JC2 Preliminary Examination

Higher 2

H2 Mathematics 9740/01

Paper 1 **13 September 2016**

3 Hours

Additional Materials: Writing paper

List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **8** printed pages.

1 A graphic calculator is not to be used for this question.

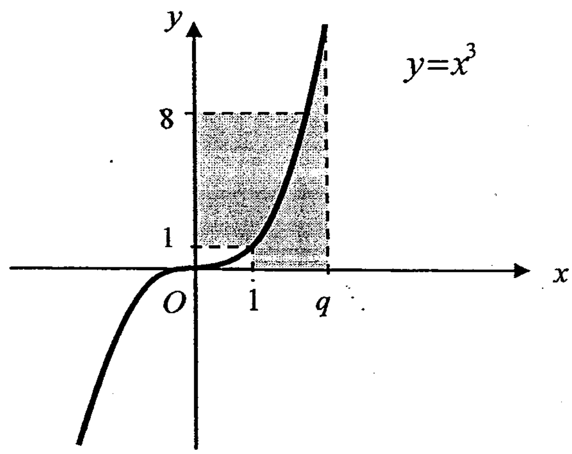
Show algebraically that $x^2 - 2x + 5$ is always positive for $x \in \mathbb{R}$, and solve the inequality

$$\frac{x}{x^2-2x+5} \le \frac{x+2}{x^3-2x^2+5x}.$$ [4]

Hence solve the inequality $\dfrac{e^x}{e^{2x}-2e^x+5} \ge \dfrac{e^x+2}{e^{3x}-2e^{2x}+5e^x}$. [2]

2 **(a)** Find, in terms of p, $\int_1^p \ln(x)\, dx$, where $p > 1$. [2]

(b)



The diagram shows the curve with the equation $y = x^3$. The area of the region bounded by the curve, the lines $x = 1$, $x = q$ and the x-axis is equal to the area of the region bounded by the curve, $y = 1$, $y = 8$ and the y-axis, where $q > 1$. Find the exact value of q in the form $a^{\frac{1}{b}}$, where a and b are integers. [4]

3 Prove by mathematical induction that

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=2\left(1-\frac{1}{n+1}\right),\ n \in \mathbf{Z}^+.$$ [5]

Hence state the value of the infinite series $1+\dfrac{1}{1+2}+\dfrac{1}{1+2+3}+\ldots+\dfrac{1}{1+2+3+\ldots+n}+\ldots$.

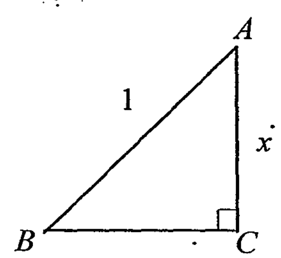
[1]

4 Let $f(x)=\cos^{-1}x$, where $-1<x<1$ and $0<f(x)<\pi$. Show that

$$\left(1-x^2\right)f''(x)=x\,f'(x).$$ [2]

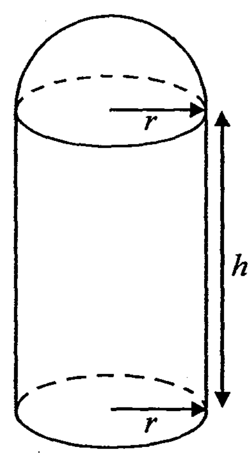
By further differentiation of this result, or otherwise, find the first three non-zero terms in the expansion of $f(x)$ in ascending powers of x. [3]

The diagram shows a triangle ABC. Given that the lengths of AB and AC are 1 and x units respectively, show that $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$. [1]



Hence find the series expansion of $\sin^{-1}x$ in ascending powers of x, up to and including the term in x^3. [1]

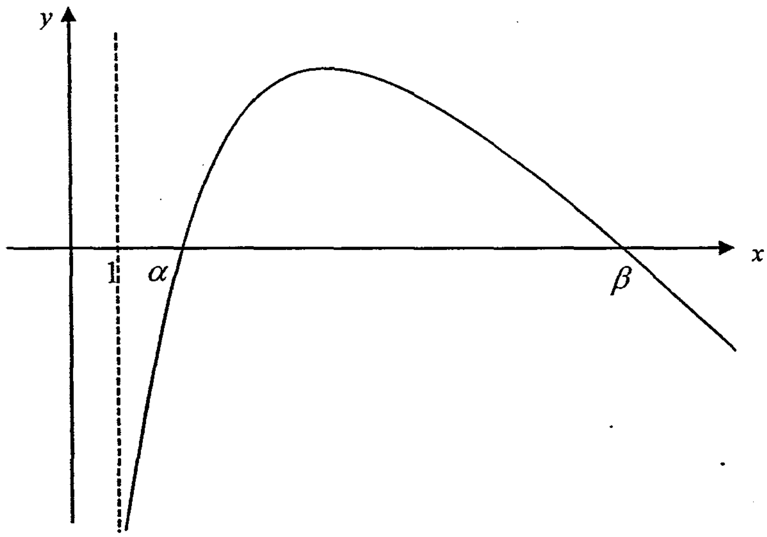
5 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]



A rice farmer wants to build a new grain silo to store his rice grains. The cylindrical section has height h m and the hemispherical roof has radius r m. After building the grain silo, the farmer will be painting its rooftop and the external curved surface. The time needed to paint the grain silo will be 20 minutes per square metre for the curved surface area of the cylinder and 35 minutes per square metre for the hemispherical roof. Given that a total time of 60 000 minutes is taken to paint the grain silo, find, using differentiation, the value of r which gives a grain silo of maximum volume. [8]

6 The diagram below shows the graph of $y=2\ln(x-1)+4-x$.

The two roots of the equation $2\ln(x-1)+4-x=0$ are denoted by α and β, where $\alpha<\beta$.



(i) Find the values of α and β, correct to 3 decimal places. [2]

A sequence of real numbers $x_1, x_2, x_3, \ldots$ where $x_n > 1$, satisfies the recurrence relation

$$x_{n+1} = \ln(x_n - 1)^2 + 4 \text{ for } n \geq 1.$$

(ii) Prove algebraically that if the sequence converges, it must converge to either α or β. [2]

(iii) Use a calculator to determine the behaviour of the sequence for each of the cases $x_1 = 3$, $x_1 = 12$. [2]

(iv) By considering $x_{n+1} - x_n$ and the graph above, prove that

$$x_{n+1} > x_n \text{ if } \alpha < x_n < \beta,$$

$$x_{n+1} < x_n \text{ if } 1 < x_n < \alpha \text{ or } x_n > \beta.$$ [2]

7 The equations of three planes are

$$x+2y+z=60$$
$$4x+5y+10z=180$$
$$2x+3y+4z=100$$

(i) It is given that all three planes meet in the line l. Find a vector equation of l. [2]

(ii) Find a cartesian equation of the plane which contains l and the origin. [3]

A technology company specialises in manufacturing circuit boards that are used for space exploration. It manufactures only 3 types of circuit boards (A, B and C). Each circuit board requires particular amounts of different raw materials for manufacturing. The amounts of raw material (in units) required for each type of circuit board and the total amounts of raw material available to the company are shown in the following table.

