and $b = 0.0307756388 = 0.0308$ (3 s.f.)

(For $a, b > 0$, $y = a + \frac{b}{x}$ decreases at a decreasing rate when x increases)

(iv)

As x is the independent variable and y is the dependent variable, we will still use the regression line $y = 30.0 - 0.0308x^2$ to estimate the value of x.
Thus, when $y = 10$, $x = 25.5$ °C (3 s.f.)
The anwer is reliable for the following reasons:

i) correlation coefficient $r = -0.995$ has absolute value close to 1
ii) the y value of 10 is within data range of the available y values.

10

(i)

$X \mapsto N(950,\ \sigma^2)$

Given that $P(X < 960) = 0.65$,

then $P(Z < \frac{960-950}{\sigma}) = 0.65$

$\Rightarrow \frac{960-950}{\sigma} = 0.3853204726$

$\Rightarrow \sigma = 25.95242327 = 26.0$ (1 decimal place)

	(ii) Let X_1 and X_2 be the amount of electricity used by the 2 randomly chosen household in a particular month. Then $X_1 - X_2 \sim \text{N}(0,\ 26.0^2 + 26.0^2)$ Thus, $\text{P}(\lvert X_1 - X_2 \rvert \le 30)$ $= \text{P}(-30 \le X_1 - X_2 \le 30)$ $= 0.585$ (iii) Let N_1, N_2 and S be the amount of electricity used by the 2 randomly chosen households in the North District and household in the South district respectively in August. Then their total electricity bill = \$ T $= \$\ 0.5\times0.22\times(N_1+N_2)+0.7\times0.22\times S$ $= \$\ 0.11N_1+0.11N_2+0.154S$ Then $\text{E}(T) = 0.11\times950\times2+0.154\times950 = 355.3$ $\text{Var}(T) = 0.11^2\times26.0^2\times2+0.154^2\times26.0^2 = 32.391216$ So, $T \sim \text{N}(355.3,\ 32.391216)$ Hence, $\text{P}(T<360) = 0.796$ (iv) Let $\overline{X} = \dfrac{X_1+X_2+X_3+......+X_n}{n} \sim \text{N}\left(950, \dfrac{26.0^2}{n}\right)$ where X_i :electricity usage for each of the randomly selected household in the month of December Then, we have $\text{P}\left(\overline{X} < 955\right) \ge 0.9$ $\Rightarrow \text{P}\left(Z < \dfrac{955-950}{26.0/\sqrt{n}}\right) \ge 0.9$ $\Rightarrow \dfrac{955-950}{26.0/\sqrt{n}} \ge 1.281551567$ $\Rightarrow \dfrac{26.0}{\sqrt{n}} \le \dfrac{5}{1.281551567} = 3.901520726$ $\Rightarrow \sqrt{n} \ge \dfrac{26}{3.901520726}$ $\Rightarrow n \ge 44.40980429$ Thus, the least value of n is 45.
11	(i) Let μ denote the population mean amount of energy released in the collisions. Test H$_0$: $\mu = 1860$ Against H$_1$: $\mu > 1860$ Using a one-tail test at 1% significance level.

Under H_0, $\overline{X} \sim N\left(1860, \frac{40^2}{n}\right)$ approx

Test statistic: $Z = \frac{\overline{X} - 1860}{40/\sqrt{n}} \sim N(0,1)$

$$z_{calc} = \frac{1864 - 1860}{40/\sqrt{n}} = \frac{\sqrt{n}}{10}$$

To reject H_0 at 1% level of significance, the critical region is:

$$z_{calc} > 2.32635$$

Hence,

$\frac{\sqrt{n}}{10} > 2.32635$

$n > 541.189$

Thus, the least value of n is 542.

(ii)

Test H₀: $\mu = 1860$

Against H_1: $\mu > 1860$

Using a one-tailed test at 1% significance level.

Under H_0, $\overline{X} \sim N\left(1860, \frac{40^2}{600}\right)$ approx

Test statistic: $Z = \frac{\overline{X} - 1860}{40/\sqrt{600}} \sim N(0,1)$

From GC,

p-value $= 0.00715$

The p-value means that the lowest level of significance at which we would reject the hypothesis that the mean amount of energy released is 1860 MeV in favour of the hypothesis that the amount is greater than 1860 MeV is 0.715 %.

(iii)

No assumption needed. This is because the sample size of 600 is large and thus by Central Limit Theorem, $\overline{X}$ follows a normal distribution.

(iv)

Let $Z \sim N(0, 1)$

$P(Z \geq 2) = 0.0228$

Hence, lowest level of significance for which the experiment meets the "two sigma" threshold is 2.28%.

Since p-value = 0.00715 < 0.0228, the result meets the "two sigma" threshold.

	<u>Alternative</u>: Under H_0, $\overline{X} \sim N\left(1860, \frac{40^2}{600}\right)$ approx $2\sigma = 2\sqrt{\frac{1600}{600}} = 3.265986$ So $\mu + 2\sigma = 1863.265986$ $P\left(\overline{X} > 1863.265986\right) \approx 0.0228$ Hence, lowest level of significance for which the experiment meets the "two sigma" threshold is 2.28%. Since $\overline{x} = 1864 > 1863.265986$ the test meets the "two sigma" threshold.