HCI Paper 1

1 The *floor function*, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. For example, $\lfloor -2.1 \rfloor = -3$ and $\lfloor 3.5 \rfloor = 3$.

The function f is defined by

$$\mathrm{f}(x) = \begin{cases} \lfloor x \rfloor & \text{for } x \in \mathbb{R}, \quad -1 \le x < 2, \\ 0 & \text{for } x \in \mathbb{R}, \quad 2 \le x < 3, \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

It is given that $\mathrm{f}(x) = \mathrm{f}(x+4)$.

(i) Find the values of $\mathrm{f}(-1.2)$ and $\mathrm{f}(3.6)$. [2]

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-2 \le x < 4$. [2]

(iii) Hence evaluate $\int_{-2}^{4} \mathrm{f}(x)\ \mathrm{d}x$. [1]

2 By writing $\sec^3 x = \sec x \sec^2 x$, find $\int \sec^3 x\ \mathrm{d}x$.

Hence find the exact value of $\int_0^{\tan^{-1} 2} \sec^3 x\ \mathrm{d}x$. [6]

3 **(i)** By first expressing $3x - x^2 - 4$ in completed square form, show that $3x - x^2 - 4$ is always negative for all real values of x. [2]

(ii) Hence, or otherwise, without the use of a calculator, solve the inequality

$$\frac{\left(3x - x^2 - 4\right)\left(x-1\right)^2}{x^2 - 2x - 5} \le 0 ,$$

leaving your answer in exact form. [4]

4 The complex number z is given by $z = r\,\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta \le \pi$. It is given that the complex number $w = \left(-\sqrt{3} - \mathrm{i}\right)z$.

(i) Find $|w|$ in terms of r, and $\arg w$ in terms of θ. [2]

(ii) Given that $\dfrac{z^8}{w^*}$ is purely imaginary, find the three smallest values of θ in terms of π. [5]

5 **(a)** It is given that three non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy the equation $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{c})=\mathbf{b}\times\mathbf{c}$, where $\mathbf{b}\neq\mathbf{c}$. Find a linear relationship between $\mathbf{a}$, $\mathbf{b}$ an $\mathbf{c}$. [3]

(b) A point A with position vector $\overrightarrow{OA}=\alpha\,\mathbf{i}+\beta\,\mathbf{j}+\gamma\,\mathbf{k}$, where α, β and γ are real constants, has direction cosines $\cos\theta$, $\cos\phi$ and $\cos\omega$, where θ, ϕ and ω are the angles $\overrightarrow{OA}$ make with the positive x, y and z-axes respectively.

(i) Express the direction cosines $\cos\theta$, $\cos\phi$ and $\cos\omega$ in terms of α, β and γ. Hence find the value of $\cos^2\theta+\cos^2\phi+\cos^2\omega$. [3]

(ii) Hence show that $\cos 2\theta+\cos 2\phi+\cos 2\omega=-1$. [2]

6 A particle moving along a path at time t, where $0<t<\frac{\pi}{3}$, is defined parametrically by

$$x=\cot 3t \quad \text{and} \quad y=2\operatorname{cosec} 3t+1 .$$

(a) The tangent to the path at the point $P\left(\cot 3p,\ 2\operatorname{cosec}3p+1\right)$ meets the y-axis at the point Q. Show that the coordinates of Q is $\left(0,\ 2\sin 3p+1\right)$. [4]

(b) The distance of the particle from the point $R(0,1)$ is denoted by s, where $s^2=x^2+(y-1)^2$. Find the exact rate of change of the particle's distance from R at time $t=\frac{\pi}{4}$. [4]

7

(i) It is given that $\ln y=2\sin x$. Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-y\ln y+\dfrac{1}{y}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$. [2]

(ii) Find the first four terms of the Maclaurin series for y in ascending powers of x. [4]

(iii) Using appropriate expansions from the List of Formulae (MF26), verify the expansion found in part **(ii)**. [2]

(iv) Given that x is sufficiently small for x^4 and higher powers of x to be neglected, deduce an approximation for $\mathrm{e}^{(2\sin x)-\ln(\sec x)}$ in ascending powers of x. [2]

8 **(a)** A curve is defined parametrically by the equations

$$x = \sin t \quad \text{and} \quad y = \cos^3 t, \quad -\pi \le t \le \pi.$$

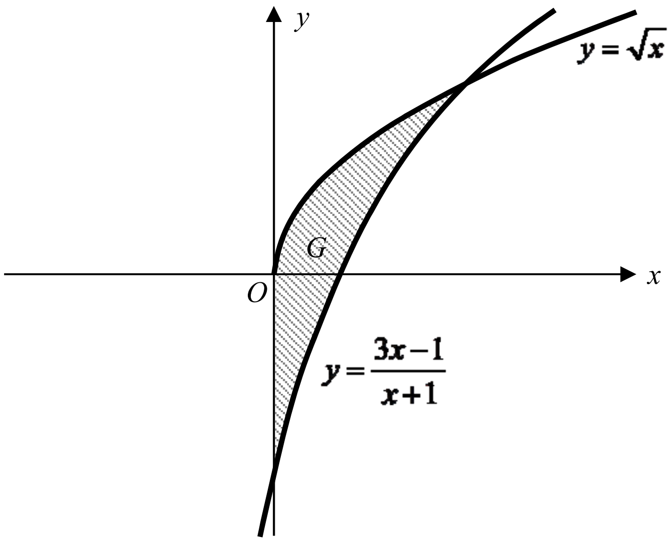
(i) Show that the area enclosed by the curve is given by

$$k\int_0^{\frac{\pi}{2}} \cos^4 t \, dt,$$

where k is a constant to be determined. [3]

(ii) Hence find the exact area enclosed by the curve. [3]

(b) In the diagram, the region G is bounded by the curves $y = \dfrac{3x-1}{x+1}$, $y = \sqrt{x}$ and the y-axis.



Find the exact volume of the solid generated when G is rotated about the y-axis through 2π radians. [6]

9 A curve C_1 has equation $y = \dfrac{ax^2 - bx}{x^2 - c}$, where a, b and c are constants. It is given that C_1 passes through the point $\left(3, \frac{9}{5}\right)$ and two of its asymptotes are $y = 2$ and $x = -2$.

(i) Find the values of a, b and c. [3]

In the rest of the question, take the values of a, b and c as found in part **(i)**.

(ii) Using an algebraic method, find the exact set of values of y that C_1 cannot take. [3]

(iii) Sketch C_1, showing clearly the equations of asymptotes and the coordinates of the turning points. [3]

(iv) It is given that the equation $e^y = x - r$, where $r \in \mathbb{Z}^+$, has exactly one real root. State the range of values of r. [1]

(v) The curve C_2 has equation $y = 2 + \dfrac{3x+5}{x^2-2x-3}$. State a sequence of transformations which transforms C_1 to C_2. [3]

10 Food energy taken in by a man goes partly to maintain the healthy functioning of his body and partly to increase body mass. The total food energy intake of the man per day is assumed to be a constant denoted by I (in joules). The food energy required to maintain the healthy functioning of his body is proportional to his body mass M (in kg). The increase of M with respect to time t (in days) is proportional to the energy not used by his body. If the man does not eat for one day, his body mass will be reduced by 1%.

(i) Show that I, M and t are related by the following differential equation:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{I - aM}{100a} \text{ , where } a \text{ is a constant.}$$

State an assumption for this model to be valid. [3]

(ii) Find the total food energy intake per day, I, of the man in terms of a and M if he wants to maintain a constant body mass. [1]

It is given that the man's initial mass is 100kg.

(iii) Solve the differential equation in part **(i)**, giving M in terms of I, a and t. [3]

(iv) Sketch the graph of M against t for the case where $I > 100a$. Interpret the shape of the graph with regard to the man's food energy intake. [3]

(v) If the man's total food energy intake per day is $50a$, find the time taken in days for the man to reduce his body mass from 100kg to 90kg. [2]

11 A manual hoist is a mechanical device used primarily for raising and lowering heavy loads, with the motive power supplied manually by hand. Three hoists, A, B and C are used to lift a load vertically.

(i) For hoist A, the first pull will raise the load by a vertical distance of 45 cm. On each subsequent pull, the load will raise 1.6 cm lesser than the vertical distance covered by the previous pull. Determine the number of pulls needed for the load to achieve maximum total height. Hence find this maximum total height. [4]

(ii) For hoist B, the first pull will raise the load by a vertical distance of 45 cm. On each subsequent pull, the vertical distance raised will be 95% of the distance covered by the previous pull. Find the theoretical maximum total height that the load can reach. [2]

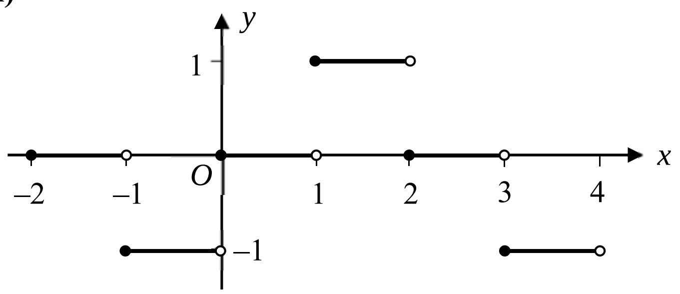
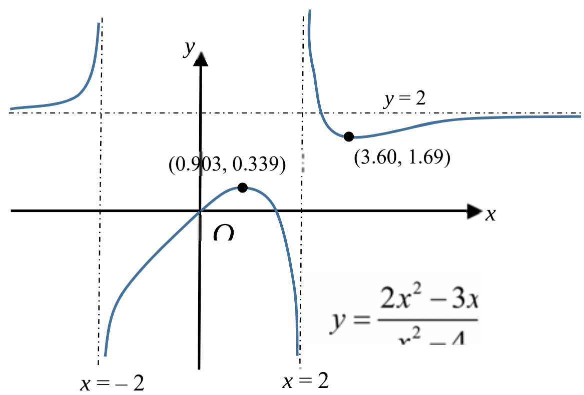
(iii) For hoist C, every pull will raise the load by a constant vertical distance of 45 cm. However, after each pull, the load will slip and drop by 2% of the total vertical height the load has reached. Show that just before the 4^{th} pull, the load would have reached a total vertical height of 130 cm, correct to 3 significant figures.

Hence show that before the $(n+1)^{\text{th}}$ pull, the load would have reached a total vertical height of $X + Y(0.98)^{n+1}$, where X and Y are integers to be determined. [5]

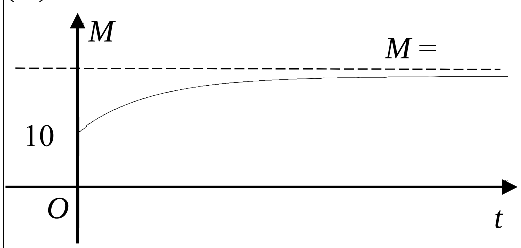
(iv) Explain clearly if hoist C can lift the load up a building of height 25 metres. [2]

ANNEX B

HCI H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Functions	**(i)** $\mathrm{f}(-1.2)=\mathrm{f}(2.8)=0$ $\mathrm{f}(3.6)=\mathrm{f}(-0.4)=-1$ **(ii)** 
2	Graphs and Transformation	**(i)** Since $y=2$ is a horizontal asymptote, $a=2$. Since $x=-2$ is a vertical asymptote, $c=4$. $\left(3,\frac{9}{5}\right)$ lies on $y=\dfrac{2x^2-bx}{x^2-4} \Rightarrow \quad b=3$ **(ii)** required set is $\left\{y\in\mathbb{R}:1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}\right\}$ **(iii)**  **(iv)** $r\geq 2$ **(v)** 1. Translation of C_1 1 unit in the negative x-direction to get $y=2+\dfrac{8-3(x+1)}{(x+1)^2-4}=2+\dfrac{-3x+5}{x^2+2x-3}$ followed by 2. Reflection of $y=2+\dfrac{-3x+5}{x^2+2x-3}$ in the y-axis to get C_2.