	Copper	Lead	Fibreglass
Circuit Board A	1	4	2
Circuit Board B	2	5	3
Circuit Board C	1	10	4
Total amount of material available (in units)	60	180	100

The company is required to use all the materials available to manufacture its circuit boards.

The vector $\mathbf{r}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is defined such that variables x, y, and z represent the number of circuit boards A, B and C that are manufactured respectively.

(iii) With the aid of your answer in part **(i)**, solve for **r**. Leave your answer clearly in the form of $\mathbf{a}+\mu\mathbf{b}$ and state the possible values for μ. [2]

(iv) Explain, in context, why your vector equation in part **(i)** is not an appropriate answer for part **(iii)**. [2]

8 Functions f and g are defined by

$$f : x \mapsto \left(\frac{1}{x+1}\right)^2, \quad x > -1,$$

$$g : x \mapsto \ln x, \quad x > 0.$$

(i) Show that gf exists and express gf in a similar form. [3]

(ii) Sketch, in a single diagram, the graphs of g and gf, labelling each graph clearly. Write down the range of gf. [3]

(iii) Describe a sequence of transformations which maps the graph of g onto the graph of gf. [4]

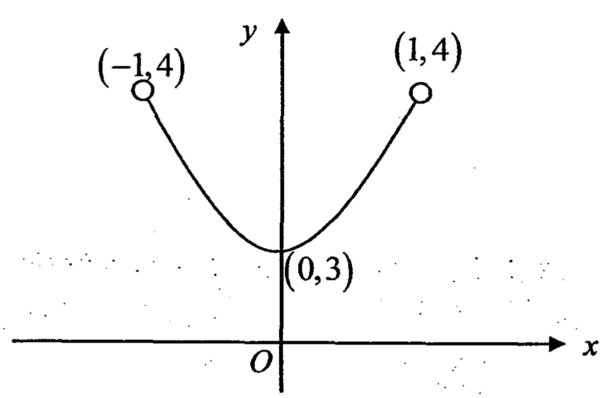
9 It is given that

$$f(x) = \frac{x}{\sqrt{(1-x^2)}}, \text{ where } -1 < x < 1.$$

(i) Show by differentiation that f is strictly increasing. [3]

(ii) Sketch the graph of $y = f(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

The diagram below shows the graph of $y = g(x)$, which is continuous and differentiable on $(-1, 1)$. It has a minimum turning point at $(0, 3)$.



(iii) It is given that $w(x) = g(x)f(x)$, where $-1 < x < 1$. By finding $w'(x)$ and using your earlier results in **(i)** and **(ii)**, determine the number of stationary points on the graph of w. [4]

10 **(a)** Solve the simultaneous equations

$$z = w + 2i - 1 \text{ and } z^2 - iw + \frac{5}{2} = 0,$$

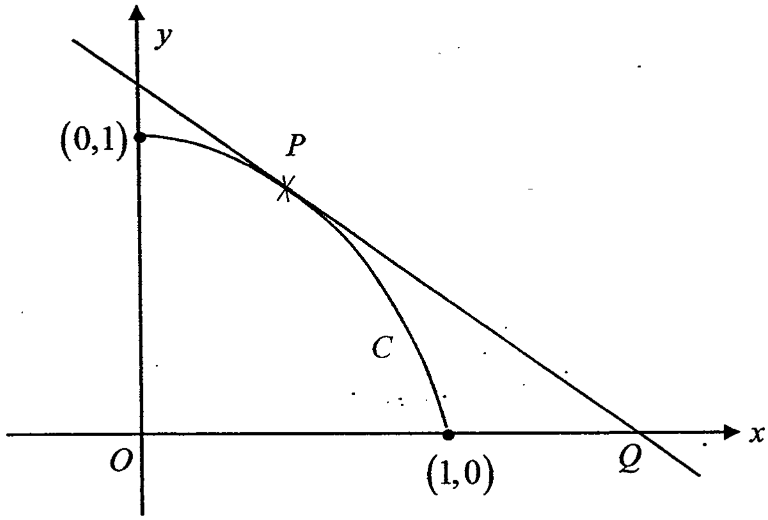
giving z and w in the form $x + yi$ where x and y are real. [5]

(b) **(i)** Given that $z = w - \frac{1}{w}$ where $w = 2(\cos\theta + i\sin\theta)$, $-\pi < \theta \leq \pi$, express the real and imaginary parts of z in terms of θ. [3]

(ii) Hence show that locus of z on an Argand diagram lies on the curve with cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where a and b are constants. [3]

(iii) Sketch this locus on an Argand diagram, indicating clearly the points of intersection with the axes. [2]

11



It is given that curve C has parametric equations

$$x = t^3, \quad y = \sqrt{(1-t^2)} \quad \text{for } 0 \le t \le 1.$$

The diagram shows the curve C and the tangent to C at P. The tangent at P meets the x-axis at Q.

(i) The point P on the curve has parameter p. Show that the equation of the tangent at P is $3p(1-p^2) - 3py\sqrt{(1-p^2)} = x - p^3$. [3]

(ii) Given further that the line $y = (4\sqrt{3})x$ meets the curve at point P, find the exact coordinates of P. [3]

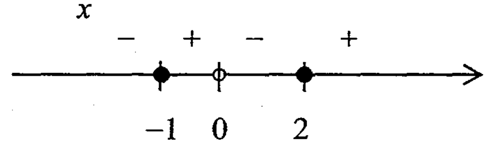
(iii) Hence find the exact coordinates of Q. [2]

(iv) Show that the area of the region bounded by C, the tangent to C at P, and the x-axis is given by $\dfrac{9\sqrt{3}}{32} - \int_{\frac{1}{2}}^{1} 3t^2\sqrt{(1-t^2)}\,\mathrm{d}t$. [3]

Show that the substitution $t = \sin u$ transforms the above integral to $\dfrac{9\sqrt{3}}{32} - \dfrac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1 - \cos 4u \,\mathrm{d}u$. Hence, evaluate this area exactly. [6]

END OF PAPER

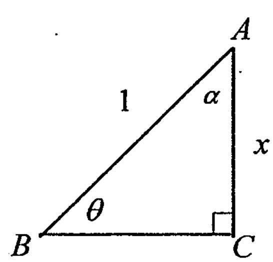
2016 H2 MATH (9740/01) JC 2 PRELIM EXAMINATION – MARKING SCHEME

Qn	Solution
1	**Inequalities**
(i)	$x^2-2x+5=(x-1)^2-1+5$ $=(x-1)^2+4\ >0\ \forall x\in \square$
(ii)	$\dfrac{x}{x^2-2x+5}\le\dfrac{x+2}{x^3-2x^2+5x},\ x>0$ $\dfrac{x}{x^2-2x+5}-\dfrac{x+2}{x\left(x^2-2x+5\right)}\le 0$ $\dfrac{x^2-x-2}{x\left(x^2-2x+5\right)}\le 0$ Since $x^2-2x+5>0$ for $\forall x\in \square$, $\dfrac{(x-2)(x+1)}{x}\le 0$ $x\le -1$ or $0<x\le 2$ 
(iii)	$\dfrac{e^x}{e^{2x}-2e^x+5}\ge\dfrac{e^x+2}{e^{3x}-2e^{2x}+5e^x}$ Replace x with e^x. $-1\le e^x<0$ (rej. $\because e^x>0$) or $e^x\ge 2$ $x\ge \ln 2$

