Preliminary Examination
Higher 2

MATHEMATICS **9740/01**
Paper 1
Friday **8am – 11am**

16 September 2016 **3 hours**

READ THESE INSTRUCTIONS FIRST

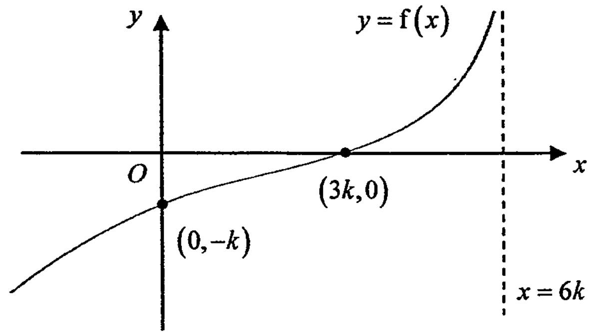
Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 6 printed pages

1



The diagram shows the curve with equation $y = \mathrm{f}(x)$, $x < 6k$, $k > \frac{1}{3}$. The curve crosses the x-axis and y-axis at the points $(3k, 0)$ and $(0, -k)$ respectively. Sketch $y = \mathrm{f}(|x|+1)$. [3]

2 Indicate on a single Argand diagram, the set of points whose complex numbers satisfy the following inequalities

$$\left|\frac{z-6-5\mathrm{i}}{2}\right| \le 4 \quad \text{and} \quad |2\mathrm{i}-4-z| \ge |z+4-10\mathrm{i}|.$$

Hence, find the least value and greatest value of $\arg(z-6+4\mathrm{i})$. [7]

3 **(a)** Without using a calculator, solve the inequality

$$\frac{4-7x}{x-3} \ge x.$$ [4]

(b) In 2016, Edwin, his father and his grandfather have an average age of 53. In the same year, the sum of one-half of his grandfather's age, one-third of his father's age and one-fourth of Edwin's age is 65. Twenty-two years ago, his grandfather's age was twice the sum of his father's age and his age. What are their respective ages in 2016? [You can assume that Edwin's age in 2016 is more than 22.] [3]

4 The function f is defined by $\mathrm{f} : x \mapsto (x-2)^2 + k,\ x \in \mathbb{R},\ x \le 2$. It is given that f^{-1} exists.

(i) When $k = 1$,

(a) define f^{-1} in a similar form, [3]

(b) sketch, on a single diagram, the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$. [3]

(ii) State the set of values of k, such that the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ has no real solutions. [1]

5 A sequence u_1, u_2, u_3,... is such that $u_1 = 0$ and

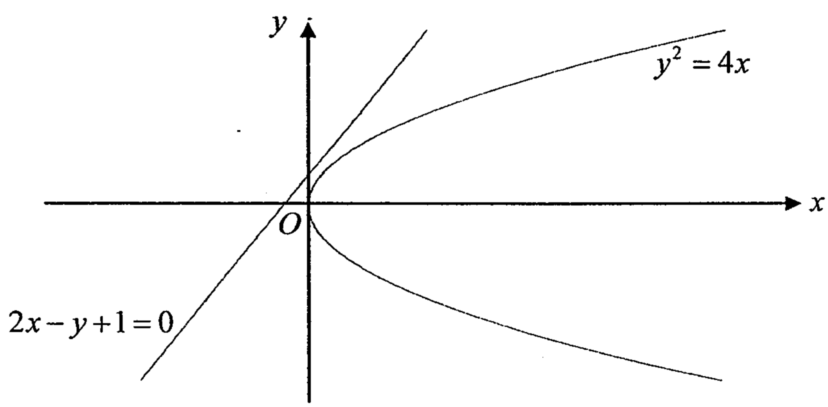
$$u_{n+1} = u_n + \frac{2-n^2}{(n+2)!}, \text{ for all } n \geq 1.$$

(i) Show that $u_2 = \frac{1}{6}$, and find the values of u_3 and u_4. [2]

(ii) Hence, give a conjecture for u_n in the form $\frac{n-1}{[f(n)]!}$, where $f(n)$ is a function of n to be determined. [1]

(iii) Use the method of mathematical induction to prove your conjecture in part **(ii)** for all positive integers n. [4]

6



A curve C has equation $y^2 = 4x$ and a line l has equation $2x - y + 1 = 0$. The diagram above shows the graphs of C and l.

$B(b, 2\sqrt{b})$ is a fixed point on C and A is an arbitrary point on l. State the geometrical relationship between the line segment AB and l if the distance from B to A is the least. [1]

Taking the coordinates of A as $(a, 2a+1)$, find an equation relating a and b for which AB is the least. [2]

Deduce that when AB is the least, $(AB)^2 = m\left(2b - 2\sqrt{b} + 1\right)^2$ where m is a constant to be found. Hence or otherwise, find the coordinates of the point on C that is nearest to l, as b varies. [5]

7 **(a)** Differentiate xe^{x^3} with respect to x. Hence, find $\int x^2\left(1+3x^3\right)e^{x^3}\,dx$. [4]

(b) The variables x and y are related by the differential equation

$$\frac{dy}{dx}=\frac{\sec^2 x}{2\sec^2 x+4\tan x+7}.$$

Using the substitution $u=\tan x$, find the general solution of the differential equation. [5]

8 **(i)** Use the method of differences to find, in terms of n,

$$\sum_{r=2}^{n}\ln\left[\frac{r(r+2)}{(r+1)^2}\right].$$ [4]

(ii) Give a reason why the series is convergent and state the sum to infinity. [2]

(iii) Given $\sum_{r=2}^{13}\ln\left[\frac{(2r)(2r+4)}{(r+1)^2}\right]=\ln\left(\frac{p}{q}\right)$, where p and q are integers and $\frac{p}{q}$ is in the simplest form, find the values of p and q. [3]

9 **(i)** Sketch the graph with equation $x^2+(y-r)^2=r^2$, where $r>0$ and $y\leq r$. [2]

A hemispherical bowl of fixed radius r cm is filled with water. Water drains out from a hole at the bottom of the bowl at a constant rate.

Use your graph in part **(i)** to show that when the depth of water is h cm (where h£ r), the volume of water in the bowl is given by

$$V=\frac{\pi h^2}{3}(3r-h),$$ [3]

(ii) Given that a full bowl of water would become empty in 24 s, find the rate of decrease, in terms of r and h, of the depth of water in the bowl at the instant when the depth of water is h cm. [3]

(iii) Without any differentiation, determine, in terms of r, the slowest rate at which the depth of water is decreasing. [1]

10 The equations of planes p_1 and p_2 are

$$x-5y+2z=13,$$
$$-2x+y+5z=1,$$

respectively.

(i) Find the acute angle between p_1 and p_2. [2]

The planes p_1 and p_2 intersect in a line l.

(ii) Find a vector equation of l. [2]

The plane p_3 is perpendicular to both p_1 and p_2. The three planes p_1, p_2 and p_3 intersect at the point $(a,0,b)$, where a and b are constants.

(iii) Show that $a=7$ and $b=3$. [2]

The plane Π is parallel to p_3 and the distance between Π and p_3 is $4\sqrt{11}$ units.

(iv) Find the two possible cartesian equations of Π. [4]

11 (a) An arithmetic progression which consists of $2n$ terms has first term a and common difference d. The third, fifth and twelfth terms of the arithmetic progression are also three distinct consecutive terms of a geometric progression. Find the sum of the even-numbered terms, i.e. the 2^{nd}, 4^{th}, … , $(2n)^{th}$ terms, of the arithmetic progression in terms of a and n. [5]

(b) To renovate his new HDB flat, Douglas is considering taking up a bank loan of \$40,000 from Citybank on 1^{st} July 2016. The bank charges a monthly interest of 0.5% on the outstanding amount owed at the end of each month.

Douglas will pay a fixed amount, \$$x$, to the bank at the beginning of each month, starting from September 2016.