3	Sigma Notation and Method of Difference	NA
4	Binomial Expansion	NA
5	AP and GP	**(i)** $n \le 29.125$ Hence number of pulls needed to achieve maximum total height is 29. Maximum total height $= \frac{29}{2}\left[2(45)+(29-1)(-1.6)\right]$ $= 655.4$ cm **(ii)** Maximum total height $= \dfrac{45}{1-0.95} = 900$ cm **(iii)** Before 4th pull, total height reached $= \dfrac{0.98(45)(1-0.98^3)}{1-0.98}$ $= 129.67164$ $= 130$ cm (3 s.f.) Before $(n+1)^{\text{th}}$ pull, total height reached $= \dfrac{0.98(45)(1-0.98^n)}{1-0.98}$ $= 2205 - 2250(0.98)^{n+1}$, where $X = 2205,\ Y = -2250$ **(iv)** Maximum total height reached by load using hoist C will approach 2205 cm. Therefore the hoist C cannot be used to lift the load up the building of 2500 cm
6	Equations and Inequalities	**(i)** $3x - x^2 - 4 = -\left(x - \frac{3}{2}\right)^2 - \frac{7}{4} \le -\frac{7}{4} < 0$ **(ii)** $x < 1-\sqrt{6}$ or $x > 1+\sqrt{6}$ or $x = 1$
7	Differentiation & Applications	**(a)** At point P, $\frac{\mathrm{d}y}{\mathrm{d}x}\big\|_{t=p} = 2\cos 3p$ Equation of tangent at P: $y - (2\operatorname{cosec} 3p + 1) = 2\cos 3p\,(x - \cot 3p)$ Hence the coordinates of Q is $(0,\ 2\sin 3p + 1)$ **(b)** $\frac{\mathrm{d}s}{\mathrm{d}t} = -5\operatorname{cosec}^2 3\left(\frac{\pi}{4}\right)\cot 3\left(\frac{\pi}{4}\right)$ $= -5(2)(-1)$ $= 10$ unit/s
8	Integration techniques	$\int \sec^3 x \, \mathrm{d}x$ $= \int \sec x \sec^2 x \, \mathrm{d}x$

		$2\int \sec^3 x \, \mathrm{d}x = \sec x \tan x + \ln\lvert\sec x + \tan x\rvert$ $\int \sec^3 x \, \mathrm{d}x = \frac{1}{2}(\sec x \tan x + \ln\lvert\sec x + \tan x\rvert) + C$ $\int_0^{\tan^{-1} 2} \sec^3 x \, \mathrm{d}x = \sqrt{5} + \frac{1}{2}\ln(\sqrt{5}+2)$
9	Application of Integration	**(a)(i)** Area $= 4\int_0^1 y \, \mathrm{d}x$ $= 4\int_0^{\frac{\pi}{2}} (\cos^3 t)\cos t \, \mathrm{d}t$ $= 4\int_0^{\frac{\pi}{2}} \cos^4 t \, \mathrm{d}t$ (shown) $\therefore\ k = 4$ **(a)(ii)** $\frac{3\pi}{4}$ unit^2 **(b)** $\frac{29\pi}{5} - 8\pi \ln 2$ unit^3
10	Maclaurin series	**(ii)** $y = 1 + 2x + 2x^2 + x^3 + \ldots$ **(iv)** $\mathrm{e}^{(2\sin x)}\cos x \approx 1 + 2x + \frac{3}{2}x^2 + \ldots$
11	Differential Equations	**(i)** Assumption (any 1 below): • The man does not exercise so that no food energy is used up through exercising. • The man does not fall sick so that no food energy is used up to help him recover from his illness. • The man does not consume weight enhancing/loss supplements that affect his food energy gain/loss other than maintaining the healthy functioning of his body and increasing body mass. **(ii)** For $\frac{\mathrm{d}M}{\mathrm{d}t}$ to be zero, $I = aM$ **(iii)** $M = \frac{I}{a} - \left(\frac{I}{a} - 100\right)\mathrm{e}^{\frac{-t}{100}}$ **(iv)** 

		Explanation (any 1 below): • The man consumes more food than is necessary for maintaining a healthy functioning body. Therefore the graph shows that his body mass will increase. • Since $I > 100a$, hence $\frac{I}{a} > 100$. The man's body mass is always less than $\frac{I}{a}$. • In the long run, the man's body mass will approach $\frac{I}{a}$. **(v)** $t = -100\ln\frac{4}{5} = 22.3$ days
12	Complex numbers	**(i)** $\lvert w\rvert = 2r$, $\arg w = -\frac{5\pi}{6} + \theta$ **(ii)** $9\theta - \frac{5\pi}{6} = \ \ldots, -\frac{3\pi}{2},\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \ldots$ The three smallest values of θ are $\frac{\pi}{27}$, $\frac{4\pi}{27}$ and $\frac{7\pi}{27}$.
13	Vectors	**(a)** Since $\underset{\sim}{a}$ is non-zero and $\underset{\sim}{b} \neq \underset{\sim}{c}$, $\therefore$ $\underset{\sim}{a}$ is parallel to $(\underset{\sim}{c} - \underset{\sim}{b})$. $\therefore$ $\underset{\sim}{a} = k(\underset{\sim}{c} - \underset{\sim}{b}),\quad k \in \mathbb{R}$. **(b)(i)** $\cos\theta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$; $\cos\phi = \frac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$; $\cos\omega = \frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$; $\cos^2\theta + \cos^2\phi + \cos^2\omega = 1$
14	P&C, Probability	NA
15	DRV	NA
16	Binomial Distribution	NA
17	Normal Distribution	NA
18	Sampling	NA
19	Hypothesis Testing	NA
20	Correlation & Linear Regression	NA

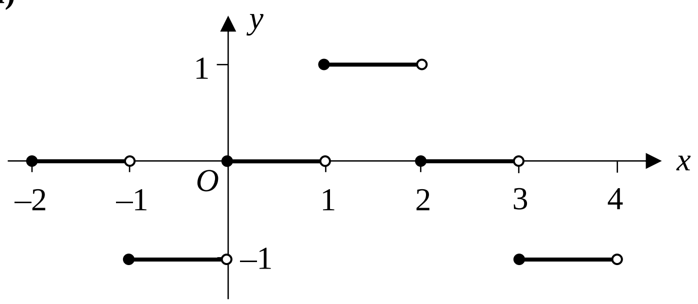
H2 Mathematics 2017 Prelim Exam Paper 1 Solution
Section A: Pure Mathematics

1

(i) $\mathrm{f}(-1.2)=\mathrm{f}(2.8)=0$

$\mathrm{f}(3.6)=\mathrm{f}(-0.4)=-1$

(ii)



(iii) $\int_{-2}^{4}\mathrm{f}(x)\,\mathrm{d}x=-1+1-1=-1$

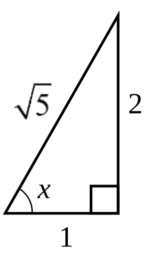
2

$u=\sec x \Rightarrow u'=\sec x\tan x$

$v'=\sec^2 x \Rightarrow v=\tan x$

$$\int \sec^3 x\ \mathrm{d}x$$
$$=\int \sec x\ \sec^2 x\ \mathrm{d}x$$
$$=\sec x\tan x-\int \sec x\tan^2 x\ \mathrm{d}x$$
$$=\sec x\tan x-\int \sec x(\sec^2 x-1)\ \mathrm{d}x$$
$$=\sec x\tan x-\int \sec^3 x-\sec x\ \mathrm{d}x$$
$$=\sec x\tan x-\int \sec^3 x\ \mathrm{d}x+\ln\left|\sec x+\tan x\right|$$
$$2\int \sec^3 x\ \mathrm{d}x=\sec x\tan x+\ln\left|\sec x+\tan x\right|$$
$$\int \sec^3 x\ \mathrm{d}x=\frac{1}{2}(\sec x\tan x+\ln\left|\sec x+\tan x\right|)+C$$

$$\int_0^{\tan^{-1}2}\sec^3 x\ \mathrm{d}x$$
$$=\frac{1}{2}\Big[\sec x\tan x+\ln\left|\sec x+\tan x\right|\Big]_0^{\tan^{-1}2}$$
$$=\frac{1}{2}\Big[\sqrt{5}\times 2+\ln(\sqrt{5}+2)\Big]$$
$$=\sqrt{5}+\frac{1}{2}\ln(\sqrt{5}+2)$$



3

(i)

$$3x-x^2-4=-\left(x^2-3x+4\right)$$
$$=-\left(\left(x-\frac{3}{2}\right)^2+\frac{7}{4}\right)$$
$$=-\left(x-\frac{3}{2}\right)^2-\frac{7}{4}$$

Since $\left(x-\frac{3}{2}\right)^2\geq 0$ for all $x\in\mathbb{R}$, $-\left(x-\frac{3}{2}\right)^2\leq 0$

Hence $3x-x^2-4=-\left(x-\frac{3}{2}\right)^2-\frac{7}{4}\leq-\frac{7}{4}<0$

$\therefore$ $3x-x^2-4$ is always negative for all values of x.

(ii)

$$\frac{\left(3x-x^2-4\right)\left(x-1\right)^2}{x^2-2x-5}\leq 0$$

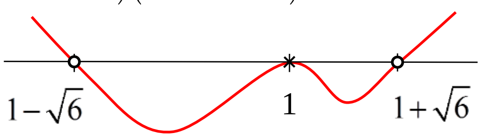
Since $3x-x^2-4$ is always negative, $\dfrac{\left(x-1\right)^2}{x^2-2x-5}\geq 0$

Method 1 (Quadratic formula)

Let $x^2-2x-5=0$

$$\therefore\ x=\frac{2\pm\sqrt{(-2)^2-4(1)(-5)}}{2(1)}=\frac{2\pm\sqrt{24}}{2}=1\pm\sqrt{6}$$

Hence $\dfrac{\left(x-1\right)^2}{\left(x-(1-\sqrt{6})\right)\left(x-(1+\sqrt{6})\right)}\geq 0$

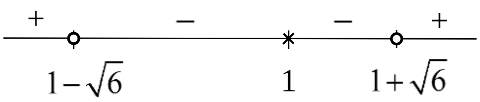


$\therefore$ $x<1-\sqrt{6}$ or $x>1+\sqrt{6}$ or $x=1$

Method 2 (Complete the square)

$$\frac{\left(x-1\right)^2}{\left(x-1\right)^2-6}\geq 0$$

$$\frac{\left(x-1\right)^2}{\left(x-(1-\sqrt{6})\right)\left(x-(1+\sqrt{6})\right)}\geq 0$$



$\therefore$ $x<1-\sqrt{6}$ or $x>1+\sqrt{6}$ or $x=1$

4

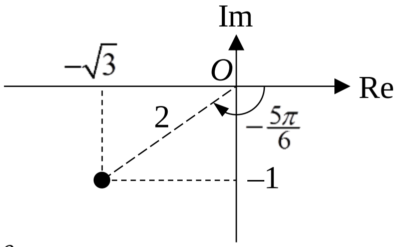
(i)

<u>Method 1</u>

$$w=\left(-\sqrt{3}-\mathrm{i}\right)z$$

$$=[2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}]r\,\mathrm{e}^{\mathrm{i}\theta}$$

$$=2r\,\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}+\theta\right)}$$



$$\therefore |w|=2r,\ \arg w=-\frac{5\pi}{6}+\theta$$

<u>Method 2</u>

$$|w|=\left|\left(-\sqrt{3}-\mathrm{i}\right)z\right|$$

$$=\left|\left(-\sqrt{3}-\mathrm{i}\right)\right||z|$$

$$=2r$$

$$\arg w=\arg\left(\left[-\sqrt{3}-\mathrm{i}\right]z\right)$$

$$=\arg\left(-\sqrt{3}-\mathrm{i}\right)+\arg z$$

$$=-\frac{5\pi}{6}+\theta$$

(ii)

<u>Method 1</u>

$$\arg\left(\frac{z^8}{w^*}\right)=\arg(z^8)-\arg(w^*) \quad \boxed{\arg(w^*)=-\arg(w)}$$

$$=8\theta+\arg w$$

$$=8\theta+\left(-\frac{5\pi}{6}+\theta\right) \quad \boxed{\text{From } \textbf{(ii)}}$$

$$=9\theta-\frac{5\pi}{6}$$

For $\dfrac{z^8}{w^*}$ to be purely imaginary,

$$\arg\left(\frac{z^8}{w^*}\right)=\ \dots,-\frac{3\pi}{2},\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \dots$$

$$\therefore\ 9\theta-\frac{5\pi}{6}=\ \dots,-\frac{3\pi}{2},\ -\frac{\pi}{2},\ \frac{\pi}{2},\ \frac{3\pi}{2},\ \dots$$

$$9\theta=\dots,-\frac{2\pi}{3},\ \frac{\pi}{3},\ \frac{4\pi}{3},\ \frac{7\pi}{3},\ \dots$$

$$\theta=\dots,-\frac{2\pi}{27},\ \frac{\pi}{27},\ \frac{4\pi}{27},\ \frac{7\pi}{27},\ \dots$$

$\therefore$ the three smallest values of θ are $\dfrac{\pi}{27}$, $\dfrac{4\pi}{27}$ and $\dfrac{7\pi}{27}$.