Qn	Solution
2	**Definite Integrals**
(a)	$\int_1^p \ln(x)\mathrm{d}x = [x\ln x]_1^p - \int_1^p \left(\frac{1}{x}\right)x\ \mathrm{d}x$ $= [(p\ln p - 0) - (p-1)]$ $= p\ln p - p + 1$
(b)	$\int_1^q x^3\ \mathrm{d}x = \int_1^8 \sqrt[3]{y}\ \mathrm{d}y$ $\left[\frac{x^4}{4}\right]_1^q = \left[\frac{3y^{\frac{4}{3}}}{4}\right]_1^8$ $\frac{q^4}{4} - \frac{1}{4} = \left(\frac{3}{4}\right)\left(8^{\frac{4}{3}} - 1\right)$ $\frac{q^4}{4} - \frac{1}{4} = \left(\frac{3}{4}\right)(16-1)$ $q^4 = 46$ $q = 46^{\frac{1}{4}}$ $\therefore a = 46,\ b = 4$

Qn	Solution
3	**Mathematical Induction + APGP**
(i)	Let P_n be the statement $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=2\left(1-\frac{1}{n+1}\right), n\in\mathbb{Z}^+$. When $n=1$, LHS $=1$. RHS $=2\left(1-\frac{1}{1+1}\right)=1=$ LHS $\therefore P_1$ is true. Assume that P_k is true for some $k\in\mathbb{Z}^+$ i.e. $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}=2\left(1-\frac{1}{k+1}\right)$ To show that P_{k+1} is also true i.e. $1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k+1}=2\left(1-\frac{1}{(k+1)+1}\right)$ When $n=k+1$, LHS $=1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k+1}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{1}{1+2+3+\ldots+k+1}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{1}{\frac{k+1}{2}(1+k+1)}$ $=2\left(1-\frac{1}{k+1}\right)+\frac{2}{(k+1)(k+2)}$ $=2+\frac{-2(k+2)+2}{(k+1)(k+2)}$ $=2+\frac{-2k-2}{(k+1)(k+2)}$ $=2+\frac{-2(k+1)}{(k+1)(k+2)}$ $=2+\frac{-2}{(k+2)}$ $=2\left(1-\frac{1}{((k+1)+1)}\right)$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$.
	$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}=2\left(1-\frac{1}{n+1}\right)$

	$\text{As } n \to \infty, \frac{1}{n+1} \to 0 \;\; \therefore 2\left(1-\frac{1}{n+1}\right) \to 2$ $\therefore 1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+n}+\ldots=2$

Qn	Solution
4	**Maclaurin Series**
	$f(x)=\cos^{-1}x$ $f'(x)=-\frac{1}{\sqrt{1-x^2}}$ $f''(x)=\frac{1}{2}\left(1-x^2\right)^{-\frac{3}{2}}(-2x)$ $\left(1-x^2\right)f''(x)=-x\left(1-x^2\right)^{-\frac{1}{2}}$ $\left(1-x^2\right)f''(x)=x\,f'(x)$ (shown)
	$\left(1-x^2\right)f''(x)=x\,f'(x)$ $\left(1-x^2\right)f'''(x)-2x\,f''(x)=x\,f''(x)+f'(x)$ $\left(1-x^2\right)f'''(x)=3x\,f''(x)+f'(x)$ $f(0)=\frac{\pi}{2},\;\; f'(0)=-1,\;\; f''(0)=0,\;\; f'''(0)=-1$ $\cos^{-1}x=\frac{\pi}{2}-x-\frac{x^3}{6}+\ldots$
	Let $\sin^{-1}x=\theta$ Let $\cos^{-1}x=\alpha$ $\theta+\alpha=\frac{\pi}{2}$ $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ (shown) 
	$\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ $\sin^{-1}x=\frac{\pi}{2}-\cos^{-1}x=\frac{\pi}{2}-\left(\frac{\pi}{2}-x-\frac{x^3}{6}+\ldots\right)$ $\sin^{-1}x=x+\frac{x^3}{6}+\ldots$

Qn	Solution
5	**Maximum/Minimum Problem**

Let volume of silo be V

$$V = \pi r^2 h + \frac{2}{3}\pi r^3$$

Time needed to paint the silo $= 20(2\pi rh) + 35(2\pi r^2)$

$$60000 = 40\pi rh + 70\pi r^2$$

$$h = \frac{60000 - 70\pi r^2}{40\pi r}$$

$$V = \pi r^2\left(\frac{60000 - 70\pi r^2}{40\pi r}\right) + \frac{2}{3}\pi r^3$$

$$= 1500r - \frac{7}{4}\pi r^3 + \frac{2}{3}\pi r^3$$

$$= 1500r - \frac{13}{12}\pi r^3$$

$$\frac{dV}{dr} = 1500 - \frac{13}{4}\pi r^2$$

For maximum V, $\frac{dV}{dr} = 0$

$$\Rightarrow 1500 - \frac{13}{4}\pi r^2 = 0$$

$$\frac{13}{4}\pi r^2 = 1500$$

$$r^2 = \frac{6000}{13\pi}$$

$$r = \pm\sqrt{\frac{6000}{13\pi}}$$

Since $r > 0$, $\therefore r = \sqrt{\frac{6000}{13\pi}} = 12.1207 = 12.1$ (3 s.f.).

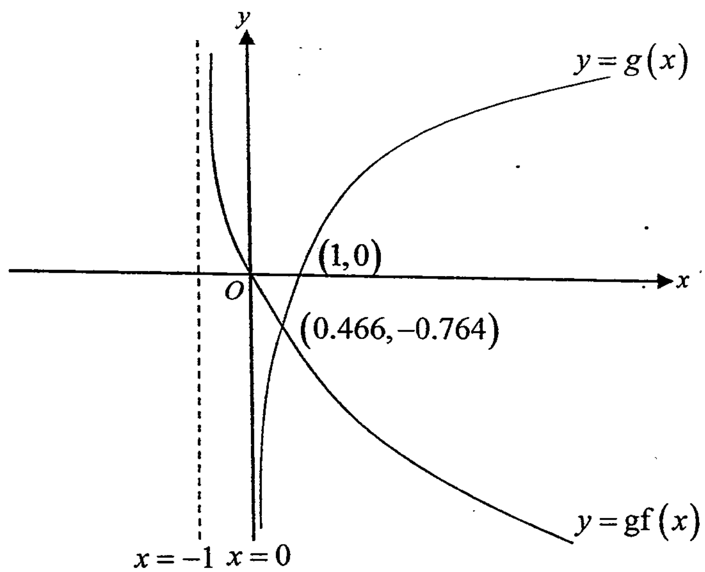
$\frac{d^2V}{dr^2} = -\frac{13}{2}\pi r < 0$, for $r = 12.1207$.