(i) Taking July 2016 as the 1^{st} month, show that the amount of money owed at the beginning of the 5^{th} month is

$$1.005^4(40000)-200x\left(1.005^3-1\right).$$ [3]

(ii) If Douglas wishes to pay up his loan within 5 years, find the minimum amount of each monthly repayment. [2]

(iii) Using the value found in part **(ii)**, calculate the interest (to the nearest dollar) that Citybank has earned in total from Douglas's loan at the end of his last repayment. [2]

12 The curve C has equation $y=\frac{f(x)}{x+a}$, where $f(x)$ is a quadratic expression, a is a constant and $a\neq\pm3$. It is given that the coordinates of the points of intersection of C with the x-axis are $(3,0)$ and $(-3,0)$, and the equation of the oblique asymptote is $y=\frac{1}{2}x+1$.

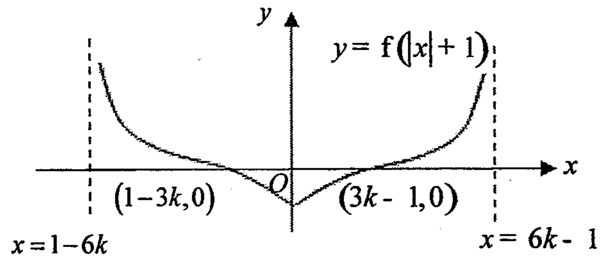
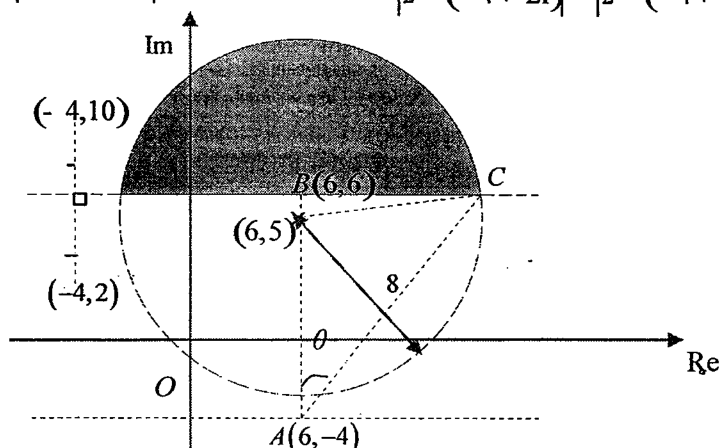
(i) Find $f(x)$, and show that $a=-2$. [5]

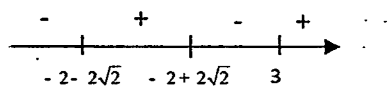
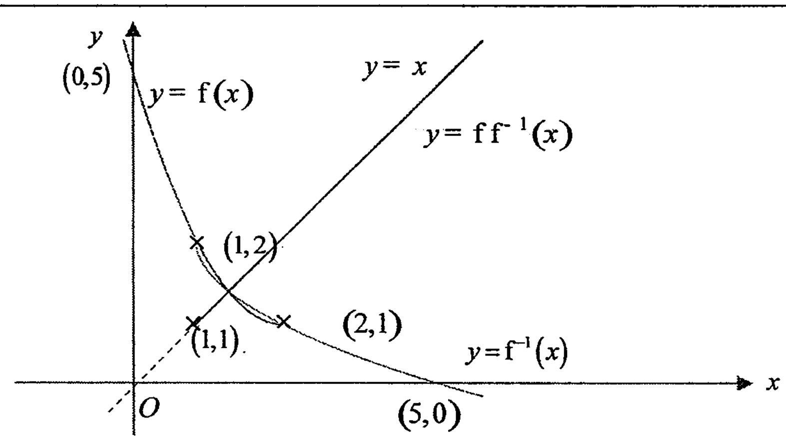
(ii) Sketch C, indicating clearly the equations of the asymptotes, and the coordinates of the points of intersection of C with the x- and y-axes. [2]

A tangent to C is parallel to the line $y=x+2$. Find the possible equations of this tangent, leaving your answer in an exact form. [5]

[End of Paper]

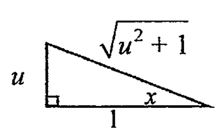
2016 JC2 Prelim Paper 1 Solutions/Comments

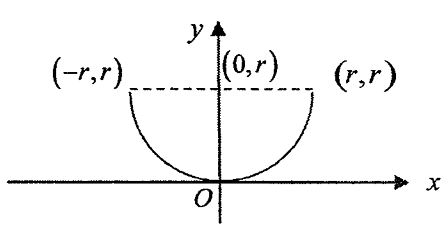
Qn	Solution
1	
2	$\left\lvert\dfrac{z-6-5i}{2}\right\rvert \le 4$ $\lvert z-(6+5i)\rvert \le 8$ $\lvert 2i-4-z\rvert \ge \lvert z+4-10i\rvert$ $\lvert z+4-2i\rvert \ge \lvert z+4-10i\rvert$ $\lvert z-(-4+2i)\rvert \ge \lvert z-(-4+10i)\rvert$  By Pythagoras theorem, $BC^2+1^2=8^2 \Rightarrow BC=\sqrt{63}$ $\theta=\tan^{-1}\dfrac{\sqrt{63}}{10}$ Min $\arg(z-6+4i)=\dfrac{\pi}{2}-\tan^{-1}\dfrac{\sqrt{63}}{10}=0.900$ Max $\arg(z-6+4i)=\dfrac{\pi}{2}+\tan^{-1}\dfrac{\sqrt{63}}{10}=2.24$
	Alternative : Equation of circle is $(x-6)^2+(y-5)^2=64$ - - (1) Equation of perpendicular bisector is $y=6$ - - - (2) Substituting (2) into (1) $(x-6)^2+(6-5)^2=64 \Rightarrow x=6\pm\sqrt{63}$ Min $\arg(z-6+4i)=\arg\left(6+\sqrt{63}+6i-6+4i\right)$ $=\arg\left(\sqrt{63}+10i\right)=\tan^{-1}\dfrac{10}{\sqrt{63}}=0.900$ Max $\arg(z-6+4i)=\arg\left(6-\sqrt{63}+6i-6+4i\right)$ $=\arg\left(-\sqrt{63}+10i\right)=\pi-\tan^{-1}\dfrac{10}{\sqrt{63}}=2.24$

Qn	Solution
3a	$\dfrac{4-7x}{x-3} \geq x$ $\dfrac{4-7x-x(x-3)}{x-3} \geq 0$ $\dfrac{x^2+4x-4}{x-3} \leq 0$ $\dfrac{(x+2)^2-8}{x-3} \leq 0$ $\dfrac{\left(x+2+2\sqrt{2}\right)\left(x+2-2\sqrt{2}\right)}{x-3} \leq 0$  $\therefore x \leq -2-2\sqrt{2}$ or $-2+2\sqrt{2} \leq x < 3$
b	Let e, f and g be the ages of Edwin, his father and his grandfather respectively. $e+f+g=53\times 3=159$ - - - - (1) $\frac{1}{4}e+\frac{1}{3}f+\frac{1}{2}g=65$ - - - - (2) $g-22=2(f-22+e-22)$ $2e+2f-g=66$ - - - - (3) From GC, $e=24, f=51, g=84$. The ages of Edwin, his father and his grandfather are 24, 51 and 84 respectively.
4ia	When $k=1$, $\mathrm{f}: x \mapsto (x-2)^2+1,\ x \in \mathbb{R}, x \leq 2$ Let $y=(x-2)^2+1$ $(x-2)^2=y-1$ $x-2=\pm\sqrt{y-1}$ $x=2\pm\sqrt{y-1}$ Since $x \leq 2$, $x=2-\sqrt{y-1}$, $\mathrm{f}^{-1}: x \mapsto 2-\sqrt{x-1},\ x \geq 1$
b	
ii	$\{k \in \mathbb{R} : k > 2\}$