<u>Method 2</u>

$$\frac{z^8}{w^*}=\frac{(r\,\mathrm{e}^{\mathrm{i}\theta})^8}{2r\,\mathrm{e}^{\mathrm{i}\left[-\left(-\frac{5\pi}{6}+\theta\right)\right]}}=\frac{r^8\,\mathrm{e}^{\mathrm{i}(8\theta)}}{2r\,\mathrm{e}^{\mathrm{i}\left(\frac{5\pi}{6}-\theta\right)}}$$

$$=\frac{r^7}{2}\mathrm{e}^{\mathrm{i}\left[8\theta-\left(\frac{5\pi}{6}-\theta\right)\right]}$$

$$=\frac{r^7}{2}\mathrm{e}^{\mathrm{i}\left(9\theta-\frac{5\pi}{6}\right)}$$

For $\dfrac{z^8}{w^*}$ to be purely imaginary,

$\arg\left(\dfrac{z^8}{w^*}\right)=\dfrac{\pi}{2}+k\pi$, where $k\in\mathbb{Z}$

$$\therefore\ 9\theta-\frac{5\pi}{6}=\frac{\pi}{2}+k\pi$$

$$9\theta=\frac{4\pi}{3}+k\pi$$

$$\theta=\frac{4\pi}{27}+\frac{k\pi}{9}$$

When $k=-2$, $\theta=-\frac{2\pi}{27}$

When $k=-1$, $\theta=\frac{\pi}{27}$

When $k=0$, $\theta=\frac{4\pi}{27}$

When $k=1$, $\theta=\frac{7\pi}{27}$

$\therefore$ the three smallest values of θ are $\dfrac{\pi}{27}$, $\dfrac{4\pi}{27}$ and $\dfrac{7\pi}{27}$.

5 **(a)**

$$(\underset{\sim}{a}+\underset{\sim}{b})\times(\underset{\sim}{a}+\underset{\sim}{c})=\underset{\sim}{b}\times\underset{\sim}{c}$$

$$(\underset{\sim}{a}\times\underset{\sim}{a})+(\underset{\sim}{a}\times\underset{\sim}{c})+(\underset{\sim}{b}\times\underset{\sim}{a})+(\underset{\sim}{b}\times\underset{\sim}{c})=\underset{\sim}{b}\times\underset{\sim}{c}$$

$$(\underset{\sim}{a}\times\underset{\sim}{c})+(\underset{\sim}{b}\times\underset{\sim}{a})=\underset{\sim}{0}$$

$$(\underset{\sim}{a}\times\underset{\sim}{c})-(\underset{\sim}{a}\times\underset{\sim}{b})=\underset{\sim}{0}$$

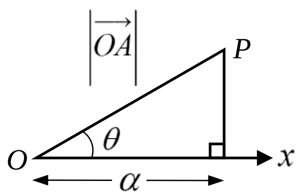
$$\underset{\sim}{a}\times(\underset{\sim}{c}-\underset{\sim}{b})=\underset{\sim}{0}$$

Since $\underset{\sim}{a}$ is non-zero and $\underset{\sim}{b}\neq\underset{\sim}{c}$,

$\therefore$ $\underset{\sim}{a}$ is parallel to $(\underset{\sim}{c}-\underset{\sim}{b})$.

$\therefore$ $\underset{\sim}{a}=k(\underset{\sim}{c}-\underset{\sim}{b})$, $k\in\mathbb{R}$.

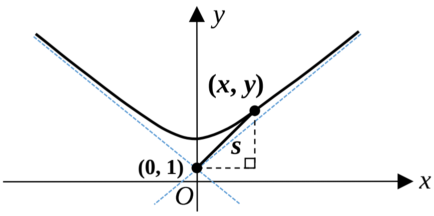
(b)(i)



$$\left|\overrightarrow{OA}\right|=\sqrt{\alpha^2+\beta^2+\gamma^2}$$

$$\therefore\quad\cos\theta=\frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

	$\cos\phi=\dfrac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$ $\cos\omega=\dfrac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$ $\cos^2\theta+\cos^2\phi+\cos^2\omega$ $=\left(\dfrac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2+\left(\dfrac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2+\left(\dfrac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2$ $=\dfrac{\alpha^2+\beta^2+\gamma^2}{\alpha^2+\beta^2+\gamma^2}$ $=1$ **(b)(ii)** $\cos 2\theta+\cos 2\phi+\cos 2\omega$ $=2\cos^2\theta-1+2\cos^2\phi-1+2\cos^2\omega-1$ $=2\left(\cos^2\theta+\cos^2\phi+\cos^2\omega\right)-3$ $=2(1)-3$ $=-1$ (shown)
6	**(a)** $x=\cot 3t \Rightarrow \dfrac{dx}{dt}=-3\operatorname{cosec}^2 3t$ $y=2\operatorname{cosec}3t+1 \Rightarrow \dfrac{dy}{dt}=-6\operatorname{cosec}3t\cot 3t$ $\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{-6\operatorname{cosec}3t\cot 3t}{-3\operatorname{cosec}^2 3t}$ $=\dfrac{2\cot 3t}{\operatorname{cosec}3t}$ $=2\cos 3t$ At point *P*, $\dfrac{dy}{dx}\Big\vert_{t=p}=2\cos 3p$ Equation of tangent at *P*: $y-\left(2\operatorname{cosec}3p+1\right)=2\cos 3p\left(x-\cot 3p\right)$ When tangent meets *y*-axis, $x=0$. Hence $y=-\left(2\cos 3p\right)\left(\cot 3p\right)+\left(2\operatorname{cosec}3p+1\right)$ $y=\dfrac{-2\left(\cos^2 3p\right)}{\sin 3p}+\dfrac{2}{\sin 3p}+1$

	$y=\frac{-2(\cos^2 3p-1)}{\sin 3p}+1$ $y=\frac{-2(-\sin^2 3p)}{\sin 3p}+1$ $y=2\sin 3p+1$ Hence the coordinates of Q is $\left(0,\ 2\sin 3p+1\right)$. (shown) **(b)**  Method 1 $s^2=x^2+\left(y-1\right)^2$ $=\cot^2 3t+\left(2\operatorname{cosec}3t+1-1\right)^2$ $=(\operatorname{cosec}^2 3t-1)+4\operatorname{cosec}^2 3t$ $=5\operatorname{cosec}^2 3t-1$ Differentiate w.r.t. t, $2s\frac{\mathrm{d}s}{\mathrm{d}t}=10\operatorname{cosec}3t(-\operatorname{cosec}3t\cot 3t)(3)$ $=-30\operatorname{cosec}^2 3t\cot 3t$ $s\frac{\mathrm{d}s}{\mathrm{d}t}=-15\operatorname{cosec}^2 3t\cot 3t$ When $t=\frac{\pi}{4}$, $s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$ $\therefore\ s=3$ (since $s>0$) $\therefore\ \frac{\mathrm{d}s}{\mathrm{d}t}=-5\operatorname{cosec}^2 3\left(\frac{\pi}{4}\right)\cot 3\left(\frac{\pi}{4}\right)$ $=-5(2)(-1)$ $=10$ unit/s Method 2 $s^2=x^2+\left(y-1\right)^2$ $=\cot^2 3t+\left(2\operatorname{cosec}3t+1-1\right)^2$ $=\cot^2 3t+4\operatorname{cosec}^2 3t$ Differentiate w.r.t. t, $2s\frac{\mathrm{d}s}{\mathrm{d}t}=2\cot 3t(-\operatorname{cosec}^2 3t)(3)+8\operatorname{cosec}3t(-\operatorname{cosec}3t\cot 3t)(3)$ $=-6\operatorname{cosec}^2 3t\cot 3t-24\operatorname{cosec}^2 3t\cot 3t$ $=-30\operatorname{cosec}^2 3t\cot 3t$ $s\frac{\mathrm{d}s}{\mathrm{d}t}=-15\operatorname{cosec}^2 3t\cot 3t$

	When $t=\frac{\pi}{4}$, $s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$ $\therefore\ s=3$ (since $s>0$) $\therefore\ \frac{ds}{dt}=-5\operatorname{cosec}^2 3\left(\frac{\pi}{4}\right)\cot 3\left(\frac{\pi}{4}\right)$ $=-5(2)(-1)$ $=10$ unit/s <u>Method 3</u> $s^2=x^2+(y-1)^2$ Differentiate w.r.t. t , $2s\frac{ds}{dt}=2x\frac{dx}{dt}+2\left(y-1\right)\frac{dy}{dt}$ $s\frac{ds}{dt}=x\frac{dx}{dt}+\left(y-1\right)\frac{dy}{dt}$ When $t=\frac{\pi}{4}$, $x=\frac{1}{\tan\left(\frac{3\pi}{4}\right)}=-1,\qquad y=\frac{2}{\sin\left(\frac{3\pi}{4}\right)}+1=2\sqrt{2}+1$ $\frac{dx}{dt}=-3\operatorname{cosec}^2 3t=\frac{-3}{\sin^2\left(\frac{3\pi}{4}\right)}=-6$ $\frac{dy}{dt}=-6\cot 3t\operatorname{cosec} 3t=\frac{-6}{\tan\left(\frac{3\pi}{4}\right)}\times\frac{1}{\sin\left(\frac{3\pi}{4}\right)}=6\sqrt{2}$ $s^2=(2\sqrt{2}+1-1)^2+(-1)^2=9$ $\therefore\ s=3$ (since $s>0$) Hence $\frac{ds}{dt}=\frac{1}{s}\left[x\frac{dx}{dt}+\left(y-1\right)\frac{dy}{dt}\right]$ $=\frac{1}{3}\left[(-1)(-6)+(2\sqrt{2})(6\sqrt{2})\right]$ $=10$ unit/s
7	**(i)** <u>Method 1</u> $\ln y=2\sin x$ $\frac{1}{y}\frac{dy}{dx}=2\cos x$ $\frac{dy}{dx}=2y\cos x$ $\frac{d^2y}{dx^2}=-2y\sin x+2\cos x\frac{dy}{dx}=-y\ln y+\frac{1}{y}\left(\frac{dy}{dx}\right)^2$ (shown) <u>Method 2</u> $y=e^{2\sin x}$ $\frac{dy}{dx}=(2\cos x)e^{2\sin x}$

	$\frac{dy}{dx}=(2\cos x)y$ $\frac{d^2y}{dx^2}=-2y\sin x+2\cos x\frac{dy}{dx}$ $\frac{d^2y}{dx^2}=-y\ln y+\frac{1}{y}\left(\frac{dy}{dx}\right)^2$ (shown) **(ii)** $\frac{d^3y}{dx^3}=-y\left(\frac{1}{y}\frac{dy}{dx}\right)-\ln y\frac{dy}{dx}-\frac{1}{y^2}\left(\frac{dy}{dx}\right)^3+\frac{2}{y}\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)$ When $x=0$, $y=1$, $\frac{dy}{dx}=2$, $\frac{d^2y}{dx^2}=4$, $\frac{d^3y}{dx^3}=6$ $y=1+2x+\frac{4x^2}{2!}+\frac{6x^3}{3!}+...$ $y=1+2x+2x^2+x^3+...$ **(iii)** <u>Method 1</u> $y=e^{2\sin x}$ $=1+(2\sin x)+\frac{(2\sin x)^2}{2}+\frac{(2\sin x)^3}{6}+...$ $=1+2(x-\frac{x^3}{6}+...)+\frac{[2(x+...)]^2}{2}+\frac{[2(x+...)]^3}{6}+...$ $=1+2x-\frac{x^3}{3}+2x^2+\frac{4x^3}{3}+...$ $=1+2x+2x^2+x^3+...$ <u>Method 2</u> $y=e^{2(x-\frac{x^3}{3!})}$ $=1+2(x-\frac{x^3}{6})+\frac{[2(x-\frac{x^3}{6})]^2}{2}+\frac{[2(x-\frac{x^3}{6})]^3}{6}+...$ $=1+2x-\frac{2x^3}{6}+\frac{4x^2}{2}+\frac{8x^3}{6}+...$ $=1+2x+2x^2+x^3+...$ **(iv)** $e^{(2\sin x)-\ln(\sec x)}=e^{(2\sin x)}e^{-\ln\sec x}=e^{(2\sin x)}e^{\ln\cos x}$ $=e^{(2\sin x)}\cos x$ <u>Method 1</u> $e^{(2\sin x)}\cos x\approx(1+2x+2x^2+x^3)(1-\frac{x^2}{2})$ $=1-\frac{x^2}{2}+2x-\frac{2x^3}{2}+2x^2+x^3+...$

$$= 1 + 2x + \tfrac{3}{2}x^2 + \ldots$$

<u>Method 2</u>

$$y = e^{2\sin x}$$

$$\frac{dy}{dx} = (2\cos x)e^{2\sin x}$$

$$\therefore \ \cos x e^{2\sin x} = \frac{1}{2}\frac{dy}{dx}$$

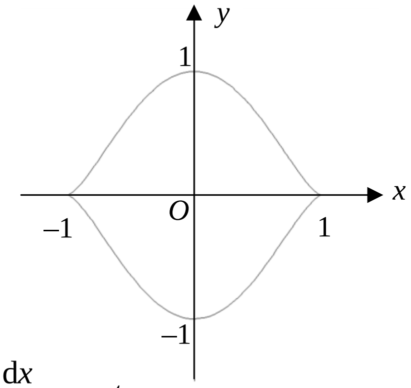
$$= \frac{1}{2}\frac{d}{dx}(1 + 2x + 2x^2 + x^3 + \ldots) \longleftarrow \boxed{\text{From } \textbf{(iii)}}$$

$$= \frac{1}{2}(2 + 4x + 3x^2 + \ldots)$$

$$= 1 + 2x + \tfrac{3}{2}x^2 + \ldots$$

8

(a)(i)



$$x = \sin t \ \Rightarrow \ \frac{dx}{dt} = \cos t$$

When $x = 0$, $t = 0$.