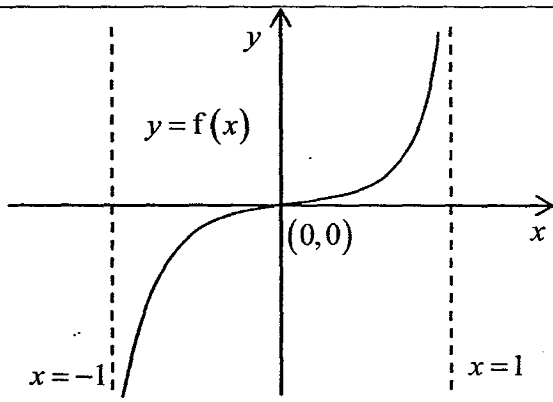
Alternative:

r	$\left(\sqrt{\frac{6000}{13\pi}}\right)^-$	$\sqrt{\frac{6000}{13\pi}}$	$\left(\sqrt{\frac{6000}{13\pi}}\right)^+$
$\frac{dV}{dr}$	+ /	0 —	− \

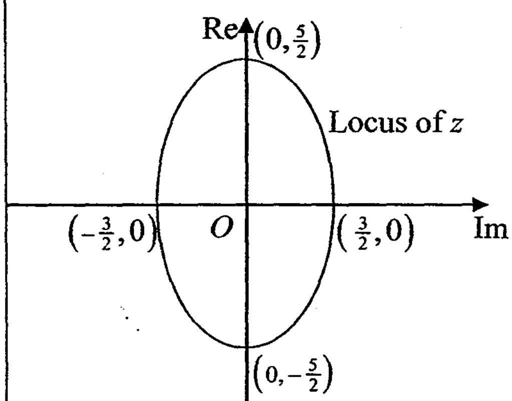
Qn	Solution
6	**Recurrence Relations**
(i)	Using GC, roots of equation are $\alpha = 1.253, \beta = 7.848$.
(ii)	As $n \to \infty, \quad x_n \to L, \quad x_{n+1} \to L$ $\therefore L = \ln(L-1)^2 + 4 \Rightarrow 2\ln(L-1) + 4 - L = 0$ Since equation is identical to $2\ln(x-1) + 4 - x = 0$ $\therefore L = 1.253 = \alpha \quad$ or $\quad L = 7.848 = \beta$ Hence the sequence converges to either α or β.
(iii)	Using GC, it can be observed that when $x_1 = 3$, the sequence increases and converges to $7.848 = \beta$. when $x_1 = 12$, the sequence decreases and converges to $7.848 = \beta$.
(iv)	$x_{n+1} - x_n = \ln(x_n - 1)^2 + 4 - x_n$ From graph, if $\alpha < x_n < \beta$, $2\ln(x_n - 1) + 4 - x_n > 0 \Rightarrow \ln(x_n - 1)^2 + 4 > x_n \Rightarrow x_{n+1} > x_n$ if $1 < x_n < \alpha$ or $x_n > \beta$, $2\ln(x_n - 1) + 4 - x_n < 0 \Rightarrow \ln(x_n - 1)^2 + 4 < x_n \Rightarrow x_{n+1} < x_n$.

Qn	Solution
7	**Vectors**
(i)	Using GC, $x = 20 - 5z$ $y = 20 + 2z$ $z = z$ $l : \mathbf{r} = \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$
(ii)	Normal Vector $\begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} \times \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ -20 \\ 140 \end{pmatrix}$ $\mathbf{r} \cdot \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 7 \end{pmatrix} = 0$ Cartesian Equation: $x - y + 7z = 0$ or equivalent
(iii)	$\mathbf{r} = \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix}$ where $\mu = 0, 1, 2, 3, 4$
(iv)	The vector equation in **(i)** allows for x, y and z to be real numbers. But the circuit boards produced is a physical quantity and must minimally be an integer.

Qn	Solution
8	**Functions & Transformation of Graphs**
(i)	Since $R_f = (0,\infty) \subseteq D_g = (0,\infty)$, gf exists. $gf : x \mapsto \ln\left(\frac{1}{x+1}\right)^2, \quad x > -1.$
(ii)	 $R_{gf} = (-\infty,\infty) = \mathbb{R}$
(iii)	Since $x > -1, x+1 > 0$, $gf(x) = -2\ln(x+1)$ From $y = \ln x$ to $y = -2\ln(x+1)$: 1. Translation of 1 unit in the negative x-direction 2. Reflection in the x-axis 3. Scaling parallel to y-axis by a factor of 2 [Accept any other possible correct sequence such as 1-3-2]

Qn	Solution
9	**Curve sketching and differentiation**
(i)	$\mathrm{f}(x)=\dfrac{x}{\sqrt{(1-x^2)}}$ $\mathrm{f}'(x)=\dfrac{\sqrt{(1-x^2)}\cdot 1-x\cdot\frac{1}{2}\cdot(1-x^2)^{-\frac{1}{2}}\cdot(-2x)}{(1-x^2)}$ $=\dfrac{1}{\sqrt{(1-x^2)}}+\dfrac{x^2}{(1-x^2)^{\frac{3}{2}}}$ $=\dfrac{1}{(1-x^2)^{\frac{3}{2}}}>0\left(\because -1<x<1,\therefore (1-x^2)^{\frac{3}{2}}>0\right)$ Since $\mathrm{f}'(x)>0$ for $-1<x<1$, f is strictly increasing.
(ii)	
(iii)	$\mathrm{w}'(x)=\mathrm{g}(x)\mathrm{f}'(x)+\mathrm{g}'(x)\mathrm{f}(x)$ From **(i)**, $\mathrm{f}'(x)>0$ for $-1<x<1$. From **(ii)**, $\mathrm{f}(x)=\begin{cases}<0 & \text{for } -1<x<0\\ =0 & \text{for } x=0\\ >0 & \text{for } 0<x<1\end{cases}$ From the given graph, $\mathrm{g}(x)>0$ for $-1<x<1$ and $\mathrm{g}'(x)=\begin{cases}<0 & \text{for } -1<x<0\\ =0 & \text{for } x=0\\ >0 & \text{for } 0<x<1\end{cases}$ Therefore, When $-1<x<0$, $\mathrm{w}'(x)>0$. When $x=0$, $\mathrm{w}'(x)>0$. When $0<x<1$, $\mathrm{w}'(x)>0$. Since $\mathrm{w}'(x)>0\neq 0$ for $-1<x<1$, there are no stationary points in the graph of w.