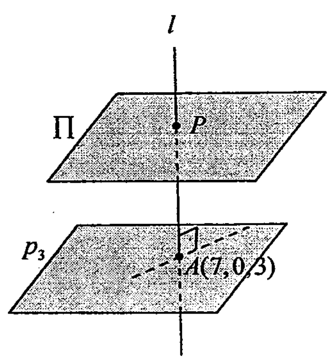
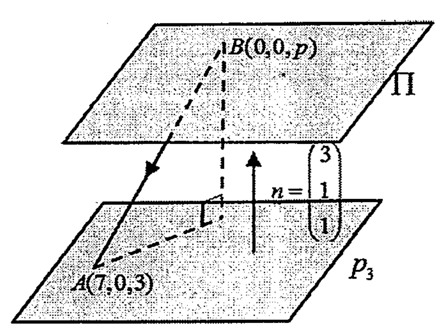
Qn	Solution
5i	$u_2 = u_1 + \dfrac{2-1^2}{(1+2)!} = 0 + \dfrac{1}{6} = \dfrac{1}{6}$ $u_3 = u_2 + \dfrac{2-2^2}{(2+2)!} = \dfrac{1}{6} - \dfrac{2}{24} = \dfrac{1}{12}$ $u_4 = u_3 + \dfrac{2-3^2}{(3+2)!} = \dfrac{1}{12} - \dfrac{7}{120} = \dfrac{1}{40}$
ii	$u_2 = \dfrac{1}{6} = \dfrac{2-1}{(2+1)!}$, $u_3 = \dfrac{1}{12} = \dfrac{2}{24} = \dfrac{3-1}{(3+1)!}$, $u_4 = \dfrac{1}{40} = \dfrac{3}{120} = \dfrac{4-1}{(4+1)!}$ By observation, a conjecture is that $u_n = \dfrac{n-1}{(n+1)!}$
iii	Let P_n be the statement $u_n = \dfrac{n-1}{(n+1)!}$, for all $n \in \mathbb{Z}^+$. Check P_1: LHS = $u_1 = 0$ RHS = $\dfrac{1-1}{(1+1)!} = 0$ $\therefore P_1$ is true Assume that P_k is true for some positive integer k i.e. $u_k = \dfrac{k-1}{(k+1)!}$ We want to show that P_{k+1} is true. i.e. $u_{k+1} = \dfrac{k}{(k+2)!}$ LHS $= u_{k+1}$ $= u_k + \dfrac{2-k^2}{(k+2)!}$ $= \dfrac{k-1}{(k+1)!} + \dfrac{2-k^2}{(k+2)!}$ $= \dfrac{(k-1)(k+2)+2-k^2}{(k+2)!}$ $= \dfrac{k^2+k-2+2-k^2}{(k+2)!}$ $= \dfrac{k}{(k+2)!} =$ RHS Since P_1 is true, and P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.
6	If the distance AB is the least, the line segment AB is perpendicular to l.
	$B(b, 2\sqrt{b})$ and $A(a, 2a+1)$ Gradient of $BA = \dfrac{2\sqrt{b}-2a-1}{b-a}$

Qn	Solution
	Since gradient of l is 2, $\frac{2\sqrt{b}-2a-1}{b-a}=-\frac{1}{2}$ $\Rightarrow 4\sqrt{b}-4a-2=a-b$ $\Rightarrow a=\frac{1}{5}\left(b+4\sqrt{b}-2\right)$
	$(AB)^2=\left(2\sqrt{b}-2a-1\right)^2+(b-a)^2$ $=\left(2\sqrt{b}-2a-1\right)^2+\left(4\sqrt{b}-4a-2\right)^2$ $=5\left(2\sqrt{b}-2a-1\right)^2$ $=5\left(2\sqrt{b}-\frac{2}{5}\left(b+4\sqrt{b}-2\right)-1\right)^2$ $=5\left(\frac{2}{5}\sqrt{b}-\frac{2}{5}b-\frac{1}{5}\right)^2$ $=\frac{1}{5}\left(2\sqrt{b}-2b-1\right)^2$ $=\frac{1}{5}\left(2b-2\sqrt{b}+1\right)^2$ $2AB\frac{\mathrm{d}AB}{\mathrm{d}b}=\frac{2}{5}\left(2b-2\sqrt{b}+1\right)\left(2-\frac{1}{\sqrt{b}}\right)$ When $\frac{\mathrm{d}AB}{\mathrm{d}b}=0$, $\frac{2}{5}\left(2b-2\sqrt{b}+1\right)\left(2-\frac{1}{\sqrt{b}}\right)=0$ Consider $\left(2b-2\sqrt{b}+1\right)=0$ Since $(-2)^2-4(2)(1)<0$, $\left(2b-2\sqrt{b}+1\right)=0$ has no real solution. $2-\frac{1}{\sqrt{b}}=0 \Rightarrow b=\frac{1}{4}$ the point on C nearest to l is $\left(b,2\sqrt{b}\right)=\left(\frac{1}{4},1\right)$.
	Alternative : $(AB)^2=\frac{1}{5}\left(2b-2\sqrt{b}+1\right)^2$ $=\frac{4}{5}\left(b-\sqrt{b}+\frac{1}{2}\right)^2$ $=\frac{4}{5}\left(\left(\sqrt{b}-\frac{1}{2}\right)^2-\frac{1}{4}+\frac{1}{2}\right)^2$ $=\frac{4}{5}\left(\left(\sqrt{b}-\frac{1}{2}\right)^2+\frac{1}{4}\right)^2$

Qn	Solution
	Since $\left(\sqrt{b}-\frac{1}{2}\right)^2 \geq 0$ for all real b, $(AB)^2$ is the least when $\sqrt{b}=\frac{1}{2}$, that is, $b=\frac{1}{4}$. Hence the point on C nearest to l is $\left(b,2\sqrt{b}\right)=\left(\frac{1}{4},1\right)$
	Alternative : When $(AB)^2$ is the least, tangent to C at B is parallel to l. i.e. gradient of tangent to $C=2$ $y^2=4x$ $2y\frac{dy}{dx}=4$ $\frac{dy}{dx}=\frac{2}{y}$ At $\left(b,2\sqrt{b}\right)$, $\frac{dy}{dx}=\frac{2}{2\sqrt{b}}=\frac{1}{\sqrt{b}}=2$ $b=\frac{1}{4}$ $\therefore$ coordinates on C nearest to l is $\left(\frac{1}{4},1\right)$.
7a	$\frac{d}{dx}xe^{x^3}=e^{x^3}+x3x^2e^{x^3}$ $=e^{x^3}(1+3x^3)$ $\int x^2\left(1+3x^3\right)e^{x^3}dx=xe^{x^3}x^2-\int xe^{x^3}2xdx$ $=xe^{x^3}x^2-\frac{2}{3}\int 3x^2e^{x^3}dx$ $=x^3e^{x^3}-\frac{2}{3}e^{x^3}+C$
b	$\frac{dy}{dx}=\frac{\sec^2 x}{2\sec^2 x+4\tan x+7}$ $y=\int \frac{\sec^2 x}{2\sec^2 x+4\tan x+7}dx$ $=\int \frac{1}{2\left(u^2+1\right)+4u+7}du$ $=\int \frac{1}{2u^2+4u+9}du$ $=\frac{1}{2}\int \frac{1}{u^2+2u+\frac{9}{2}}du$ $=\frac{1}{2}\int \frac{1}{(u+1)^2+\frac{7}{2}}du$ $u=\tan x$ $\frac{du}{dx}=\sec^2 x$  $\sec x=\sqrt{u^2+1}$ $\sec^2 x=u^2+1$ Or $\sec^2 x=\tan^2 x+1$ $=u^2+1$

Qn	Solution
	$=\frac{1}{2}\left(\frac{\sqrt{2}}{\sqrt{7}}\right)\tan^{-1}\left(\frac{\sqrt{2}(u+1)}{\sqrt{7}}\right)+C$ $=\frac{1}{\sqrt{14}}\tan^{-1}\left(\frac{\sqrt{2}(\tan x+1)}{\sqrt{7}}\right)+C$
8i	$\sum_{r=2}^{n}\ln\left[\frac{r(r+2)}{(r+1)^2}\right]$ $=\sum_{r=2}^{n}\left(\ln r-2\ln(r+1)+\ln(r+2)\right)$ $\begin{array}{ccccccc} = & \ln 2 & - & 2\ln 3 & + & \ln 4 \\ + & \ln 3 & - & 2\ln 4 & + & \ln 5 \\ + & \ln 4 & - & 2\ln 5 & + & \ln 6 \\ & & & \vdots & & \\ + & \ln(n-2) & - & 2\ln(n-1) & + & \ln n \\ + & \ln(n-1) & - & 2\ln n & + & \ln(n+1) \\ + & \ln n & - & 2\ln(n+1) & + & \ln(n+2) \end{array}$ $=\ln 2-\ln 3-\ln(n+1)+\ln(n+2)$ $=\ln\frac{2}{3}+\ln\frac{n+2}{n+1}$
ii	As $n\to\infty$, $\ln\frac{n+2}{n+1}\to\ln 1=0$, $\ln\frac{2}{3}+\ln\frac{n+2}{n+1}\to\ln\frac{2}{3}$. Since the series tends to a constant, it converges. The sum to infinity is $\ln\frac{2}{3}$.
iii	$\sum_{r=2}^{13}\ln\left[\frac{(2r)(2r+4)}{(r+1)^2}\right]=\sum_{r=2}^{13}\left(\ln 4+\ln\left[\frac{r(r+2)}{(r+1)^2}\right]\right)$ $=12\ln 4+\ln\frac{2}{3}+\ln\frac{15}{14}$ $=\ln\frac{83886080}{7}$
9i	