When $x = 1$, $t = \frac{\pi}{2}$.

$$\text{Area} = 4\int_0^1 y \ dx$$

$$= 4\int_0^{\frac{\pi}{2}} (\cos^3 t)\cos t \ dt$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^4 t \ dt \qquad \text{(shown)}$$

$$\therefore \ k = 4$$

(a)(ii)

$$\text{Area} = 4\int_0^{\frac{\pi}{2}} \cos^4 t \ dt$$

$$= \int_0^{\frac{\pi}{2}} (2\cos^2 t)^2 \ dt$$

	$=\int_0^{\frac{\pi}{2}}(1+\cos 2t)^2\ \mathrm{d}t$ $=\int_0^{\frac{\pi}{2}}1+2\cos 2t+\cos^2 2t\ \mathrm{d}t$ $=\int_0^{\frac{\pi}{2}}1+2\cos 2t+\frac{1+\cos 4t}{2}\ \mathrm{d}t$ $=\int_0^{\frac{\pi}{2}}\frac{3}{2}+2\cos 2t+\frac{\cos 4t}{2}\ \mathrm{d}t$ $=\left[\frac{3t}{2}+\sin 2t+\frac{\sin 4t}{8}\right]_0^{\frac{\pi}{2}}$ $=\frac{3\pi}{4}\ \text{unit}^2$ **(b)** From GC, coordinates of intersection = (1, 1) <u>Method 1</u> $y=\frac{3x-1}{x+1}\ \Rightarrow\ xy+y=3x-1\ \Rightarrow\ x=\frac{1+y}{3-y}$ Required volume $=\pi\int_{-1}^{1}\left(\frac{1+y}{3-y}\right)^2\ \mathrm{d}y-\pi\int_0^1(y^2)^2\ \mathrm{d}y$ $=\pi\int_{-1}^{1}\left(\frac{4}{3-y}-1\right)^2\ \mathrm{d}y-\pi\int_0^1 y^4\ \mathrm{d}y$ $=\pi\int_{-1}^{1}\left(\frac{16}{(3-y)^2}-\frac{8}{3-y}+1\right)\mathrm{d}y-\pi\left[\frac{y^5}{5}\right]_0^1$ $=\pi\left[\frac{16}{3-y}+8\ln\lvert 3-y\rvert+y\right]_{-1}^{1}-\frac{\pi}{5}$ $=\pi\left[8+8\ln 2+1-(4+8\ln 4-1)\right]-\frac{\pi}{5}$ $=\pi\left[6+8\ln 2-16\ln 2\right]-\frac{\pi}{5}$ $=\frac{29\pi}{5}-8\pi\ln 2\quad \text{unit}^3$ <u>Method 2</u> $y=\frac{3x-1}{x+1}\ \Rightarrow\ xy+y=3x-1\ \Rightarrow\ x=\frac{1+y}{3-y}$ Required volume

$$= \pi \int_{-1}^{1} \left(\frac{1+y}{3-y} \right)^2 \, \mathrm{d}y - \pi \int_{0}^{1} (y^2)^2 \, \mathrm{d}y$$

$$= \pi \int_{-1}^{1} \frac{y^2+2y+1}{y^2-6y+9} \, \mathrm{d}y - \pi \int_{0}^{1} y^4 \, \mathrm{d}y$$

$$= \pi \int_{-1}^{1} 1 + \frac{8y-8}{y^2-6y+9} \, \mathrm{d}y - \pi \left[\frac{y^5}{5} \right]_0^1$$

$$= \pi [y]_{-1}^{1} + 4\pi \int_{-1}^{1} \frac{2y-6}{y^2-6y+9} \, \mathrm{d}y + \pi \int_{-1}^{1} \frac{16}{(y-3)^2} \, \mathrm{d}y - \frac{\pi}{5}$$

$$= 2\pi + 4\pi \left[\ln \left| y^2 - 6y + 9 \right| \right]_{-1}^{1} + 16\pi \left[\frac{(y-3)^{-1}}{-1} \right]_{-1}^{1} - \frac{\pi}{5}$$

$$= \frac{9\pi}{5} + 4\pi [\ln 4 - \ln 16] + 16\pi \left[\frac{1}{3-y} \right]_{-1}^{1}$$

$$= \frac{9\pi}{5} + 4\pi \ln \frac{1}{4} + 16\pi \left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= \frac{9\pi}{5} - 4\pi \ln 4 + 4\pi$$

$$= \frac{29\pi}{5} - 8\pi \ln 2 \quad \text{unit}^3$$

9 **(i)**

$$y = \frac{ax^2 - bx}{x^2 - c}$$

Since $y = 2$ is a horizontal asymptote, $a = 2$.
Since $x = -2$ is a vertical asymptote, $c = 4$.

$\left(3, \frac{9}{5}\right)$ lies on $y = \dfrac{2x^2 - bx}{x^2 - 4}$

$\therefore \ \frac{9}{5} = \frac{2(3)^2 - b(3)}{(3)^2 - 4} \quad \Rightarrow \quad b = 3$

(ii)

$$y = \frac{2x^2 - 3x}{x^2 - 4}$$

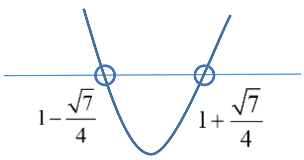
$$y\left(x^2 - 4\right) = 2x^2 - 3x$$

$$(y-2)x^2 + 3x - 4y = 0$$

For no real roots,

$$(3)^2 - 4(y-2)(-4y) < 0$$

$$16y^2 - 32y + 9 < 0$$



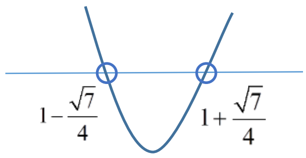
Method 1

$$\therefore\ y=\frac{32\pm\sqrt{(32)^2-4(16)(9)}}{2(16)}=\frac{32\pm\sqrt{448}}{32}=1\pm\frac{\sqrt{7}}{4}$$

$\therefore$ required set is $\left\{y\in\mathbb{R}:1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}\right\}$.

Method 2 (completing the square)

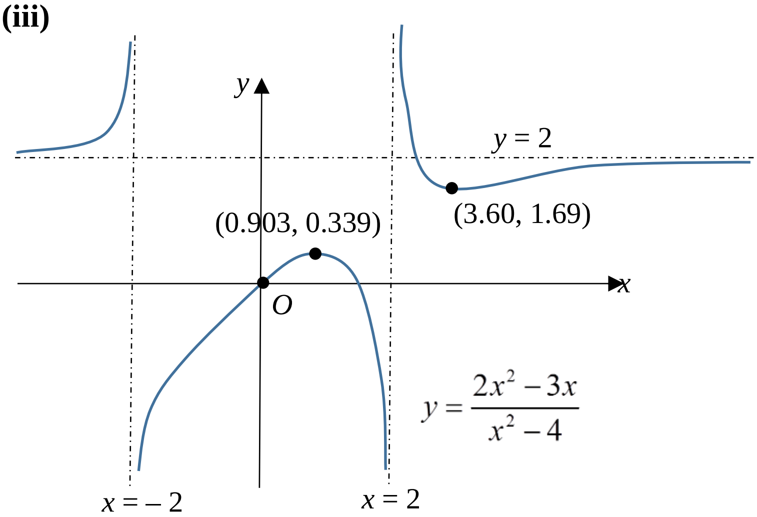
$$16y^2-32y+9<0$$
$$y^2-2y+\tfrac{9}{16}<0$$
$$(y-1)^2-\tfrac{7}{16}<0$$
$$\left(y-1+\tfrac{\sqrt{7}}{4}\right)\left(y-1-\tfrac{\sqrt{7}}{4}\right)<0$$



$$\therefore\ 1-\tfrac{\sqrt{7}}{4}<y<1+\tfrac{\sqrt{7}}{4}$$

$\therefore$ required set is $\left\{y\in\mathbb{R}:1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}\right\}$

(iii)

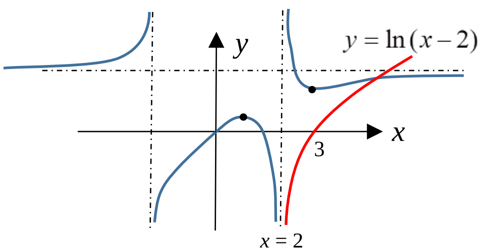


(iv)

$e^y=x-r$

$y=\ln(x-r)$

$r\geq 2$



(v)

$$C_1 : y = \frac{2x^2 - 3x}{x^2 - 4} = 2 + \frac{8 - 3x}{x^2 - 4}$$

$$\begin{array}{r} 2 \\ x^2 - 4 \overline{\smash{)}\, 2x^2 - 3x} \\ \underline{2x^2 - 8} \\ -3x + 8 \end{array}$$

$$C_2 : y = 2 + \frac{3x+5}{x^2 - 2x - 3}$$

$$= 2 + \frac{3x+5}{(x-1)^2 - 4}$$

$$= 2 + \frac{8 - 3(1-x)}{(1-x)^2 - 4}$$

<u>Method 1</u>

Transformation: $x \to x+1 \to -x+1$

1. Translation of C_1 1 unit in the negative x-direction to get

$y = 2 + \dfrac{8 - 3(x+1)}{(x+1)^2 - 4} = 2 + \dfrac{-3x+5}{x^2 + 2x - 3}$ followed by

2. Reflection of $y = 2 + \dfrac{-3x+5}{x^2 + 2x - 3}$ in the y-axis to get C_2.

<u>Method 2</u>

Transformation: $x \to -x \to -(x-1) = -x+1$

1. Reflection of C_1 in the y-axis to get $y = 2 + \dfrac{8+3x}{x^2 - 4}$

followed by

2. Translation of $y = 2 + \dfrac{8+3x}{x^2 - 4}$ 1 unit in the positive x-direction to get C_2.

---

10 **(i)**

$\dfrac{\mathrm{d}M}{\mathrm{d}t} \propto I - kM$, where k is a positive constant.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = b(I - kM)$$

If $I = 0$, $-\dfrac{1}{100}M = b(0 - kM)$

$$-\frac{M}{100} = -bkM$$

$$b = \frac{1}{100k}$$

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{1}{100k}(I - kM) = \frac{I - kM}{100k}$$

$= \dfrac{I - aM}{100a}$, where $a = k$ (shown)

Assumption (any 1 below):

- The man does not exercise so that no food energy is used up through exercising.
- The man does not fall sick so that no food energy is used up to help him recover from his illness.

- The man does not consume weight enhancing/loss supplements that affect his food energy gain/loss other than maintaining the healthy functioning of his body and increasing body mass.

(ii) For $\dfrac{\mathrm{d}M}{\mathrm{d}t}$ to be zero, $I = aM$

(iii)

$$\int \frac{a}{I - aM}\,\mathrm{d}M = \int \frac{1}{100}\,\mathrm{d}t$$

$$-\ln\left|I - aM\right| = \frac{t}{100} + C$$

$$\ln\left|I - aM\right| = \frac{-t}{100} - C$$

$$I - aM = \pm \mathrm{e}^{\frac{-t}{100}}\mathrm{e}^{-C} = A\mathrm{e}^{\frac{-t}{100}}, \text{ where } A = \pm \mathrm{e}^{-C}$$

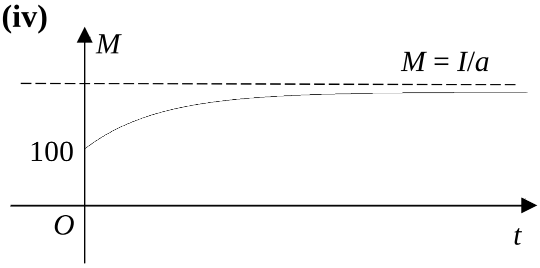
When $t = 0$, $M = 100 \Rightarrow A = I - 100a$

$$I - aM = (I - 100a)\mathrm{e}^{\frac{-t}{100}}$$

$$aM = I - (I - 100a)\mathrm{e}^{\frac{-t}{100}}$$

$$M = \frac{I}{a} - \left(\frac{I}{a} - 100\right)\mathrm{e}^{\frac{-t}{100}}$$

(iv)



Explanation (any 1 below):

- The man consumes more food than is necessary for maintaining a healthy functioning body. Therefore the graph shows that his body mass will increase.
- Since $I > 100a$, hence $\frac{I}{a} > 100$. The man's body mass is always less than $\frac{I}{a}$.