Qn	Solution
10	**Complex Numbers**
(a)	$z = w + 2\mathrm{i} - 1$ --- (1) $z^2 - \mathrm{i}w + \frac{5}{2} = 0$ --- (2) **Method 1** From (1): $w = z - 2\mathrm{i} + 1$ --- (3) Substitute (3) into (2): $z^2 - \mathrm{i}(z - 2\mathrm{i} + 1) + \frac{5}{2} = 0$ $z^2 - \mathrm{i}z - \mathrm{i} + \frac{1}{2} = 0$ $z = \frac{-(-\mathrm{i}) \pm \sqrt{(-\mathrm{i})^2 - 4(1)\left(-\mathrm{i} + \frac{1}{2}\right)}}{2(1)}$ $= \frac{\mathrm{i} \pm \sqrt{-3+4\mathrm{i}}}{2}$ $= \frac{\mathrm{i} \pm (1+2\mathrm{i})}{2}$ $z = \frac{1}{2} + \frac{3}{2}\mathrm{i},\ w = \frac{3}{2} - \frac{1}{2}\mathrm{i}$, or $z = -\frac{1}{2} - \frac{1}{2}\mathrm{i},\ w = \frac{1}{2} - \frac{5}{2}\mathrm{i}$, ---- **Method 2** Substitute (1) into (2): $(w + 2\mathrm{i} - 1)^2 - \mathrm{i}w + \frac{5}{2} = 0$ $w^2 + (2\mathrm{i} - 1)^2 + 2(2\mathrm{i} - 1)w - \mathrm{i}w + \frac{5}{2} = 0$ $w^2 + w(3\mathrm{i} - 2) - \frac{1}{2} - 4\mathrm{i} = 0$ $w = \frac{-(3\mathrm{i} - 2) \pm \sqrt{(3\mathrm{i} - 2)^2 - 4(1)\left(-\frac{1}{2} - 4\mathrm{i}\right)}}{2(1)}$ $w = \frac{-(3\mathrm{i} - 2) \pm (1 + 2\mathrm{i})}{2}$ $w = \frac{3}{2} - \frac{1}{2}\mathrm{i},\ z = \frac{1}{2} + \frac{3}{2}\mathrm{i}$ or $w = \frac{1}{2} - \frac{5}{2}\mathrm{i},\ z = -\frac{1}{2} - \frac{1}{2}\mathrm{i}$
(i)	$z = w - \frac{1}{w} = 2\cos\theta + 2\mathrm{i}\sin\theta - \left(\frac{1}{2}\cos\theta - \frac{1}{2}\mathrm{i}\sin\theta\right) = \frac{3}{2}\cos\theta + \frac{5}{2}\mathrm{i}\sin\theta$

	$\text{Re}(z) = \frac{3}{2}\cos\theta, \quad \text{Im}(z) = \frac{5}{2}\sin\theta$
(ii)	$x = \frac{3}{2}\cos\theta, \quad y = \frac{5}{2}\sin\theta$ $\therefore \cos\theta = \frac{2}{3}x, \quad \sin\theta = \frac{2}{5}y$ Since $\cos^2\theta + \sin^2\theta = 1$ $\left(\frac{2}{3}x\right)^2 + \left(\frac{2}{5}y\right)^2 = 1$ $\frac{x^2}{\left(\frac{3}{2}\right)^2} + \frac{y^2}{\left(\frac{5}{2}\right)^2} = 1$ $\therefore a = \frac{3}{2}, \quad b = \frac{5}{2}$ 

Qn	Solution
11	**Parametric Equations + Applications of Differentiation and Integration**
(i)	$\frac{dx}{dt}=3t^2, \quad \frac{dy}{dt}=-\frac{t}{\sqrt{1-t^2}}$ $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$ $=-\frac{t}{\sqrt{1-t^2}}\times\frac{1}{3t^2}$ $=-\frac{1}{3t\sqrt{1-t^2}}$ At point P, $x=p^3$, $y=\sqrt{1-p^3}$. Equation of tangent at P: $y-\sqrt{1-p^2}=-\frac{1}{3p\sqrt{1-p^2}}(x-p^3)$ $3p(1-p^2)-3py\surd(1-p^2)=x-p^3$ (Shown)
(ii)	Substitute $x=p^3$, $y=\sqrt{1-p^2}$ into $y=4\sqrt{3}\,x$ $\sqrt{1-p^2}=4\sqrt{3}\,p^3$ $1-p^2=48p^6$ $48p^6+p^2-1=0$ Using GC, $p=-\frac{1}{2}$ or $p=\frac{1}{2}$ (N.A. since $0\le p\le 1$) $\therefore$ Exact coordinates of p are $\left(\frac{1}{8},\frac{\sqrt{3}}{2}\right)$.

(iii) Equation of tangent to C at P:

$$3\left(\frac{1}{2}\right)\left[1-\left(\frac{1}{2}\right)^2\right]-\frac{3}{2}y\sqrt{1-\left(\frac{1}{2}\right)^2}=x-\left(\frac{1}{2}\right)^3$$

$$\frac{9}{8}-\frac{3y}{2}\sqrt{\frac{3}{4}}=x-\frac{1}{8}$$

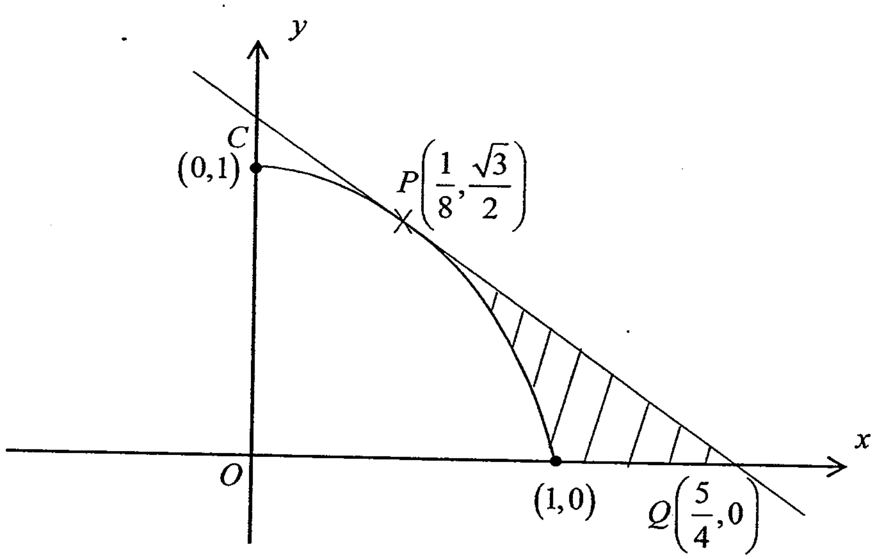
When $y=0$,

$$\frac{9}{8}=x-\frac{1}{8}$$

$$x=\frac{5}{4}$$

$\therefore$ Exact coordinates of Q are $\left(\frac{5}{4},0\right)$.

(iv)



When $x=\frac{1}{8}$, $t=\frac{1}{2}$.

When $x=1$, $t=1$.

Area of shaded region = Area of required region

$$=\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\left(\frac{5}{4}-\frac{1}{8}\right)-\int_{\frac{1}{8}}^{1} y_C \, dx$$

$$=\frac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1}\sqrt{1-t^2}\left(\frac{dx}{dt}\right)dt$$

$$=\frac{9\sqrt{3}}{32}-\int_{\frac{1}{2}}^{1}3t^2\sqrt{1-t^2}\,dt$$

(Shown).

Let $t = \sin u$, $\frac{dt}{du} = \cos u$.

When $t = \frac{1}{2}$, $u = \frac{\pi}{6}$.