Qn	Solution
ii	$V = \pi\int_0^h x^2\,\mathrm{d}y$ $= \pi\int_0^h \left(r^2 - (y-r)^2\right)\mathrm{d}y$ $= \pi\left[r^2 y - \frac{(y-r)^3}{3}\right]_0^h$ $= \pi\left(r^2 h - \frac{(h-r)^3}{3} - \frac{r^3}{3}\right)$ $= \pi\left(r^2 h - \frac{h^3}{3} + h^2 r - hr^2 + \frac{r^3}{3} - \frac{r^3}{3}\right)$ $= \pi\left(h^2 r - \frac{h^3}{3}\right)$ $= \frac{\pi h^2}{3}(3r - h)$ Alternative: $= \pi\int_0^h \left(r^2 - (y^2 - 2ry + r^2)\right)\mathrm{d}y$ $= \pi\int_0^h \left(2ry - y^2\right)\mathrm{d}y$ $= \pi\left[ry^2 - \frac{1}{3}y^3\right]_0^h$ $= \vdots$
iii	$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\frac{2}{3}\pi r^3}{24} = -\frac{\pi r^3}{36}$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \pi\left(2hr - h^2\right)$ $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{\pi\left(2hr - h^2\right)}\left(-\frac{\pi r^3}{36}\right)$ $= -\frac{r^3}{36\left(2hr - h^2\right)}$ Rate of decrease is $\frac{r^3}{36\left(2hr - h^2\right)}$ $\text{cm}^3\ \text{s}^{-1}$.
iv	The rate of decrease of the depth is the least when the bowl is full, i.e. $h = r$. $\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{r^3}{36r(2r - r)} = -\frac{r}{36}$ The slowest rate at which the depth of water is decreasing is $\frac{r}{36}$ cm s^{-1}.
10i	Let θ be the angle between p_1 and p_2. $\cos\theta = \frac{\begin{pmatrix}1\\-5\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\5\end{pmatrix}}{\sqrt{30}\sqrt{30}} = \frac{3}{30} \Rightarrow \theta = 84.3^\circ$
ii	$x - 5y + 2z = 13,$ $-2x + y + 5z = 1,$ From GC, $x = -2 + 3\lambda, y = -3 + \lambda, z = \lambda$

Qn	Solution
	$\therefore$ equation of l is $\underset{\sim}{r} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$
iii	The point of intersection of p_1, p_2 and p_3 is the point of intersection of l and p_3. $(a,0,b)$ is a point on l. $\begin{pmatrix} a \\ 0 \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \Rightarrow$ $a = -2 + 3\lambda$, $\lambda = 3$, $b = \lambda$ $\therefore a = -2 + 3 \times 3 = 7$ and $b = 3$ Alternatively, subst $x = a, y = 0, z = b$ into equation of planes $a + 2b = 13 \text{ --- } (1)$ $-2a + 5b = 1 \text{ --- } (2)$ $(1) \times 2 \quad 2a + 4b = 26 \text{ --- } (3)$ $(2) + (3) \quad 9b = 27 \Rightarrow b = 3$ $a = 7$
iv	l is perpendicular to p_3 and intersect p_3 at $A(7,0,3)$. Let P be a point on l such that $AP = 4\sqrt{11}$, then P lies in Π. $\overrightarrow{AP} = \pm 4\sqrt{11} \times \frac{1}{\sqrt{11}} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \pm 4 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ $\overrightarrow{OP} = \overrightarrow{OP} + \overrightarrow{AP}$ $= \begin{pmatrix} 7 \\ 0 \\ 3 \end{pmatrix} \pm 4 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 19 \\ 4 \\ 7 \end{pmatrix}$  $\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = -20$ or $\underset{\sim}{r} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 19 \\ 4 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 68$ two possible cartesian equations of Π are $3x + y + z = -20$ and $3x + y + z = 68$. **Alternatively,** The cartesian equation of Π is of the form $3x + y + z = p$. $x = 0, y = 0$ and $z = p$ satisfy $3x + y + z = p$, $B(0,0,p)$ is a point in Π.  Distance between Π and p_3 is $4\sqrt{11}$.

Qn	Solution

$$\left|\overrightarrow{BA}\cdot\hat{n}\right| = 4\sqrt{11}$$

$$\left|\begin{pmatrix}7\\0\\3-p\end{pmatrix}\cdot\frac{1}{\sqrt{11}}\begin{pmatrix}3\\1\\1\end{pmatrix}\right| = 4\sqrt{11}$$

$$\frac{1}{\sqrt{11}}\left|24-p\right| = 4\sqrt{11}$$

$$\left|p-24\right| = 44$$

$$p-24 = -44 \text{ or } 44$$

$$p = -20 \text{ or } 68$$

two possible cartesian equations of π are

$3x+y+z=-20$ and $3x+y+z=68$.

11a

$$\frac{a+4d}{a+2d} = \frac{a+11d}{a+4d}$$

$$a^2+8ad+16d^2 = a^2+13ad+22d^2$$

$$5ad = -6d^2$$

Since the terms are distinct, $d \neq 0$, $d = -\frac{5}{6}a$

$$\text{Required sum} = \frac{n}{2}\left((a+d)+(a+(2n-1)d)\right)$$

$$= \frac{n}{2}(2a+2nd) = n(a+nd)$$

bi

n	Beginning	End
1	40000	40000(1.005)
2	40000(1.005)	$40000(1.005)^2$
3	$40000(1.005)^2 - x$	$40000(1.005)^3 - 1.005x$
4	$40000(1.005)^3 - 1.005x - x$	$40000(1.005)^3 - 1.005^2x - 1.005x$
5	$40000(1.005)^4 - 1.005^2x - 1.005x - x$	

Amount at the beginning of 5th month

$= 40000(1.005)^4 - 1.005^2x - 1.005x - x$

$$= 40000(1.005)^4 - \frac{x\left(1.005^3-1\right)}{1.005-1}$$

$$= 40000(1.005)^4 - 200x\left(1.005^3-1\right)$$

ii

He wishes to repay his in 5 years, $n = 60$

$$40000(1.005)^{59} - 200x\left(1.005^{58}-1\right) \le 0$$

$$x \ge \frac{40000(1.005)^{59}}{200\left(1.005^{58}-1\right)}$$

$$x \ge 800.17$$

His minimum repayment is \$800.17

iii

Amount interest bank earned = \$(800.17(58) – 40000)

= \$6410.06 = \$6410 (nearest dollar)

Qn	Solution
12	Since f(x) is a quadratic expression and $f(3)=f(-3)=0$, $f(x)=k(x^2-9)$.