  In the long run, the man's body mass will approach $\frac{I}{a}$.

(v)

Given $I = 50a$,

$$90 = 50 - (50 - 100)\mathrm{e}^{\frac{-t}{100}}$$

Using equation found in **(iii)**

$$50\mathrm{e}^{\frac{-t}{100}} = 40$$

$$\mathrm{e}^{\frac{-t}{100}} = \frac{4}{5}$$

$$\frac{-t}{100} = \ln\frac{4}{5}$$

	$\therefore\ t = -100\ln\dfrac{4}{5} = 22.3$ days (3 s.f.)
11	**(i)** Method 1 Distance covered at the n^{th} pull $= 45 + (n-1)(-1.6)$ $= 46.6 - 1.6n$ $46.6 - 1.6n \geq 0$ $n \leq 29.125$ Hence number of pulls needed to achieve maximum total height is 29. Maximum total height $= \frac{29}{2}\left[2(45) + (29-1)(-1.6)\right]$ $= 655.4$ cm Method 2 Distance covered at the n^{th} pull, $u_n = 45 + (n-1)(-1.6)$ $= 46.6 - 1.6n$ Using GC, n | u_n 29 | 0.2 30 | –1.4 Hence number of pulls needed to achieve maximum total height is 29. Maximum total height $= \frac{29}{2}\left(45 + 0.2\right) = 655.4$ cm Method 3 Distance covered at the n^{th} pull $= 45 + (n-1)(-1.6) = 0$ $\Rightarrow n = 29.125$ n | u_n 29 | 0.2 30 | –1.4 Hence number of pulls needed to achieve maximum total height is 29. Maximum total height $= \frac{29}{2}\left(45 + 0.2\right) = 655.4$ cm Method 4 Total height after n pulls, $S_n = \dfrac{n}{2}\left[2(45) + (n-1)(-1.6)\right] = 45.8n - 0.8n^2$ Using GC, n | S_n 28 | 655.2 29 | 655.4 30 | 654 Hence the number of pulls needed to achieve maximum total height is 29, and the maximum total height covered is 655.4 cm.

(ii)
Since $r = 0.95 < 1$, sum to infinity of G.P. exists.
$\therefore$ maximum total height $= \dfrac{45}{1-0.95} = 900$ cm
(iii)

	Total height reached
Before 2nd pull	$0.98(45)$
Before 3rd pull	$0.98\left(0.98(45)+45\right)$ $= 0.98^2(45) + 0.98(45)$
Before 4th pull	$0.98\left(0.98^2(45)+0.98(45)+45\right)$ $= 0.98^3(45) + 0.98^2(45) + 0.98(45)$
$\vdots$	$\vdots$
Before $(n+1)$th pull	$0.98^n(45) + 0.98^{n-1}(45) + \ldots + 0.98(45)$ $= \dfrac{0.98(45)(1-0.98^n)}{1-0.98}$ [sum of G.P. with $a = 45$, $r = 0.98$]

$\therefore$ before 4th pull, total height reached
$= \dfrac{0.98(45)(1-0.98^3)}{1-0.98}$
$= 129.67164$
$= 130$ cm (3 s.f.)
Before $(n+1)^{\text{th}}$ pull, total height reached
$= \dfrac{0.98(45)(1-0.98^n)}{1-0.98}$
$= 2205 - 2250(0.98)^{n+1}$, where $X = 2205$, $Y = -2250$

(iv)
From **(iii)**,
Total height reached by load using hoist C $= 2205 - 2250(0.98)^{n+1}$
As $n \to \infty$, $(0.98)^{n+1} \to 0$.
Hence maximum total height $\to 2205$.
Therefore maximum total height reached by load using hoist C will approach 2205 cm. Therefore the hoist C cannot be used to lift the load up the building of 2500 cm

HCI Paper 2

1 The sum, S_n , of the first n terms of a sequence u_1, u_2, u_3, … is given by

$$S_n = b - \frac{3a}{(n+1)!} ,$$

where a and b are constants.

(i) It is given that $u_1 = k$ and $u_2 = \frac{2}{3}k$, where k is a constant. Find a and b in terms of k. [3]

(ii) Find a formula for u_n in terms of k, giving your answer in its simplest form. [2]

(iii) Determine, with a reason, if the series $\sum_{r=1}^{\infty} u_r$ converges. [1]

2 The complex numbers z and w satisfy the following equations

$$2z + 3w = 20 ,$$

$$w - z\,w^* = 6 + 22\mathrm{i} .$$

(i) Find z and w in the form $a + b\mathrm{i}$, where a and b are real, $a \neq 0$. [5]

(ii) Show z and w on a single Argand diagram, indicating clearly their modulus. State the relationship between z and w with reference to the origin O. [2]

3 The function f is defined by

$$\mathrm{f} : x \mapsto \sqrt{3}\sin x + \cos x , \qquad x \in \mathbb{R} ,\ -\pi < x < \tfrac{\pi}{6} .$$

(i) Express f in the form $R\sin(x+\alpha)$, where R and α are exact constants to be determined, $R > 0$, $0 \le \alpha \le \frac{\pi}{2}$. [2]

(ii) Sketch f, giving the exact coordinates of the turning point and the end-points. Deduce the exact range of f. [4]

(iii) The function g is defined by

$$\mathrm{g} : x \mapsto \frac{1}{2} - |x-1| , \qquad x \in \mathbb{R},\ -\frac{5}{2} \le x \le \frac{1}{2} .$$

Explain why the composite function fg exists. Find the range of fg . [3]

(iv) The domain of f is restricted such that the function f^{-1} exists. Find the largest domain of f for which f^{-1} exists. Define f^{-1} in a similar form. [4]

4 Referred to the origin O, the position vector of a point A is $-\mathbf{i}+2\mathbf{j}+4\mathbf{k}$. A plane p contains A and is parallel to the vectors $4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+\mathbf{k}$.

(i) Find a cartesian equation of p. [2]

(ii) A plane q has equation $x-2y+z=2$. Find a vector equation of the line l where p and q meet. [1]

A point B lies on l such that AB is perpendicular to l.

(iii) Find the position vector of B. [3]

(iv) Find the length of projection of AB on q. [2]

(v) A point C lies on q such that AC is perpendicular to q. Find the position vector of C. Hence find a cartesian equation of the line of reflection of AB in q. [6]

5 The independent random variables X and Y are normally distributed with the same mean 7 but different variances $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$, respectively. It is given that $\mathrm{P}(X<10)=\mathrm{P}(Y>6)$.

(i) Show that $\mathrm{Var}(X)=9\mathrm{Var}(Y)$. [3]

(ii) If $\mathrm{Var}(Y)=1$, find $\mathrm{P}(X<9)$. [2]

6 A biased tetrahedral (4-sided) die has its faces numbered '-1', '0', '2' and '3'. It is thrown onto a table and the random variable X denotes the number on the face in contact with the table. The probability distribution of X is as shown.

x	-1	0	2	3
$\mathrm{P}(X=x)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$

(i) The random variable Y is defined by X_1+X_2, where X_1 and X_2 are 2 independent observations of X. Show that $\mathrm{P}(Y=2)=\frac{3}{16}$. [2]

(ii) In a game, a player pays $2 to throw two such biased tetrahedral dice simultaneously on a table. For each die, the number on the face in contact with the table is the score of the die. The player receives $16 if the maximum of the two scores is -1, and receives $3 if the sum of the two scores is prime. For all other cases, the player receives nothing. Find the player's expected gain in the game. [4]

7 Mandy has 10 beads, of which 5 are spherical and 5 are cubical, each of different colours. She wishes to decorate a card by forming a circle using 8 of the 10 beads. Find the number of ways Mandy can arrange the beads if

(i) there are no restrictions, [1]

(ii) 3 particular beads are used and not all are next to one another, [3]

(iii) spherical beads and cubic beads must alternate. [3]

8 A man wishes to buy a 4-digit number lottery. He plays by randomly choosing any number from 0000 to 9999. It is assumed that each number is equally likely to be chosen. Find the probability that a randomly chosen 4-digit number has

(i) four different digits, [1]

(ii) exactly one of the first three digits is the same as the last digit, and the last digit is even, [3]

(iii) four different digits with the first digit greater than 6, given that the 4-digit number has odd and even digits that alternate. [4]

9 In a large shipment of glass stones used for the *Go* board game, a proportion p of the glass stones is chipped. The glass stones are sold in boxes of 361 pieces each. Let X denote the number of chipped glass stones in a box.

(i) Based on this context, state two assumptions in order for X to be well modelled by a binomial distribution. [2]

In the rest of the question, assume that X follows a binomial distribution.

(ii) It is known that the probability of a box containing at most 2 chipped glass stones is 0.90409. Find p. [2]

(iii) A box is deemed to be of inferior quality if it contains more than 2 chipped glass stones. Find the probability that, in a batch of 20 boxes of glass stones, there are more than 5 boxes of inferior quality in the batch. [3]

(iv) Each week, a distributor purchases 50 batches of glass stones, each batch consisting of 20 boxes of glass stones. A batch will be rejected if it contains more than 5 boxes of inferior quality. The distributor will receive a compensation of \$10 for each rejected batch in the first 20 weeks of a year, and a compensation of \$20 for each rejected batch in the remaining weeks of the year. Assuming that there are 52 weeks in a year, find the probability that the total compensation in a year is more than \$250. [5]

10 A large cohort of students sat for a mathematics examination. Based on selected data of the examination results, the following table shows y, the proportions of students who scored x marks.

x	20	30	40	50	60	70	80	90
y	0.00029	0.00174	0.00663	0.0161	0.0252	0.0252	0.0161	0.00663

(i) Draw a scatter diagram for these values, labelling the axes. [2]

(ii) Explain why, in this context, a linear model is not appropriate. [1]

It is decided to fit a model of the form $\ln y = -a(x-m)^2 + b$, where $a > 0$ and m is a suitable constant, to the data. The product moment correlation coefficient between $(x-m)^2$ and $\ln y$ is denoted by r. The table below gives values of r for some possible values of m.

m	62.5	65	67.5
r	□□0.9899292		□□0.9938968

(iii) Calculate the value of r for $m = 65$, giving your answer correct to 7 decimal places. [1]

(iv) Use the table and your answer in part **(iii)** to suggest with a reason which of 62.5, 65 or 67.5 is the most appropriate value for m. [1]

(v) Using the value of m found in part **(iv)**, calculate the values of a and b, and use them to predict the proportion of students who scored 45 marks.
Comment on the reliability of your prediction. [5]

11 Yummy Berries Farm produces blueberries and raspberries packed in boxes.

(a) Yummy Berries Farm claims that the mass, x grams, of each box of blueberries is no less than 125 grams. After receiving ~~a~~ complaints from consumers, the Consumers Association of Singapore (CASE) took a random sample of 50 boxes of blueberries from Yummy Berries Farm and the mass of each box was recorded. The data obtained are summarised in the table.

x (grams)	120	121	122	123	124	125	126	127	128	129	130
No. of boxes	3	6	6	6	3	10	3	4	6	2	1

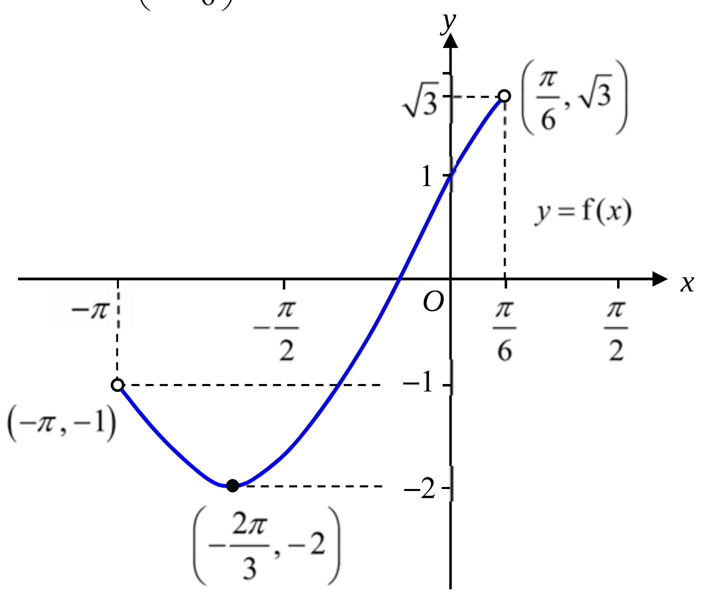
(i) Find unbiased estimates of the population mean and variance. [2]