When $t = 1$, $u = \frac{\pi}{2}$.

$$\begin{aligned}\frac{9\sqrt{3}}{32} - \int_{\frac{1}{2}}^{1} 3t^2\sqrt{1-t^2}\, dt &= \frac{9\sqrt{3}}{32} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 u\sqrt{1-\sin^2 u}\,\cos u\, du \\ &= \frac{9\sqrt{3}}{32} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 3\sin^2 u\cos^2 u\, du \\ &= \frac{9\sqrt{3}}{32} - \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2\sin u\cos u)^2\, du \\ &= \frac{9\sqrt{3}}{32} - \frac{3}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 2u\, du \\ &= \frac{9\sqrt{3}}{32} - \frac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u\, du\end{aligned}$$

$$\begin{aligned}\frac{9\sqrt{3}}{32} - \frac{3}{8}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1-\cos 4u\, du &= \frac{9\sqrt{3}}{32} - \frac{3}{8}\left[u - \frac{1}{4}\sin 4u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} du \\ &= \frac{9\sqrt{3}}{32} - \frac{3}{8}\left[\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - \left(\frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}\right)\right] \\ &= \frac{9\sqrt{3}}{32} - \frac{3}{8}\left[\frac{\pi}{2} - \frac{\pi}{6} + \frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)\right] \\ &= \frac{9\sqrt{3}}{32} - \frac{3}{8}\left[\frac{\pi}{3} + \frac{\sqrt{3}}{8}\right] \\ &= \frac{9\sqrt{3}}{32} - \frac{3\sqrt{3}}{64} - \frac{\pi}{8} \\ &= \frac{15\sqrt{3}}{64} - \frac{\pi}{8}\end{aligned}$$

JC2 Preliminary Examination

Higher 2

H2 Mathematics 9740/02

Paper 2 **21 September 2016**

3 Hours

Additional Materials: Writing paper
List of Formulae (MF 15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 Referred to the origin O, points A and B have position vectors **a** and **b** respectively, where **a** and **b** are non-zero vectors that are neither perpendicular nor parallel to each other. The length of projection of **a** onto **b** and the length of projection of **b** onto **a** are equal. Show that $|\mathbf{a}| = |\mathbf{b}|$. [2]

Hence state the geometrical interpretation of $|\mathbf{a} \times \mathbf{b}|$. [1]

It is further given that $\mathbf{a} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + p\mathbf{k}$, where $p < 0$.

(i) Find the exact value of p. [2]

(ii) A circle with centre O passes through A and B. Find the area of the minor sector OAB. [4]

2 **(a)** The roots of the equation $z^3 - z^2 - z - 15 = 0$ are denoted by z_1, z_2 and z_3 where $\arg(z_1) = 0$, and $\arg(z_2) > \arg(z_3)$. Find z_1, z_2 and z_3 and show these roots on an Argand diagram. [3]

Explain why the locus of all points z such that $|z+1| = 2$ passes through the roots represented by z_2 and z_3 Draw this locus on the same Argand diagram. [3]

(b) **(i)** Show that $1 + e^{i\theta} = 2e^{i\frac{\theta}{2}} \cos\frac{\theta}{2}$. [2]

(ii) Hence find, in trigonometric form, the imaginary part of the complex number

$$w = \frac{e^{i\theta}}{1+e^{i\theta}}.$$ [2]

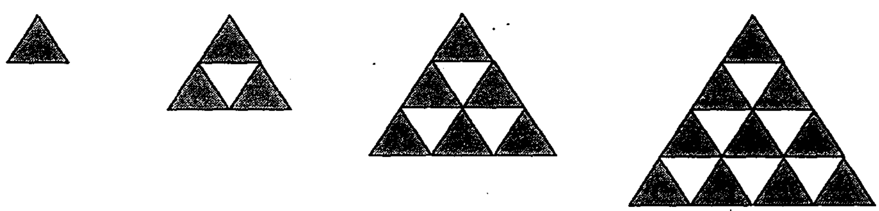
3 Newton's law of cooling states that the rate of decrease of temperature of a hot body is proportional to the difference in temperature between the body and its surroundings. Using t for time in minutes, θ for temperature of the body in °C and α for the temperature of the surroundings (assumed constant), express the law in the form of a differential equation. [1]

(i) Show that the general solution of the differential equation may be expressed in the form $\theta = \alpha + Ae^{-kt}$ where A and k are constants. [3]

(ii) Given that $\theta = 9\alpha$ when $t = 0$ and that $\theta = 5\alpha$ when $t = T$, find, in terms of T, the value of t when $\theta = 2\alpha$. [3]

(iii) State what happens to θ for large values of t and sketch the solution curve of θ against t. [3]

4 **(a)** Judith is making a pattern consisting of rows of matchstick triangles as shown. She uses three matchsticks to complete a triangle. She adds two more triangles in the second row, three more triangles in the third row and four more triangles in the fourth row.



Judith has completed $n - 1$ rows in the pattern. How many matchsticks does she need in order to form the n^{th} row? [1]

Show that the total number of matchsticks used in making a pattern with n rows is $\frac{3n(n+1)}{2}$. Hence find the maximum number of complete rows she is able to make with two thousand matchsticks. [4]

(b) A geometric progression has first term a and second term b, where a and b are non-zero constants. Given that the sum to infinity of the series is $a+2b$, find the common ratio. [2]

The sum of the first n terms is denoted by G_n. Find G_n in terms of a and n. Hence show that $\sum_{n=1}^{N} G_n = 2aN - G_N$. [4]

Section B: Statistics [60 marks]

5 **(i)** Describe what is meant by 'systematic sampling'. [2]

(ii) A bakery wishes to gather feedback on what residents in the neighbourhood think of its new salted egg lava buns. A surveyor is hired to survey a sample of 150 residents who visit the bakery during the evening rush hour using systematic sampling. State, in this context, one advantage and one disadvantage of this procedure. [2]

6 A box consists of a very large number of balls, of which 20% are red and 80% are white.

A game consists of a player drawing n balls at random from the box and counting the number of red balls drawn. If at most one red ball is drawn, the player wins. If more than two red balls are drawn, the player loses. If exactly two red balls are drawn, the player draws another n balls and if none of these n balls drawn are red, the player wins. Otherwise, the player loses.

Show that the probability that a randomly chosen player wins is P where

$$P=(0.8+0.2n)(0.8)^{n-1}+\binom{n}{2}(0.2)^2(0.8)^{2n-2}.$$ [3]

(i) Given that the probability that a randomly chosen player wins is less than 0.1, write down an inequality in terms of n to represent this information. Hence find the least possible value of n. [3]

(ii) Given instead that $P=0.3$, find the probability that out of 100 games played, at least 40 games are won. [2]

7 An overseas study revealed that school children sleep an average of 6.5 hours each night. Ms Patricia believes that the children in her school sleep even fewer than that. She took a random sample of 8 children from her school. The number of hours of sleep each child gets at night was reported as:

5.9 6 6.1 6.2 6.3 6.5 6.7 6.9

Test, at the 8% level of significance, whether this evidence supports Ms Patricia's belief, stating clearly any assumption made. [5]

Ms Patricia conducted a further study involving a random sample of 15 children from another school and the number of hours of sleep each child gets at night is recorded. The sample mean is $\bar{x}$ and the sample variance is 0.849. Find the set of values of $\bar{x}$ for which the null hypothesis would be rejected at the 8% level of significance. [3]

8 The mass of a randomly chosen bar of body soap manufactured by a factory has a normal distribution with mean 110 grams and standard deviation 1.5 grams.