$$\frac{k(x^2-9)}{x+a}=\frac{1}{2}x+1+\frac{b}{x+a}$$

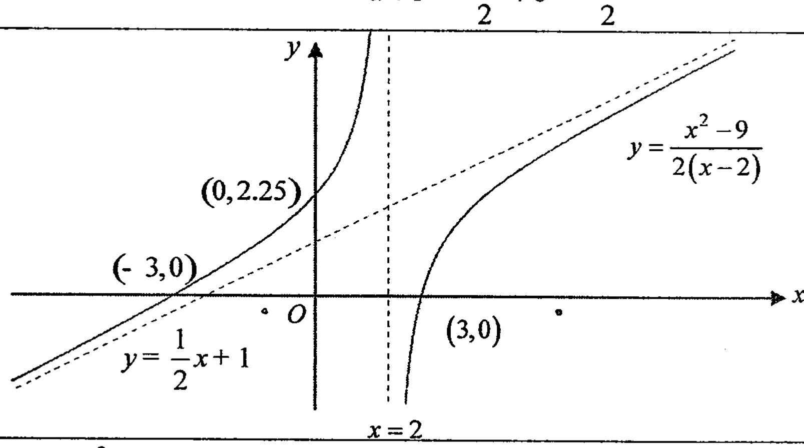
$$\frac{kx^2-9k}{x+a}=\frac{\left(\frac{1}{2}x+1\right)(x+a)+b}{x+a}$$

$$=\frac{\frac{1}{2}x^2+\left(1+\frac{1}{2}a\right)x+a+b}{x+1}$$

Comparing coefficients,

$k=\frac{1}{2}\Rightarrow f(x)=\frac{1}{2}(x^2-9)$ $\therefore 1+\frac{1}{2}a=0\Rightarrow a=-2$ (shown)

$$a+b=-\frac{9}{2}\Rightarrow b=-\frac{5}{2}$$



$$y=\frac{x^2-9}{2(x-2)}=\frac{1}{2}x+1-\frac{5}{2(x-2)}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}+\frac{5}{2(x-2)^2}$$

When $\frac{dy}{dx}=1$, $\frac{1}{2}+\frac{5}{2(x-2)^2}=1\Rightarrow \frac{5}{2(x-2)^2}=\frac{1}{2}$

$$(x-2)^2=5$$

$$x=2\pm\sqrt{5}$$

When $x=2+\sqrt{5}$,

$$y=\frac{1}{2}(2+\sqrt{5})+1-\frac{5}{2\sqrt{5}}=2$$

When $x=2-\sqrt{5}$,

$$y=\frac{1}{2}(2-\sqrt{5})+1-\frac{5}{2(-\sqrt{5})}=2$$

The equations of tangent are

$$y-2=x-\left(2\pm\sqrt{5}\right)$$

$y=x-\sqrt{5}$ or $y=x+\sqrt{5}$

Preliminary Examination
Higher 2

MATHEMATICS **9740/02**
Paper 1
Wednesday **8am – 11am**

21 September 2016 **3 hours**

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 When an object moves through a fluid, it experiences a force that slows it down. This force is called the drag force. At low speeds, it is known that the drag force causes the rate of change in the speed of the object to be proportional to its speed. You may assume that the experiment described below is carried out at low speeds and the only factor that affects the speed is the drag force.

An experiment is conducted to find out how the speed of an object changes as it moves through a certain fluid. When the speed of the object slows down to a speed of D m s^{-1}, a sensor is triggered and the subsequent speeds of the object are recorded.

(i) Show that the speed of the object, v m s^{-1}, at t s after the sensor is triggered, is given by

$v = De^{-pt}$, where p is a positive constant. [4]

(ii) On a single diagram, sketch the curves, C_1 and C_2, of v against t corresponding to $p = \mathrm{e}$ and $p = \frac{1}{\mathrm{e}}$.

State a single transformation the maps C_1 onto C_2. [3]

2 Given that $y = \sqrt{\left(\mathrm{e}^x \cos^2 x\right)}$, show that $2y\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - \mathrm{e}^x \sin 2x$. [2]

(i) Find the series expansion of y in ascending powers of x up to and including the term in x^2. [3]

(ii) Hence, or otherwise, find the series expansion of $\frac{1}{\sqrt{\left(\mathrm{e}^x \cos^2 x\right)}}$ in ascending powers of x up to and including the term in x^2. [3]

3 **(a)** The points A, B, C and D represent the complex numbers $-2+5\mathrm{i}$, z_1, $4+\mathrm{i}$ and z_2 respectively. Given that $ABCD$ is a square, labelled in an anti-clockwise direction, show that $z_1 = -1$. Find z_2. [4]

(b) Show that the equations $z^5 = z^*$ and $|z| = 1$ can be reduced to $z^n = 1$, where n is a positive integer to be determined. Find all possible values of z in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [4]

Given further that $0 < \arg(z) < \frac{\pi}{2}$, find the smallest positive real number k for $\frac{(1+\mathrm{i})}{z^k}$ to be purely imaginary. [4]

4 **(a)** Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The points C on AB and D on OB are such that $2AC = CB$ and $2OD = 3DB$. Show that a vector equation of the line m passing through C and D can be written as

$$\mathbf{r} = \frac{3}{5}\mathbf{b} + \lambda(5\mathbf{a} - 2\mathbf{b}),\ \lambda \in \mathbb{R}\ . \qquad [4]$$

It is given that $|\mathbf{a}| = 2, |\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$. The point F on m is such that F is nearest to O. Show that the position vector of F can be written as $k(5\mathbf{a} + 2\mathbf{b})$, where k is a constant to be found. [4]

(b) Plane π has equation $3x + 2y + 5z = 45$.

Obtain a vector equation of π in the form

$$\mathbf{r} = \mathbf{t} + \lambda\mathbf{u} + \mu\mathbf{v},\ \lambda, \mu \in \mathbb{R}\ ,$$

given that $\mathbf{t}$ and $\mathbf{u}$ are of the form $p\mathbf{i} + p\mathbf{j}$ and $2\mathbf{i} + q\mathbf{j}$ respectively, where p and q are constants to be determined, and $\mathbf{u}$ is perpendicular to $\mathbf{v}$. [5]

Section B: Statistics [60 marks]

5 The head of the Physical Education department of a school wants to gather students' views about the school's efforts in promoting student participation in physical activities. On a particular afternoon, he surveys the first 30 students who turn up at the school gymnasium.

(i) Explain why the above method may not be suitable for the purpose of his survey. [1]

(ii) Describe another sampling method that would yield a sample that is more appropriate in this context. [2]

6 Numbers in this question are formed using only the digits 1, 2, 6, 7 and 9.

(i) How many 4-digit numbers can be formed if repetition of digits is allowed? [1]

(ii) How many even numbers between 10,000 and 30,000 can be formed, if each digit can only be used once? [2]

(iii) A "trick" number is a 6-digit number formed using exactly 3 different digits, and that each digit is smaller than or equal to the following digit. How many "trick" numbers can be formed? [e.g. 127777 and 667799 are "trick" numbers, 111122 and 192992 are not "trick" numbers.] [5]

7 Box A contains 10 red, 8 blue and 7 green balls. Box B contains 2 white and 3 black balls. All the balls are indistinguishable except for their colours. Three balls are taken from Box A and two balls are taken from Box B, at random and without replacement.

Mr Wong guesses that there are at least 1 red ball and exactly 2 black balls taken, while Mr Tan guesses that all the balls taken are of different colours.

(i) Show that the probability that Mr Wong is correct is 0.241, correct to 3 significant figures. [3]

(ii) Find the probability that Mr Tan is correct. [2]

(iii) Find the probability that Mr Wong is correct, given that Mr Tan is wrong. [3]

8 A shop sells two brands of refrigerators which are in the same price range. The number of Tahichi refrigerators sold per week is a random variable with the distribution Po(1.3) and the number of Sungsam refrigerators sold per week is a random variable with the distribution Po(1.1).

(i) Show that the probability of a total of at least 10 refrigerators being sold in a randomly chosen 4-week period is 0.491, correct to 3 significant figures. [3]

(ii) A 4-week period is called a "good" period if at least 10 refrigerators are sold. Find, using a suitable approximation, the probability that, in 52 randomly chosen 4-week periods, there are more than 25 but at most 32 "good" periods. [4]

(iii) State, in the context of this question, two assumptions needed for your calculations in part **(i)** to be valid. Explain why one of these assumptions may not hold in this context. [3]

9 The masses of grade A durians from a plantation are normally distributed with mean 1.96 kg and standard deviation 0.24 kg and the masses of grade B durians from the same plantation are normally distributed with mean 1.00 kg and standard deviation σ kg.

The probability that a randomly chosen grade B durian has a mass of more than 0.8 kg is 0.95. Show that $\sigma = 0.122$, correct to 3 significant figures. [3]

(i) 50 grade A and 1 grade B durians are randomly picked from this plantation. Find the probability that the average mass of the 50 grade A durians is more than twice the mass of the grade B durian. Explain whether there is a need to use Central Limit Theorem in your working. [4]

(ii) A wholesaler buys 50 grade B durians. Using a suitable approximation, find the probability that more than 47 of the durians will have a mass of more than 0.8 kg. [4]

10 An ice-cream shop owner in Singapore wishes to find out how the daily sales of ice-cream depend on the daily average temperature. The following data are collected over 10 days.