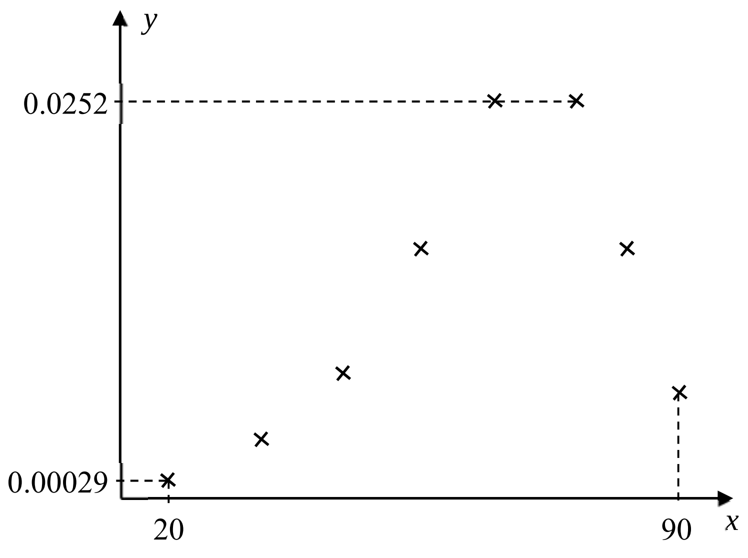
(ii) Test, at the 10% level of significance, whether Yummy Berries Farm has overstated its claim.
State, giving a reason, whether any assumptions about the masses of boxes of blueberries are needed in order for the test to be valid. [6]

(b) The masses of boxes of raspberries, each of y grams, are assumed to have a mean of 170 grams with standard deviation 15 grams. CASE took a random sample of n boxes of raspberries and the mean mass of boxes of raspberries from the sample is found to be 165 grams. A test is to be carried out at the 5% level of significance to determine if the mean mass of the boxes of raspberries is not 170 grams. Find the minimum number of boxes of raspberries to be taken for which the result of the test would be to reject the null hypothesis. [4]

ANNEX B

2017 HCI H2 Maths Preliminary Examination Paper 2

Qn /No	Topic Set	Answers
1	AP and GP	(i) $a=\frac{2}{3}k$, $b=2k$ (ii) $U_n=\frac{2k}{n!}\left(\frac{n}{n+1}\right)$ (iii) $S_n \to 2k$, $\sum_{r=1}^{\infty} u_r$ converges.
2	Complex Numbers	(i) $w=6+2\mathrm{i}$, $z=1-3\mathrm{i}$ (ii) $\angle WOZ$ is $90°$
3	Functions	(i) $\mathrm{f}(x)=2\sin\left(x+\frac{\pi}{6}\right)$ (ii)  $R_{\mathrm{f}}=[-2,\sqrt{3})$ (iii) $R_{\mathrm{fg}}=[-2,1]$ (iv) largest $D_{\mathrm{f}}=[-\frac{2\pi}{3},\frac{\pi}{6})$, $\mathrm{f}^{-1}:x\mapsto \sin^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{6}$, $x\in\mathbb{R}$, $-2\le x<\sqrt{3}$

4	Vectors	(i) $x+y-2z=-7$; (ii) $\underset{\sim}{r}=\begin{pmatrix}-4\\-3\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix},\alpha\in\mathbb{R}$ (iii) $\overrightarrow{OB}=\begin{pmatrix}0\\1\\4\end{pmatrix}$; (iv) $\frac{\sqrt{2}}{2}$; (v) $\overrightarrow{OC}=\begin{pmatrix}-\frac{1}{2}\\1\\\frac{9}{2}\end{pmatrix}$, $x=0$, $y=5-z$.
5	Normal Distribution	(ii) $\mathrm{P}(X<9)=0.748$
6	DRV	(ii) $-\$0.25$
7	P&C	(i) 226800; (ii) 90720; (iii) 3600
8	Probability	(i) $\frac{63}{125}$; (ii) $\frac{243}{2000}$; (iii) $\frac{3}{125}$
9	Binomial Distribution; Sampling	(i) Assumptions • The probability of a randomly chosen glass stone being chipped is constant. • Whether a glass stone is chipped or not is independent of that of any other glass stones. (ii) $p=0.00300$; (iii) 0.00923; (iv) 0.953
10	Correlation & Linear Regression	(i)  (ii) The scatter diagram displays a curvilinear relationship which suggests the presence of a maximum point. Hence a linear model is inappropriate. (iii) $r=-0.9999984$ (7 decimal places)

		(iv) $m = 65$. Of the 3 negative r values, the r value corresponding to $m = 65$ is closest to -1 (v) $a \approx 0.00222$, $b \approx -3.63$, $\ln y = -0.00222(x-65)^2 - 3.63$, 0.0109, Since $x = 45$ is within data range and $r = -0.9999984$ is very close to -1, the prediction is reliable.
11	Hypothesis Testing	(a) (i) $\overline{x} = 124.4$, $s^2 = 7.43$; (ii) p-value $= 0.0598 < 0.1$, we reject $\mathrm{H_0}$ and conclude that at the 10% level of significance, there is sufficient evidence that Yummy Berries Farm has overstated its claim. No assumptions about masses of boxes of blueberries are needed. Since $n = 50$ is sufficiently large, by Central Limit Theorem, the <u>mean</u> mass of boxes of raspberries will follows a normal distribution approximately. (b) least n is 35

1	**(i)** $S_1 = b - \frac{3a}{2!} = b - \frac{3a}{2} = k \quad \ldots(1)$ $S_2 = b - \frac{3a}{3!} = b - \frac{a}{2} = k + \frac{2}{3}k = \frac{5}{3}k \quad \ldots(2)$ $(2)-(1)$, $-\frac{a}{2} - \left(-\frac{3a}{2}\right) = \frac{5}{3}k - k$ $\therefore \quad a = \frac{2}{3}k$ $\therefore \; b = k + \frac{3a}{2} = k + \frac{3}{2}\left(\frac{2}{3}k\right) = 2k$ **(ii)** $S_n = 2k - \frac{2k}{(n+1)!}$ $u_n = S_n - S_{n-1}$ $= \left(2k - \frac{2k}{(n+1)!}\right) - \left(2k - \frac{2k}{n!}\right)$ $= \frac{2k}{n!} - \frac{2k}{(n+1)!}$ $= \frac{2k}{n!}\left(1 - \frac{1}{n+1}\right)$ $= \frac{2k}{n!}\left(\frac{n}{n+1}\right)$ $= \frac{2kn}{(n+1)!}$ **(iii)** $\sum_{r=1}^{n} u_r = S_n = 2k - \frac{2k}{(n+1)!}$ As $n \to \infty$, $\frac{1}{(n+1)!} \to 0$. $\therefore \; S_n = 2k - \frac{2k}{(n+1)!} \to 2k$ Hence the series $\sum_{r=1}^{\infty} u_r$ converges.

2	**(i)**

$$2z + 3w = 20 \quad ...(1)$$
$$w - zw^* = 6 + 22\mathrm{i} \quad ...(2)$$

From (1), $z = \dfrac{20-3w}{2}$

Substitute into (2),

$$w - \left(\frac{20-3w}{2}\right)w^* = 6 + 22\mathrm{i}$$
$$2w - (20-3w)w^* = 12 + 44\mathrm{i}$$
$$2w - 20w^* + 3ww^* = 12 + 44\mathrm{i}$$

Let $w = a + b\mathrm{i}$

$$2(a+b\mathrm{i}) - 20(a-b\mathrm{i}) + 3(a+b\mathrm{i})(a-b\mathrm{i}) = 12 + 44\mathrm{i}$$
$$2a + 2b\mathrm{i} - 20a + 20b\mathrm{i} + 3(\mathrm{a}^2 + \mathrm{b}^2) = 12 + 44\mathrm{i}$$
$$\left(3a^2 - 18a + 3b^2\right) + (22b)\mathrm{i} = 12 + 44\mathrm{i}$$

Comparing real and imaginary parts,

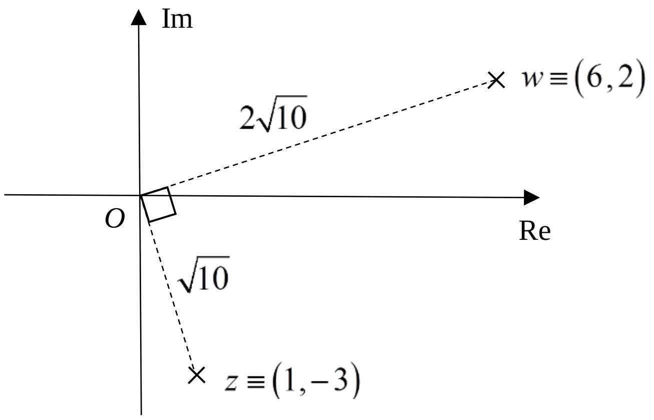
$$22b = 44$$
$$\therefore\ b = 2$$
$$3a^2 - 18a + 3(2)^2 = 12$$
$$3a^2 - 18a + 12 = 12$$
$$3a(a-6) = 0$$

$a = 0$ (rejected since $a \neq 0$), $a = 6$

$\therefore w = 6 + 2\mathrm{i}$

$$z = \frac{20 - 3(6+2\mathrm{i})}{2}$$
$$z = 1 - 3\mathrm{i}$$

(ii)



$\angle WOZ$ is $90°$

3	**(i)**

$\mathrm{f}(x) = \sqrt{3}\sin x + \cos x$

$R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = \sqrt{3} \quad \ldots(1)$

$R\sin\alpha = 1 \quad \ldots(2)$

$(1)^2 + (2)^2$,

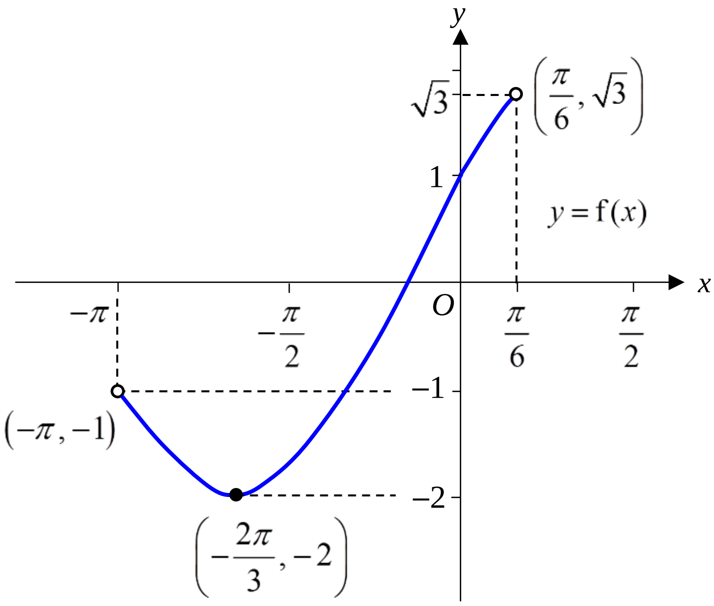
$\therefore\ R = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$(1)/(2),\quad \tan\alpha = \dfrac{1}{\sqrt{3}}$,

$\therefore\ \alpha = \dfrac{\pi}{6}$

Hence $\mathrm{f}(x) = 2\sin\left(x + \dfrac{\pi}{6}\right)$

(ii)



When $y = -2$,

$2\sin(x + \frac{\pi}{6}) = -2$

$\sin(x + \frac{\pi}{6}) = -1$

$x + \dfrac{\pi}{6} = -\dfrac{\pi}{2} \Rightarrow x = -\dfrac{2\pi}{3}$

$\therefore$ turning point is $\left(-\dfrac{2\pi}{3}, -2\right)$.