(i) Find the probability that the difference in sample means between any two random samples of 20 bars of body soap each, is within 0.5 grams. [4]

(ii) Five randomly chosen bars of body soap are liquefied and separated into four equal portions, which are each placed into a bottle. Find the probability that the mass of liquid body soap in a randomly chosen bottle exceeds 140 grams. [4]

The factory ventured into the manufacturing of coconut oil soap as its new product and the mass of a randomly chosen bar of coconut oil soap has a normal distribution. A random sample of 15 bars of coconut oil soap is taken and the mass, u grams, of each bar is measured. The results are summarised by $\sum u = 1590$, $\sum u^2 = 169046$.

(iii) Find unbiased estimates of population mean and variance. [2]

9 **(a)** For events A and B, it is given that $\mathrm{P}(A)=\frac{1}{4}$ and $\mathrm{P}(B)=\frac{1}{2}$.

(i) Given that $\mathrm{P}(A'\mid B)=\frac{3}{4}$, determine whether events A and B are independent and calculate $\mathrm{P}(A\cup B)$. [3]

(ii) For a third event C, it is given that $\mathrm{P}(C\mid A)=\frac{2}{3}$. Find the value of $\mathrm{P}(A\cap C)$. [2]

(b) Find the number of ways in which the word EVERYDAY can be arranged if

(i) all the vowels (A, E) must be together and the two 'Y's must be separated, [3]

(ii) the repeated letters E and Y must appear symmetrical about the centre of the word (e.g. EVRYYDAE, YVERDEAY). [2]

10 **(i)** A bakery sells cookies in tins and keeps track of the number of tins sold per week. State two conditions under which a Poisson distribution would be a suitable probability model for the number of tins sold in a week. [2]

(ii) Two types of cookies, chocolate and raisin, are sold. The mean number of tins for chocolate cookies sold in a week is 2.4. The mean number of tins for raisin cookies sold in a week is 1.8. Use a Poisson distribution to find the probability that in a given week, the total number of tins sold is more than 9. [3]

(iii) Use a normal approximation to the Poisson distribution to find the probability that the total number of tins sold in 4 weeks is at least 15 but not more than 25. [4]

(iv) Explain why the Poisson distribution may not be a good model for the number of cookies sold in a year. [1]

11 The table gives the population y, in thousands, for a particular species of mammal over 10 years.

Year, x	1	2	3	4	5	6	7	8	9	10
Population, y (in thousands)	10.8	8.7	6.9	5.5	4.4	3.5	2.8	2.3	1.8	1.4

(i) Find the equation of the regression line of y on x, giving your answer to 3 decimal places. [1]

(ii) Let Y be the value obtained by substituting a value of x into the equation of the regression line of y on x found in **(i)**. Find $\sum(y-Y)^2$. [1]

(iii) For each of the values of x, Y' is given by $Y' = A + Bx$, where A and B are any constants. What can you say about the value of $\sum(y-Y')^2$? [1]

(iv) Draw a scatter diagram to illustrate the data. [2]

An animal conservationist suggested the model $\ln y = c + dx$ for this set of data.

(v) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(a) x and y,

(b) x and $\ln y$. [2]

(vi) Use your answers to parts **(iv)** and **(v)** to explain which of $y = a + bx$ or $\ln y = c + dx$ is the better model. [1]

(vii) Using the better model found in **(vi)**, predict the population of this species in the 20th year. [2]

END OF PAPER

Qn	Solution
1	**Vectors**

$$\frac{\left|\mathbf{a} \square \mathbf{b}\right|}{\left|\mathbf{b}\right|} = \frac{\left|\mathbf{b} \square \mathbf{a}\right|}{\left|\mathbf{a}\right|}$$

$\left|\mathbf{a}\right| = \left|\mathbf{b}\right|$ as $\left|\mathbf{a} \square \mathbf{b}\right| \neq 0$ (as $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular)

$\left|\mathbf{a} \times \mathbf{b}\right|$ is the area of rhombus with adjacent sides OA and OB.

(i)

$\left|\mathbf{a}\right| = \left|\mathbf{i} - \mathbf{j} + 3\mathbf{k}\right| = \sqrt{11}$ $\qquad$ $\left|\mathbf{b}\right| = \left|-2\mathbf{i} + \mathbf{j} + p\mathbf{k}\right| = \sqrt{5 + p^2}$

As $\left|\mathbf{a}\right| = \left|\mathbf{b}\right|$,

$\Rightarrow 11 = 5 + p^2$

$\Rightarrow p^2 = 6$

Since $p < 0, p = -\sqrt{6}$

(ii)

Let θ denote angle AOB.

$$\cos\theta = \frac{\mathbf{a} \square \mathbf{b}}{\left|\mathbf{a}\right|\left|\mathbf{b}\right|}$$

$$= \frac{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \square \begin{pmatrix} -2 \\ 1 \\ -\sqrt{6} \end{pmatrix}}{\left(\sqrt{11}\right)\left(\sqrt{11}\right)}$$

$$= \frac{-3 - 3\sqrt{6}}{11}$$

$$\theta = \cos^{-1}\left(\frac{-3 - 3\sqrt{6}}{11}\right) = 2.79569 \text{ rad} \quad \text{or} \quad 160.18126 \text{ degrees}$$

Area of minor sector OAB

$$= \frac{1}{2}r^2\theta \qquad\qquad = \left(\frac{\theta}{360}\right)\left(\pi r^2\right)$$

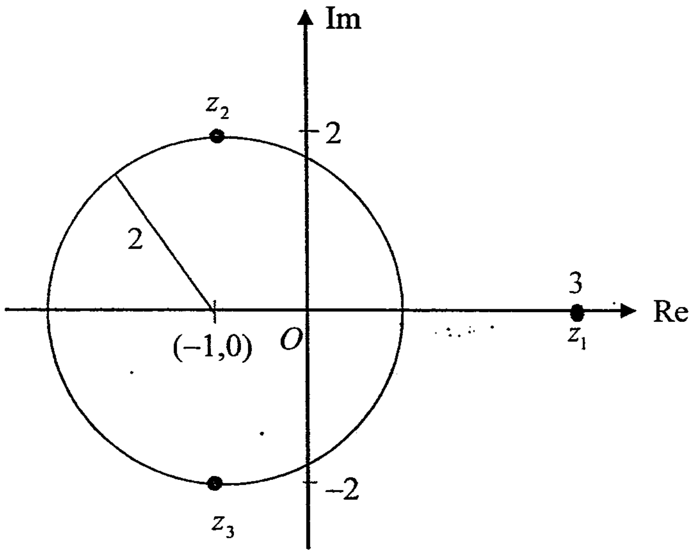
$$= \frac{1}{2}(11)(2.79569) \qquad \text{OR} \qquad = \left(\frac{160.18126}{360}\right)(11\pi)$$

$$= 15.4 \text{ units}^2 \qquad\qquad = 15.4 \text{ units}^2$$

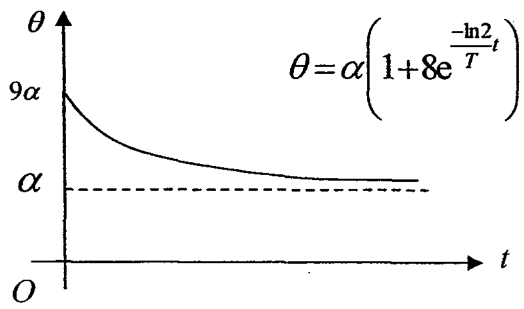
$$= \left(\frac{2.79569}{2\pi}\right)\left(\pi r^2\right)$$