Day	1	2	3	4	5	6	7	8	9	10
Daily average temperature, t °C	24.0	25.1	26.2	31.0	28.4	34.0	27.2	32.9	33.5	29.5
No. of cups of ice creams sold in one day, x	100	130	140	171	158	179	150	176	178	163

(i) Without calculating the equation of the regression line of x on t, find the coordinates of a point that will lie on this line. [2]

(ii) Draw a scatter diagram to illustrate the data and find the product moment correlation coefficient between x and t. [2]

(iii) **Without any calculations**, explain whether a quadratic model is more appropriate than a linear model to fit the data. [1]

(iv) The model $x = a(34.2 - t)^2 + b$ is used to fit the data. Calculate the least squares estimates of a and b. [2]

(v) By using the values found in part **(iv)**, estimate the expected number of cups of ice creams sold in 1 day if the daily average temperature is 31.0°C. [1]

11 The mass X g, of one loaf of "Gardener" wholemeal bread is a random variable with mean μ g, which is claimed to be 400g. A random sample of 5 loaves of wholemeal bread has masses in g as follows,

371.3, 399.4, 402.3, 388.3, 400.4.

Carry out a test at the 5% significance level to determine whether this claim is valid, stating clearly any assumption made. [4]

Another random sample of 50 loaves of wholemeal bread is taken, with results summarised below,

$$\sum(x-400) = -102.4, \sum(x-400)^2 = 8030.2.$$

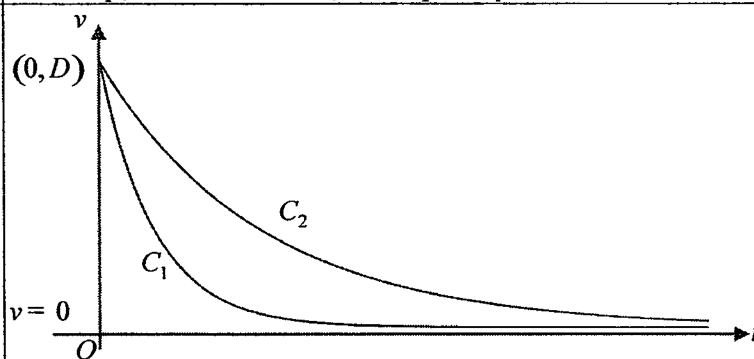
Using the second sample, another test was carried out at the k% significance level to determine the validity of the claim. Find the set of possible values of k for which the test concludes that the claim is incorrect.
Explain, in the context of the question, the meaning of "k% significance level". [5]

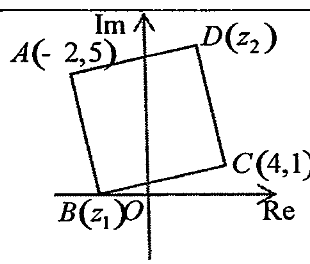
n hypothesis tests are carried out at 4% level of significance to test the validity of the claim.

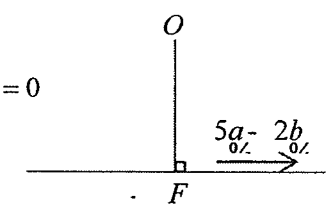
Given that μ is indeed 400g, find the least value of n such that the probability of at most 1 test making a wrong conclusion is less than 0.05. [3]

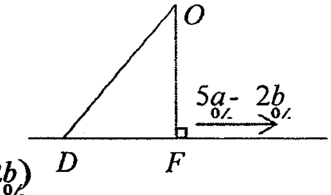
[End of Paper]

2016 JC2 Prelim Paper 2 Solutions/Comments

Qn	Solution
1i	Since speed is decreasing and v is positive, $\frac{\mathrm{d}v}{\mathrm{d}t} = -kv$, where k is a positive constant $\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}t} = -k$ $\int \frac{1}{v}\,\mathrm{d}v = \int -k\,\mathrm{d}t$ $\ln v = -kt + C \quad \because v > 0$ $v = Be^{-kt}$ When $t = 0$s, $v = D$ m s^{-1} $B = D$ Let $k = p$, hence $v = De^{-pt}$, where p is a positive constant.
ii	 Stretch C_1 parallel to the t-axis, factor e^2, v-axis is invariant.
2	$y = \sqrt{e^x \cos^2 x}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x \cos^2 x - 2e^x \sin x \cos x}{2\sqrt{e^x \cos^2 x}}$ $= \frac{y^2 - e^x \sin 2x}{2y}$ $2y\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - e^x \sin 2x$ **Alternative Solution** $y = \sqrt{e^x \cos^2 x}$ $y^2 = e^x \cos^2 x$ $2y\frac{\mathrm{d}y}{\mathrm{d}x} = e^x \cos^2 x - 2e^x \sin x \cos x$ $= y^2 - e^x \sin 2x$
i	$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y\frac{\mathrm{d}y}{\mathrm{d}x} - e^x \sin 2x - 2e^x \cos 2x$ When $x = 0$, $y = \sqrt{e^0 \cos^2 0} = 1$

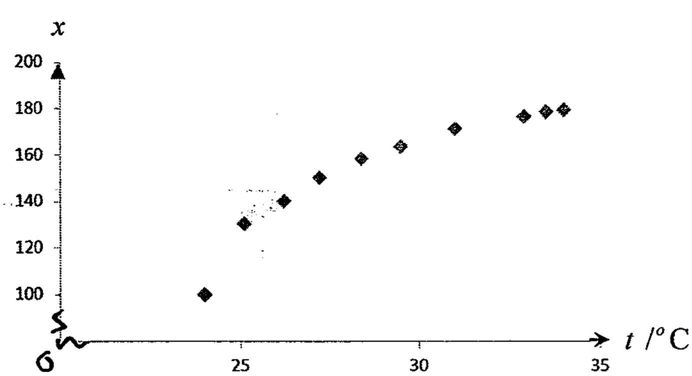
Qn	Solution
	$2\frac{\mathrm{d}y}{\mathrm{d}x}=1-0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}$ $2\left(\frac{1}{2}\right)^2+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(\frac{1}{2}\right)-0-2\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{3}{4}$ $y=1+\frac{1}{2}x-\frac{3}{4}\left(\frac{x^2}{2!}\right)+\ldots$ $\quad=1+\frac{1}{2}x-\frac{3}{8}x^2+\ldots$
ii	$\frac{1}{\sqrt{\mathrm{e}^x\cos^2 x}}=\left(1+\frac{1}{2}x-\frac{3}{8}x^2+\ldots\right)^{-1}$ $=1+(-1)\left(\frac{1}{2}x-\frac{3}{8}x^2+\ldots\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{2}x+\ldots\right)^2+\ldots$ $=1-\frac{1}{2}x+\frac{3}{8}x^2+\frac{1}{4}x^2+\ldots$ $=1-\frac{1}{2}x+\frac{5}{8}x^2+\ldots$
3a	From the diagram, $\arg(4+\mathrm{i}-z_1)+\frac{\pi}{2}=\arg(-2+5\mathrm{i}-z_1)$ $\mathrm{i}(4+\mathrm{i}-z_1)=(-2+5\mathrm{i}-z_1)$ $4\mathrm{i}-1-\mathrm{i}z_1=-2+5\mathrm{i}-z_1$ $(1-\mathrm{i})z_1=-1+\mathrm{i}$ $z_1=-1$ 
	Midpoint of AC is $\left(\frac{-2+4}{2},\frac{5+1}{2}\right)=(1,3)$ Let $z_2=x+\mathrm{i}y$ Since the diagonals of a square bisect other, Midpoint of BD is $(1,3)$ $\left(\frac{x-1}{2},\frac{y+0}{2}\right)=(1,3)$ $\therefore x=3, y=6$ $z_2=3+6\mathrm{i}$
bi	Let $z=\mathrm{e}^{\mathrm{i}\theta}\Rightarrow z^*=\mathrm{e}^{-\mathrm{i}\theta}=\frac{1}{z}$ $z^5=z^{-1}$ $z^6=1$ **Alternatively** $z^5=z^*$ $z^6=zz^*=\lvert z\rvert^2$ $z^6=1$
	$z^6=\mathrm{e}^{2k\pi\mathrm{i}}, k\in\mathbb{Z}$ $z=\mathrm{e}^{\frac{k\pi\mathrm{i}}{3}}, k=0,1,2,3,4,5$ $=1,\mathrm{e}^{\frac{\pi}{3}\mathrm{i}},\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}},-1,\mathrm{e}^{\frac{4\pi}{3}\mathrm{i}},\mathrm{e}^{\frac{5\pi}{3}\mathrm{i}}$