$R_{\mathrm{f}} = [-2, \sqrt{3})$

(iii)

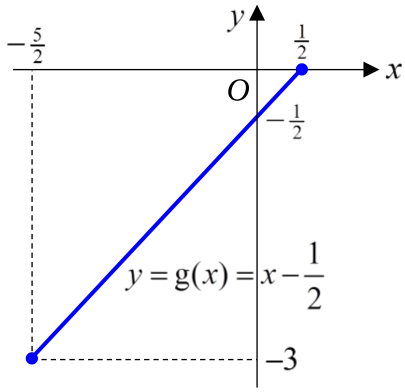
Since $-\frac{5}{2} \le x \le \frac{1}{2}$,

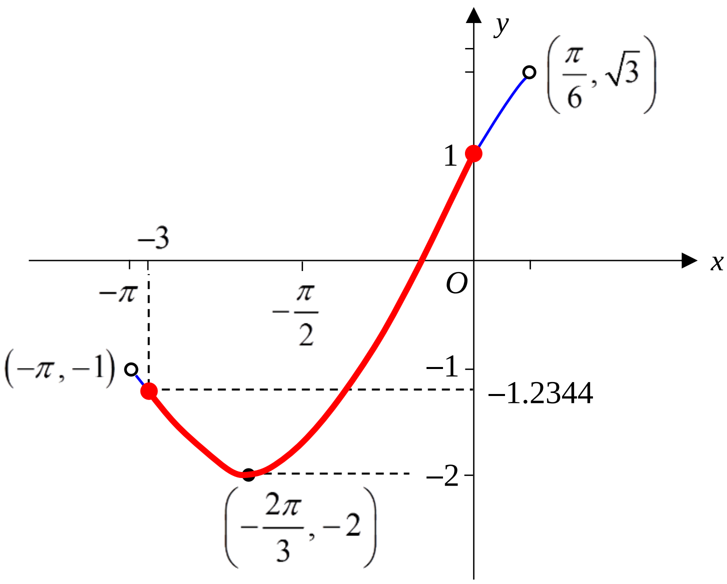
$g(x) = \frac{1}{2} + x - 1 = x - \frac{1}{2}$

$R_g = [-3, 0]$

$D_f = (-\pi, \frac{\pi}{6})$

Since $R_g \subset D_f$, fg exists





$$\overbrace{[-\tfrac{5}{2}, \tfrac{1}{2}]}^{D_g} \rightarrow \overset{R_g = \text{restricted } D_f}{[-3, 0]} \rightarrow \overset{\text{restricted } R_f = R_{fg}}{[-2, 1]}$$

$\therefore\ R_{fg} = [-2, 1]$

(iv)

From the graph,

largest domain for f $= [-\frac{2\pi}{3}, \frac{\pi}{6})$

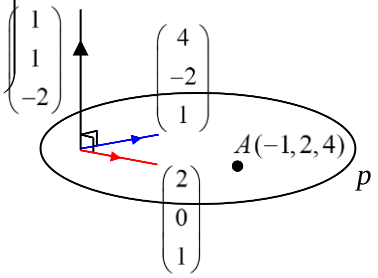
Let $y = 2\sin(x + \frac{\pi}{6})$

$x = \sin^{-1}\left(\frac{y}{2}\right) - \frac{\pi}{6}$

$f^{-1} : x \mapsto \sin^{-1}\left(\frac{x}{2}\right) - \frac{\pi}{6}$, $x \in \mathbb{R}$, $-2 \le x < \sqrt{3}$.

4

(i) $\begin{pmatrix}4\\-2\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\\4\end{pmatrix}=-2\begin{pmatrix}1\\1\\-2\end{pmatrix}$



$\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}-1\\2\\4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-7$

$\therefore$ Cartesian equation of p is $x+y-2z=-7$.

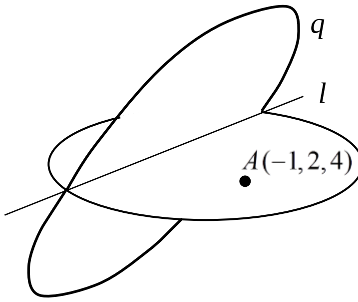
(ii) $x+y-2z=-7$

$x-2y+z=2$

Using GC,

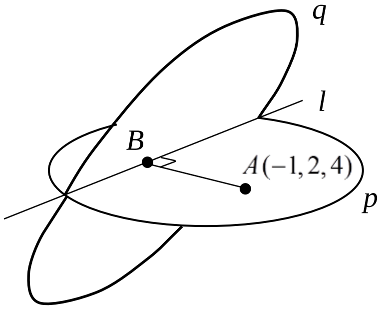
a vector equation of l is

$\underset{\sim}{r}=\begin{pmatrix}-4\\-3\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix},\alpha\in\mathbb{R}.$



(iii)

$\overrightarrow{OB}=\begin{pmatrix}-4+\alpha\\-3+\alpha\\\alpha\end{pmatrix}$



$\overrightarrow{AB}=\begin{pmatrix}-4+\alpha\\-3+\alpha\\\alpha\end{pmatrix}-\begin{pmatrix}-1\\2\\4\end{pmatrix}$

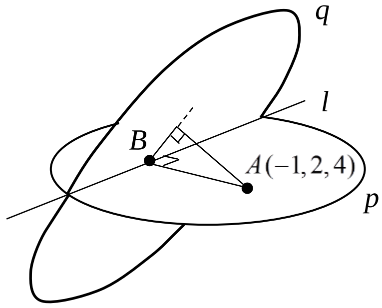
$=\begin{pmatrix}\alpha-3\\\alpha-5\\\alpha-4\end{pmatrix}$

$\begin{pmatrix}\alpha-3\\\alpha-5\\\alpha-4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=0$

$\alpha-3+\alpha-5+\alpha-4=0 \quad\Rightarrow\quad \alpha=4$

$\therefore\ \overrightarrow{OB}=\begin{pmatrix}0\\1\\4\end{pmatrix}=\underset{\sim}{j}+4\underset{\sim}{k}$

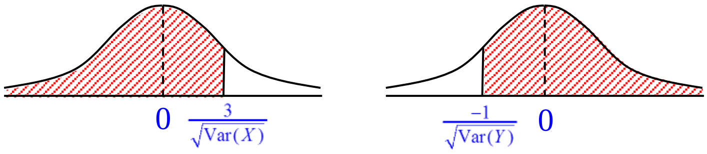
(iv) Equation of q: $\underset{\sim}{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=2$



$\overrightarrow{AB}=\begin{pmatrix}4-3\\4-5\\4-4\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$

$\therefore$ length of projection of AB on q is

$\left|\overrightarrow{AB}\times\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left|\begin{pmatrix}1\\-1\\0\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left|\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left(\sqrt{3}\right)=\frac{\sqrt{2}}{2}$

5	**(i)** $X \sim \mathrm{N}(7, \mathrm{Var}(X))$ $Y \sim \mathrm{N}(7, \mathrm{Var}(Y))$ $\mathrm{P}(X < 10) = \mathrm{P}(Y > 6)$ $\mathrm{P}\left(Z < \dfrac{10-7}{\sqrt{\mathrm{Var}(X)}}\right) = \mathrm{P}\left(Z > \dfrac{6-7}{\sqrt{\mathrm{Var}(Y)}}\right)$ $\mathrm{P}\left(Z < \dfrac{3}{\sqrt{\mathrm{Var}(X)}}\right) = \mathrm{P}\left(Z > \dfrac{-1}{\sqrt{\mathrm{Var}(Y)}}\right)$  $\therefore \dfrac{3}{\sqrt{\mathrm{Var}(X)}} = -\left(\dfrac{-1}{\sqrt{\mathrm{Var}(Y)}}\right)$ $3\sqrt{\mathrm{Var}(Y)} = \sqrt{\mathrm{Var}(X)}$ Hence $\mathrm{Var}(X) = 9\mathrm{Var}(Y)$ (shown) **(ii)** $\mathrm{Var}(X) = 9(1) = 9$ $X \sim \mathrm{N}(7, 9)$ $\therefore \mathrm{P}(X < 9) = 0.748$

6	**(i)** $\mathrm{P}(Y=2)$ $=2\mathrm{P}(X_1=2 \text{ and } X_2=0)+2\mathrm{P}(X_1=3 \text{ and } X_2=-1)$ $=2\mathrm{P}(X_1=2)\mathrm{P}(X_2=0)+2\mathrm{P}(X_1=3)\mathrm{P}(X_2=-1)$ $=2\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)+2\left(\frac{1}{8}\right)\left(\frac{1}{4}\right)$ $=\frac{3}{16}$ **(ii)** $\mathrm{P}(\text{max of 2 scores}=-1)$ $=\mathrm{P}(X_1=-1)\mathrm{P}(X_2=-1)$ $=\left(\frac{1}{8}\right)^2$ $=\frac{1}{64}$ When sum of scores is prime, then Y = 2, 3 or 5 . From (i), $\mathrm{P}(Y=2)=\frac{3}{16}$ $\mathrm{P}(Y=3)=2\mathrm{P}(X_1=0)\mathrm{P}(X_2=3)$ $=2\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)$ $=\frac{1}{4}$ $\mathrm{P}(Y=5)=2\mathrm{P}(X_1=3)\mathrm{P}(X_2=2)$ $=2\left(\frac{1}{8}\right)\left(\frac{1}{4}\right)$ $=\frac{1}{16}$ $\therefore$ Expected gain $=16\left(\frac{1}{64}\right)+3\left(\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)-2$ $=-0.25$ Hence expected gain is $-\$0.25$. [Or expected loss is $\$0.25$.] Alternatively, $\therefore$ Expected gain $=(16-2)\left(\frac{1}{64}\right)+(3-2)\left(\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)-2\left[1-\left(\frac{1}{64}+\frac{3}{16}+\frac{1}{4}+\frac{1}{16}\right)\right]$ $=-0.25$ Hence expected gain is $-\$0.25$. [Or expected loss is $\$0.25$.]

7

(i)

No. of ways $= {}^{10}C_8\,(8-1)! = 226800$

(ii)

Method 1: (method of complementation)

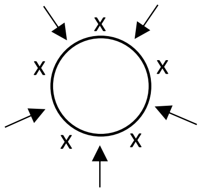
$$\text{No. of ways} = \underbrace{{}^{7}C_5\,(8-1)!}_{\substack{\text{No. of ways}\\\text{without restriction}\\\text{[5 other beads, and}\\\text{together with 3}\\\text{particular beads}\\\text{arranged in a circle]}}} - \underbrace{{}^{7}C_5\,3!(6-1)!}_{\substack{\text{No. of ways}\\\text{with 3 particular beads}\\\text{all together}\\\text{[5 other beads, with 3}\\\text{particular beads grouped}\\\text{as 1 unit and 3! ways to}\\\text{arrange among themselves,}\\\text{and all 6 units arranged}\\\text{in a circle]}}} = 90720$$

Method 2: (method of slotting)

Case 1: (all not next to one another)

No. of ways

$$= \underbrace{{}^{7}C_5\,(5-1)!}_{\substack{\text{5 other beads used}\\\text{as seperators, and}\\\text{arranged in a circle}}} \times \underbrace{{}^{5}C_3\,3!}_{\substack{\text{3 out of 5 slots}\\\text{for 3 particular}\\\text{beads, and 3! ways}\\\text{to arrange among}\\\text{themselves}}} = 30240$$



Case 2: (2 together, 1 not)

$$\text{No. of ways} = \underbrace{{}^{7}C_5\,(5-1)!}_{\substack{\text{5 other beads used}\\\text{as seperators, and}\\\text{arranged in a circle}}} \times \underbrace{{}^{3}C_2\,2!}_{\substack{\text{2 of 3 particular}\\\text{beads together,}\\\text{and 2! ways to}\\\text{arrange among}\\\text{themselves}}} \times \underbrace{{}^{5}C_2\,2!}_{\substack{\text{2 out of 5 slots}\\\text{for 3 particular}\\\text{beads grouped}\\\text{as 2 units (2}\\\text{together, 1 not),}\\\text{and 2! ways to}\\\text{arrange among}\\\text{themselves}}} = 60480$$

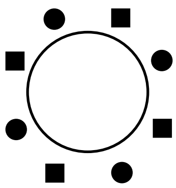
$\therefore$ total no. of ways $= 30240 + 60480 = 90720$

(iii)

If spherical beads and cubic beads alternate, then there must be 4 spherical beads and 4 cubic beads.