Qn	Solution
ii	Since $0 < \arg(z) < \dfrac{\pi}{2}, z = e^{\frac{\pi}{3}i} \Rightarrow z^k = e^{\frac{k\pi}{3}i}$ $\dfrac{(1+i)}{z^k} = \sqrt{2}e^{i\frac{\pi}{4}}.e^{-\frac{k\pi}{3}i} = \sqrt{2}e^{\left(\frac{\pi}{4}-\frac{k\pi}{3}\right)i}$ If $\dfrac{(1+i)}{z^k}$ is purely imaginary, $\dfrac{\pi}{4} - \dfrac{k\pi}{3} = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \ldots$ Since k is positive, $\dfrac{\pi}{4} - \dfrac{k\pi}{3} = -\dfrac{\pi}{2}, -\dfrac{3\pi}{2}, \ldots$ $\dfrac{k\pi}{3} = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}, \ldots$ Smallest positive k when $\dfrac{k\pi}{3} = \dfrac{3\pi}{4}$ Smallest positive $k = \dfrac{9}{4}$
4a	$\overrightarrow{OC} = \dfrac{1}{3}(2\underset{\sim}{a} + \underset{\sim}{b}), \overrightarrow{OD} = \dfrac{3}{5}\underset{\sim}{b}$ $\overrightarrow{CD} = \dfrac{3}{5}\underset{\sim}{b} - \dfrac{1}{3}(2\underset{\sim}{a} + \underset{\sim}{b}) = -\dfrac{2}{15}(5\underset{\sim}{a} - 2\underset{\sim}{b})$ Since line m passes through D and is parallel to CD, $\underset{\sim}{r} = \overrightarrow{OD} + \mu\overrightarrow{CD}$ $= \dfrac{3}{5}\underset{\sim}{b} + \dfrac{2}{15}\mu(2\underset{\sim}{b} - 5\underset{\sim}{a})$ $= \dfrac{3}{5}\underset{\sim}{b} - \dfrac{2}{15}\mu(5\underset{\sim}{a} - 2\underset{\sim}{b})$ $\underset{\sim}{r} = \dfrac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a} - 2\underset{\sim}{b}), \lambda \in \mathbb{R}$ Equation of m is $\underset{\sim}{r} = \dfrac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a} - 2\underset{\sim}{b}), \lambda \in \mathbb{R}$.
	F is a point on m $\therefore \overrightarrow{OF} = \dfrac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a} - 2\underset{\sim}{b})$ for a value of λ  $\overrightarrow{OF}$ is perpendicular to $l \Rightarrow \overrightarrow{OF} \cdot (5\underset{\sim}{a} - 2\underset{\sim}{b}) = 0$ $\Rightarrow \left[\dfrac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a} - 2\underset{\sim}{b})\right] \cdot (5\underset{\sim}{a} - 2\underset{\sim}{b}) = 0$ $\Rightarrow 3(\underset{\sim}{a} \cdot \underset{\sim}{b}) - \dfrac{6}{5}(\underset{\sim}{b} \cdot \underset{\sim}{b}) + \lambda[25(\underset{\sim}{a} \cdot \underset{\sim}{a}) - 20(\underset{\sim}{a} \cdot \underset{\sim}{b}) + 4(\underset{\sim}{b} \cdot \underset{\sim}{b})] = 0$ $\Rightarrow 3(\underset{\sim}{a} \cdot \underset{\sim}{b}) - \dfrac{6}{5}\lvert\underset{\sim}{b}\rvert^2 + \lambda\left[25\lvert\underset{\sim}{a}\rvert^2 - 20(\underset{\sim}{a} \cdot \underset{\sim}{b}) + 4\lvert\underset{\sim}{b}\rvert^2\right] = 0$ Since $\underset{\sim}{a} \cdot \underset{\sim}{b} = \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos 60^\circ = 2 \times 5 \times \dfrac{1}{2} = 5$ $\therefore 3(5) - \dfrac{6}{5}(5)^2 + \lambda\left[25(2)^2 - 20(5) + 4(5)^2\right] = 0$ $\Rightarrow \lambda = \dfrac{3}{20}$

Qn	Solution
	$\therefore \overrightarrow{OF}=\frac{3}{5}\underset{\sim}{b}+\frac{3}{20}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)=\frac{3}{20}\left(5\underset{\sim}{a}+2\underset{\sim}{b}\right)$
	Alternative Method  $\overrightarrow{DF}=\left(\overrightarrow{DO}\cdot\frac{5\underset{\sim}{a}-2\underset{\sim}{b}}{\lvert 5\underset{\sim}{a}-2\underset{\sim}{b}\rvert}\right)\frac{5\underset{\sim}{a}-2\underset{\sim}{b}}{\lvert 5\underset{\sim}{a}-2\underset{\sim}{b}\rvert}$ $=\frac{1}{\lvert 5\underset{\sim}{a}-2\underset{\sim}{b}\rvert^2}\left(\frac{3}{5}\underset{\sim}{b}\cdot\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)\right)\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $=\frac{-3\underset{\sim}{a}\cdot\underset{\sim}{b}+\frac{6}{5}\lvert\underset{\sim}{b}\rvert^2}{\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)\cdot\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $=\frac{-3(5)+\frac{6}{5}(5)^2}{25\lvert\underset{\sim}{a}\rvert^2-20\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)+4\lvert\underset{\sim}{b}\rvert^2}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $=\frac{15}{25(2)^2-20(5)+4(5)^2}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $=\frac{3}{20}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)$ $\therefore \overrightarrow{OF}=\frac{3}{5}\underset{\sim}{b}+\frac{3}{20}\left(5\underset{\sim}{a}-2\underset{\sim}{b}\right)=\frac{3}{20}\left(5\underset{\sim}{a}+2\underset{\sim}{b}\right)$
b	The equation of the plane π is $3x+2y+5z=45$. $(p,p,0)$ lies in $\pi\Rightarrow 3p+2p+0=45\Rightarrow p=9$ $\begin{pmatrix}2\\q\\0\end{pmatrix}$ is perpendicular to $\begin{pmatrix}3\\2\\5\end{pmatrix}\Rightarrow\begin{pmatrix}2\\q\\0\end{pmatrix}\cdot\begin{pmatrix}3\\2\\5\end{pmatrix}=0$ $6+2q=0\Rightarrow q=-3$ Since $\underset{\sim}{v}$ is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{n}$, $\underset{\sim}{u}\times\underset{\sim}{n}=\begin{pmatrix}3\\2\\5\end{pmatrix}\times\begin{pmatrix}2\\-3\\0\end{pmatrix}=\begin{pmatrix}15\\10\\-13\end{pmatrix}$ $\underset{\sim}{r}=\begin{pmatrix}9\\9\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}15\\10\\-13\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$ **Alternative method to find $\underset{\sim}{v}$** Let $\underset{\sim}{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$ $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}3\\2\\5\end{pmatrix}=0$ and $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-3\\0\end{pmatrix}=0$

Qn	Solution
	$3x + 2y + 5z = 0$ and $2x - 3y = 0$ $x = -\frac{15}{13}z,\ y = -\frac{10}{13}z,\ z = z$ Let $z = 13$ (any non-zero number will work) $\mathbf{v} = \begin{pmatrix} -15 \\ -10 \\ 13 \end{pmatrix}$
5i	He will not get to survey the students who do not go to the school gymnasium. Hence, the sample obtained is biased.
ii	He can obtain a numbered list of all the students (labelled 1 to N) in the school. Using a random number generator, he generates 30 **distinct** numbers. He will survey the students corresponding the numbers generated. **Alternatively.** Let the total number of students be N Sampling interval = $\frac{N}{30}$ He can obtain a numbered list of all the students (labelled 1 to N) in the school. Using a random number generator, select a starting number k where $1 \le k \le \frac{N}{30}$. He can interview the students corresponding to the numbers $k,\ k+\frac{N}{30}, k+\frac{N}{15}, \ldots, k+\frac{29N}{30}$.
6i	Number of 4-digit numbers = $5^4 = 625$
ii	Case1: Starts with 1 (1 _ _ _ 6 with 1 or 2 above first and last) No. of ways = 2(3!) = 12 1/1 _ _ _ 2/6 Case 2: starts with 2 No. of ways = 3! = 6 2 _ _ _ 6 Total number of ways = 18
iii	Case 1: XXXXYZ No. of ways = ${}^5C_3({}^3C_1) = 30$ Case 2: XXXYYZ No. of ways = ${}^5C_3({}^3C_1)({}^2C_1) = 60$ Case 3: XXYYZZ No. of ways = ${}^5C_3 = 10$ Total number of ways = 100
7i	P(Mr Wong is correct) = $\left(1 - \frac{{}^{15}C_3}{{}^{25}C_3}\right) \frac{{}^3C_2}{{}^5C_2} = 0.24065 = 0.241$
ii	P(Mr Tan is correct) = $\frac{{}^{10}C_1 \times {}^8C_1 \times {}^7C_1}{{}^{25}C_3} \times \frac{{}^3C_1 \times {}^2C_1}{{}^5C_2} = \frac{84}{575} = 0.146$ <u>Alternative method:</u> P(Mr Tan is correct) = $\frac{10 \times 8 \times 7}{25 \times 24 \times 23} \times 3! \times \frac{3 \times 2}{5 \times 4} \times 2!$

Qn	Solution
	$= \dfrac{84}{575} = 0.146$
iii	P(Mr Wong's guess is right, given that Mr Tan's guess is wrong) $= \dfrac{\text{P(Mr Wong is correct and Mr Tan is wrong)}}{\text{P(Mr Tan is wrong)}}$ $= \dfrac{0.24065}{1-0.14609}$ $= 0.282$
8i	Let T be the total number of refrigerators sold in a 4-week period. $T \sim \text{Po}((1.3 + 1.1) \times 4)$ $T \sim \text{Po}(9.6)$ $\text{P}(T \geq 10) = 1 - \text{P}(T \leq 9) = 0.49114 = 0.491$ (3sf)
ii	Let X be number of good periods out of 52. $X \sim \text{B}(52, 0.491)$ or $X \sim B(52,\ 0.49114)$ Since $np = 25.532 > 5$ and $np(1-p) = 26.468 > 5$ $X \sim \text{N}(25.532, 12.996)$ approx. or $X \sim N(25.539, 12.996)$ approx. $\text{P}(25 < X \leq 32) = \text{P}(25.5 < X \leq 32.5) = 0.477$ (or 0.478)
iii	We need to assume that the sales of **all** the refrigerators are independent of one another. We also need to assume that the **average rate** of refrigerators being sold is constant. The first assumption may not hold as the two brands of refrigerator are in the same price range and they can be competing in terms of sales. OR The average rate of refrigerators sold is unlikely to be a constant due to sale, festive seasons, economic conditions etc.
9	Let A kg and B kg be masses of a randomly chosen grade A and grade B durian respectively. $A \sim \text{N}(1.96, 0.24^2)$ and $B \sim \text{N}(1.00, \sigma^2)$ $\text{P}(B > 0.8) = 0.95$ $\text{P}\left(Z > \dfrac{0.8-1.00}{\sigma}\right) = 0.95$ $\text{P}\left(Z \leq \dfrac{0.8-1.00}{\sigma}\right) = 0.05$ $\dfrac{-0.2}{\sigma} = -1.64485 \Rightarrow \sigma = 0.12159 \approx 0.122$
i	$A \sim \text{N}(1.96, 0.24^2)$ and $B \sim \text{N}(1.00, \sigma^2)$ $\bar{A} \sim N\left(1.96, \dfrac{0.24^2}{50}\right)$ and

Qn	Solution
	$2B \sqcup \mathrm{N}(2.00, 2^2(0.122^2))$ or $2B \square N(2.00, 2^2(0.12159)^2$ $\bar{A} - 2B \sim N(-0.04, 0.060688)$ or $\bar{A} - 2B \sim \mathrm{N}(-0.04, 0.060290)$ $P(\bar{A} - 2B > 0) = 0.436$ (or 0.435) Central limit theorem is not needed because the masses of grade *A* durians follow a normal distribution.
ii	Let Y be the number of grade *B* durians with a mass of more than 0.8 kg out of 50 durians. $Y \sqcup \mathrm{B}(50, 0.95)$ $np = 50 \times 0.95 = 47.5 > 5$ and $n(1-p) = 50 \times 0.05 = 2.5 < 5$ Let Y' be the number of grade *B* durians with a mass $\leq$ 0.8 kg out of 50 durians. $Y' \sqcup \mathrm{Po}\ (2.5)$ approx. $\mathrm{P}(Y > 47) = \mathrm{P}(50 - Y' > 47)$ $= \mathrm{P}(Y' \leq 2)$ $= 0.544$
10i	$\bar{t}$ and $\bar{x}$ $\bar{t} = 29.18, \bar{x} = 154.5$ Hence, (29.18, 154.5) lies on the regression line x on t.
ii	 $r = 0.934$ (3s.f.)
iii	From the scatter diagram, x increases by decreasing amounts as t increases. Hence, a quadratic model might be more appropriate.
iv	By GC, $a = -0.673$ (3sf), $b = 179$ (3sf)
v	Substituting $t = 31.0$, $x = -0.67342(34.2 - 31.0)^2 + 179.28$ $= 172.388$

Qn	Solution
	Expected number of cups of ice cream sold is 172.

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$H_0: \mu = 400$

$H_1: \mu \neq 400$

Level of significance: 5%

Test Statistic: When H_0 is true, $T = \dfrac{\overline{X} - 400}{S/\sqrt{5}}$

Computation: $\nu = 5 - 1 = 4$.

By GC, $\bar{x} = 392.34, s = 12.971$, p-value $= 0.257$ (3sf)

Conclusion: Since p-value $= 0.257 > 0.05$, H_0 is not rejected at 5% level of significance. So there is insufficient evidence to conclude that the claim is invalid.

It is assumed that the masses of loaves of "Gardener" wholemeal bread follow a normal distribution.

$\bar{x} = 400 - \dfrac{102.4}{50} = 397.952$

$s^2 = \dfrac{1}{49}\left(8030.2 - \dfrac{(-102.4)^2}{50}\right) = 159.60$

$H_0: \mu = 400$

$H_1: \mu \neq 400$

Level of significance: k%

Test Statistic: When H_0 is true, $Z = \dfrac{\overline{X} - 400}{\sqrt{159.6017306}\big/\sqrt{55}}$

Computation:

By GC, $\bar{x} = 397.952$, p-value $= 0.252$ (3sf)

For H_0 to be rejected at k% level of significance,

$p\text{-value} \leq \dfrac{k}{100} \Rightarrow k \geq 25.2$

Set of values $= \{k \in \mathbb{R} : k \geq 25.2\}$

"k% significance level" in this context means there is a probability of $\dfrac{k}{100}$ (or k%) that the test will conclude that the mean mass of "Gardener" wholemeal bread is not 400g, when it is actually 400g.

Let Y be the number of wrong conclusions out of n hypothesis tests

$Y \sim B(n, 0.04)$

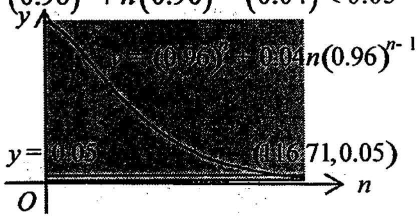
$P(Y \leq 1) < 0.05$

By GC,

n	$P(Y \leq 1)$
116	0.05121
117	0.04952

Least $n = 117$

Alternatively

Qn	Solution
	$P(Y \leq 1) < 0.05$ $(0.96)^n + n(0.96)^{n-1}(0.04) < 0.05$  Least $n = 117$