No. of ways

$$= \underbrace{{}^{5}C_4\,(4-1)!}_{\substack{\text{4 spherical beads and}\\\text{arranged in a circle}}} \times \underbrace{{}^{5}C_4\,4!}_{\substack{\text{4 cubic beads,}\\\text{and 4! ways to}\\\text{arrange among}\\\text{themselves}}} = 3600$$



8 **(i)**

<u>Method 1</u>: (using permutations)

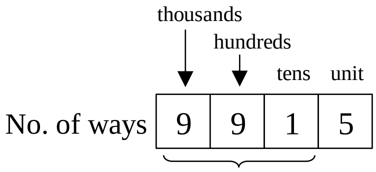
Probability $=\dfrac{10\times 9\times 8\times 7}{10^4}=\dfrac{63}{125}$ [or 0.504]

<u>Method 2</u>: (using probability)

Probability $=\dfrac{10}{10}\times\dfrac{9}{10}\times\dfrac{8}{10}\times\dfrac{7}{10}=\dfrac{63}{125}$ [or 0.504]

(ii)

<u>Method 1</u>: (using permutations)



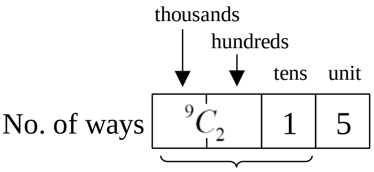
3 ways to arrange digit same as last even digit

Required probability $=\dfrac{[(9\times 9\times 1)\times 3]\times 5}{10^4}$

$=\dfrac{243}{2000}$ [or 0.1215]

<u>Method 2</u>: (using permutations and combinations)

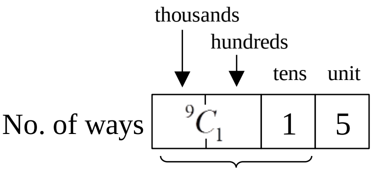
Case 1: The other 2 digits are different



3! ways to arrange

Probability $=\dfrac{[({}^9C_2\times 1)\times 3!]\times 5}{10^4}=\dfrac{27}{250}$ [or 0.108]

Case 2: The other 2 digits are the same



$\frac{3!}{2!}$ ways to arrange

Probability $=\dfrac{[({}^9C_1\times 1)\times \frac{3!}{2!}]\times 5}{10^4}=\dfrac{27}{2000}$ [or 0.0135]

Required probability $=\dfrac{27}{250}+\dfrac{27}{2000}=\dfrac{243}{2000}$ [or 0.1215]

<u>Method 3</u>: (using probability)

Case 1: The other 2 digits are different

$$\text{Probability } = \frac{9}{10}\times\frac{8}{10}\times\frac{1}{10}\times\frac{5}{10}\times\frac{3!}{2!} = \frac{27}{250} \quad \text{[or 0.108]}$$

Case 2: The other 2 digits are the same

$$\text{Probability} = \frac{9}{10}\times\frac{1}{10}\times\frac{1}{10}\times\frac{5}{10}\times\frac{3!}{2!} = \frac{27}{2000} \quad \text{[or 0.0135]}$$

$$\text{Required probability } = \frac{27}{250}+\frac{27}{2000} = \frac{243}{2000} \quad \text{[or 0.1215]}$$

(iii)

Let A be the event '4 different digits with 1st digit greater than 6'.
Let B be the event 'odd and even digits that alternate'.

Method 1: (using permutations)

Case 1: 1st digit is even, i.e. 8, and odd and even digits alternate

	thousands	hundreds	tens	unit
No. of ways	1	5	4	4

$$\text{Probability } = \frac{1\times5\times4\times4}{10^4} = \frac{1}{125} \quad \text{[or 0.008]}$$

Case 2: 1st digit is odd, i.e. 7 or 9, and odd and even digits alternate

	thousands	hundreds	tens	unit
No. of ways	2	5	4	4

$$\text{Probability } = \frac{2\times5\times4\times4}{10^4} = \frac{2}{125} \quad \text{[or 0.016]}$$

$$\text{Hence } \mathrm{P}\left(A\cap B\right) = \frac{1}{125}+\frac{2}{125} = \frac{3}{125} \quad \text{[or 0.024]}$$

$$\begin{aligned}\mathrm{P}\left(B\right) &= \mathrm{P}\left(\text{'odd,even,odd,even' or 'even,odd,even,odd'}\right)\\ &= \frac{2\times(5\times5\times5\times5)}{10^4}\\ &= \frac{1}{8} \quad \text{[or 0.125]}\end{aligned}$$

$$\therefore\ \mathrm{P}\left(A\,|\,B\right) = \frac{\mathrm{P}\left(A\cap B\right)}{\mathrm{P}\left(B\right)} = \frac{\frac{3}{125}}{\frac{1}{8}} = \frac{24}{125} \quad \text{[or 0.192]}$$

Method 2: (using probability)

Case 1: 1st digit is even, i.e. 8, and odd and even digits alternate

$$\text{Probability } = \frac{1}{10}\times\frac{5}{10}\times\frac{4}{10}\times\frac{4}{10} = 0.008$$

Case 2: 1st digit is odd, i.e. 7 or 9, and odd and even digits alternate

$$\text{Probability } = \frac{2}{10}\times\frac{5}{10}\times\frac{4}{10}\times\frac{4}{10} = 0.016$$

Hence $\mathrm{P}(A\cap B) = \frac{1}{125}+\frac{2}{125} = \frac{3}{125}$ [or 0.024]

$$\mathrm{P}(B) = \mathrm{P}(\text{'odd,even,odd,even' or 'even,odd,even,odd'})$$

$$= 2\times\left(\frac{5}{10}\times\frac{5}{10}\times\frac{5}{10}\times\frac{5}{10}\right)$$

$$= 0.125$$

$\therefore\ \mathrm{P}(A\,|\,B) = \frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)} = \frac{\frac{3}{125}}{\frac{1}{8}} = \frac{24}{125}$ [or 0.192]

9	

9

(i)

Assumptions

- The probability of a randomly chosen glass stone being chipped is constant.
- Whether a glass stone is chipped or not is independent of that of any other glass stones.

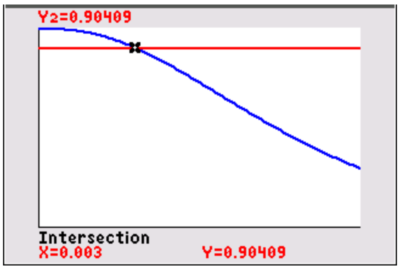
(ii)

$X \sim \mathrm{B}(361, p)$

$\mathrm{P}(X \le 2) = 0.90409$

Using GC,

$p = 0.00300$



(iii)

$P(X > 2) = 1 - P(X \le 2) = 1 - 0.90409 = 0.09591$

Let Y be number of boxes with more than 2 chipped glass stones, out of 20 boxes.

$$Y \sim \mathrm{B}(20, 0.09591)$$

$$\begin{aligned}\mathrm{P}(Y > 5) &= 1 - \mathrm{P}(Y \le 5)\\ &= 1 - 0.9907736392\\ &= 0.0092263608\\ &\approx 0.00923\end{aligned}$$

(iv)

Let A be the number of rejected batches, out of 50 batches.

$$A \sim \mathrm{B}(50, 0.0092263608)$$

$\mathrm{E}(A) = 50(0.0092264) = 0.46132$

$\mathrm{Var}(A) = 50(0.0092264)(1 - 0.0092264) = 0.45706$

Let $M_1 = A_1 + \ldots + A_{20}$

Since $n = 20$ is sufficiently large, by CLT,

$$\begin{aligned}M_1 &\sim \mathrm{N}(20 \times 0.46132,\ 20 \times 0.45706)\\ &= \mathrm{N}(9.2264,\ 9.1412) \quad \text{approximately}\end{aligned}$$

Let $M_2 = A_{21} + \ldots + A_{52}$

Since $n = 32$ is sufficiently large, by CLT,

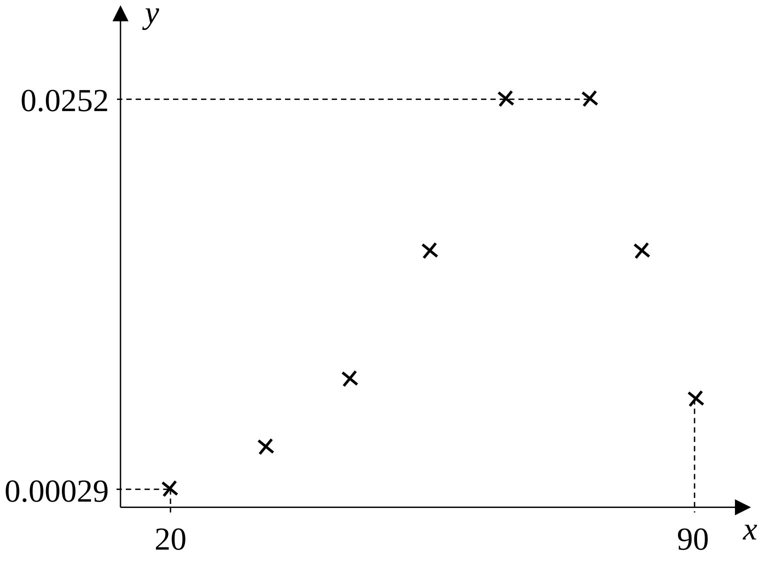
$$\begin{aligned}M_2 &\sim \mathrm{N}(32 \times 0.46132,\ 32 \times 0.45706)\\ &= \mathrm{N}(14.76224,\ 14.62592) \quad \text{approximately}\end{aligned}$$

Let $T = 10M_1 + 20M_2$

Hence $$\begin{aligned}T &\sim \mathrm{N}\left(10(9.2264) + 20(14.76224),\ 10^2(9.1412) + 20^2(14.62592)\right)\\ &= \mathrm{N}(387.5088, 6764.488) \quad \text{approximately}\end{aligned}$$

	$\therefore\ \mathrm{P}\left(T > 250\right) = 0.952729 \approx 0.953$

10	**(i)**



(ii)
The scatter diagram displays a curvilinear relationship which suggests the presence of a maximum point. Hence a linear model is inappropriate.

(iii)
$r = -0.9999984$ (7 decimal places)

(iv)
$m = 65$. Of the 3 negative r values, the r value corresponding to $m = 65$ is closest to -1 .

(v)
Using GC with $m = 65$,
$a \approx 0.0022230 \approx 0.00222$ (3 s.f.)
$b \approx -3.6269 \approx -3.63$ (3 s.f.)
$\therefore\ \ln y = -0.002223(x-65)^2 - 3.6269$
or $\ln y = -0.00222(x-65)^2 - 3.63$
When $x = 45$,
$y \approx 0.0109$ (3 s.f.)

Since $x = 45$ is within data range and $r = -0.9999984$ is very close to -1 , the prediction is reliable.

11

(a)(i)

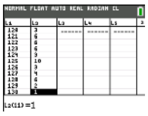
Using GC,

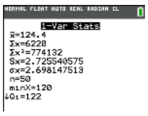
Unbiased estimate of the population mean,

$\bar{x} = 124.4$ g

Unbiased estimate of the population variance,

$$s^2 = 2.725540575^2$$
$$= 7.428571429$$
$$= 7.43 \quad \text{(3 s.f.)}$$





(a)(ii)

Let μ g be the population mean mass of a box of blueberries.

$H_0 : \mu = 125$

$H_1 : \mu < 125$

Under H_0, test statistic

$$Z = \frac{\bar{X} - 125}{\sqrt{\frac{7.428571429}{50}}} \sim \mathrm{N}(0,1) \text{ approximately by CLT}$$

Level of significance: 10%

Critical region: Reject H_0 if p-value ≤ 0.1

Since p-value $= 0.0598 < 0.1$, we reject H_0 and conclude that at the 10% level of significance, there is sufficient evidence that Yummy Berries Farm has overstated its claim.
No assumptions about masses of boxes of blueberries are needed. Since $n = 50$ is sufficiently large, by Central Limit Theorem, the mean mass of boxes of raspberries will follows a normal distribution approximately.

(b)
Let μ_1 g be the population mean mass of a box of raspberries.

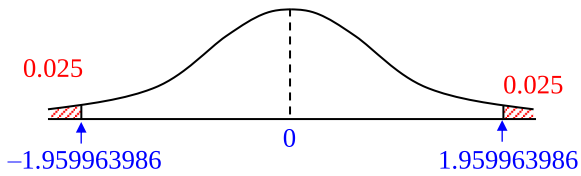
$H_0 : \mu_1 = 170$

$H_1 : \mu_1 \neq 170$

Under H_0, assuming n is large,

test statistic $Z = \dfrac{\overline{Y} - 170}{\frac{15}{\sqrt{n}}} \sim N(0,1)$ approximately by CLT

Level of significance: 5%
Critical region: Reject H_0 if p-value $\leq$ 0.05
i.e. Reject H_0 if z-value ≤ -1.959963986 or z-value ≥ 1.959963986



$$\frac{165-170}{\frac{15}{\sqrt{n}}} \leq -1.959963986 \quad \text{or} \quad \frac{165-170}{\frac{15}{\sqrt{n}}} \geq 1.959963986$$

$$\sqrt{n} \geq 5.87989 \quad \text{or} \quad \sqrt{n} \leq -5.87989 \text{ (rejected)}$$

$\therefore\ n \geq 34.573$
Hence least n is 35.