OR $$= \left(\frac{2.79569}{2\pi}\right)(11\pi)$$

$$= 15.4 \text{ units}^2$$

Qn	Solution
2	**Complex Numbers**
(a)	Using GC, $z_1=3,\ z_2=-1+2\mathrm{i},\ z_3=-1-2\mathrm{i}.$  The locus of points given by $\lvert z+1\rvert=2$ passes through the roots represented by z_2 and z_3 since $\lvert z_2+1\rvert=\lvert -1+2\mathrm{i}+1\rvert=\lvert 2\mathrm{i}\rvert=2$ and $\lvert z_3+1\rvert=\lvert -1-2\mathrm{i}+1\rvert=\lvert -2\mathrm{i}\rvert=2$.
(b) (i)	$1+\mathrm{e}^{\mathrm{i}\theta}=\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}+\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right)$ $=\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\cos\frac{\theta}{2}-\mathrm{i}\sin\frac{\theta}{2}+\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)$ $=\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(2\cos\frac{\theta}{2}\right)=2\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\cos\frac{\theta}{2}$
(ii)	$w=\frac{\mathrm{e}^{\mathrm{i}\theta}}{1+\mathrm{e}^{\mathrm{i}\theta}}$ $=\frac{\mathrm{e}^{\mathrm{i}\theta}}{2\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\cos\frac{\theta}{2}}=\frac{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}}{2\cos\frac{\theta}{2}}$ $=\frac{\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}=\frac{1}{2}+\frac{1}{2}\mathrm{i}\tan\frac{\theta}{2}$ $\therefore \mathrm{Im}(w)=\frac{1}{2}\tan\frac{\theta}{2}$

Qn	Solution
3	**Differential Equations**
	$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-\alpha)$, k is a positive constant
(i)	$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-k(\theta-\alpha), k>0$ $\int\frac{1}{\theta-\alpha}\,\mathrm{d}\theta=\int -k\,\mathrm{d}t$ $\ln(\theta-\alpha)=-kt+c$ since $\theta>\alpha$ $\theta-\alpha=\mathrm{e}^{-kt+c}$ $\theta-\alpha=A\mathrm{e}^{-kt}, A=\mathrm{e}^{c}$ $\theta=\alpha+A\mathrm{e}^{-kt}$ (shown)
(ii)	When $t=0, \theta=9\alpha$ $\therefore 9\alpha=\alpha+A \quad \therefore A=8\alpha$ When $t=T, \theta=5\alpha$ $5\alpha=\alpha+8\alpha\mathrm{e}^{-kT}$ $\mathrm{e}^{-kT}=\frac{1}{2}$ $kT=\ln 2$ $k=\frac{\ln 2}{T}$ $\theta=\alpha+8\alpha\mathrm{e}^{\frac{-\ln 2}{T}t}$ $\theta=\alpha\left(1+8\mathrm{e}^{\frac{-\ln 2}{T}t}\right)$ When $\theta=2\alpha$ $2\alpha=\alpha\left(1+8\mathrm{e}^{\frac{-\ln 2}{T}t}\right)$ $\mathrm{e}^{\frac{-\ln 2}{T}t}=\frac{1}{8}$ $-\frac{\ln 2}{T}t=-\ln 8$ $t=\frac{\ln 8}{\ln 2}T=3T$

(iii)	For large values of t, $e^{\frac{-\ln 2}{T}t} \to 0, \theta \to \alpha$, θ decreases and approaches to α. 

Qn	Solution
4	**APGP + Summation**
(a)	1^{st} row: number of matches = 3 2^{nd} row: number of matches = 6 3^{rd} row: number of matches = 9 n^{th} row: number of matches = $3+(n-1)(3)=3n$
	1 row: total number of matches = 3 2 rows: total number of matches = 3 + 6 3 rows: total number of matches = 3 + 6 + 9 n rows: total number of matches $=\frac{n}{2}(3+3n)$ $=\frac{3n(n+1)}{2}$ (shown) OR $=\frac{n}{2}[2(3)+(n-1)(3)]$ $=\frac{3n(n+1)}{2}$ (shown) $\frac{3n(n+1)}{2}\le 2000$ Using GC, When $n=36$, $\frac{3n(n+1)}{2}=1998<2000$ When $n=37$, $\frac{3n(n+1)}{2}=2109>2000$ Maximum number of complete rows = 36.
(b)	Let $r=\frac{b}{a}$ $\frac{a}{1-r}=a+2b$ $\frac{1}{1-r}=1+2r$ $1=1+r-2r^2$ $2r^2-r=0$ $r(2r-1)=0$ $r=0$ (rejected $\frac{b}{a}\ne 0$) or $r=\frac{1}{2}$ $\therefore$ common ratio = $\frac{1}{2}$ $G_n=\frac{a\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ $G_n=2a\left(1-\left(\frac{1}{2}\right)^n\right)$

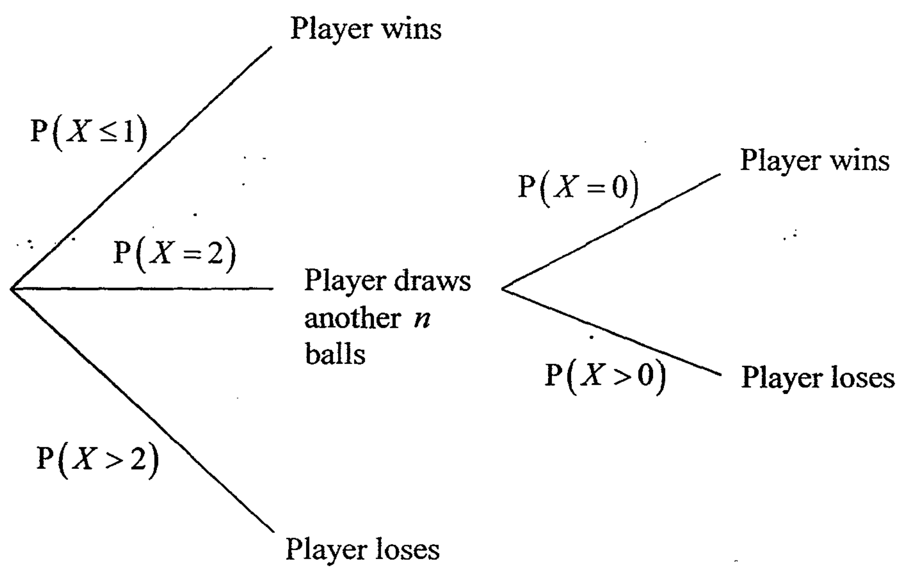
	$\sum_{n=1}^{N} G_n = 2a\sum_{n=1}^{N}\left(1-\left(\frac{1}{2}\right)^n\right)$ $=2a\left[N-\sum_{n=1}^{N}\left(\frac{1}{2}\right)^n\right]$ $=2a\left[N-\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^N\right)}{1-\frac{1}{2}}\right]$ $=2a\left(N-\left(1-\left(\frac{1}{2}\right)^N\right)\right)$ $=2aN-2a\left(1-\left(\frac{1}{2}\right)^N\right)$ $=2aN-G_N$

Qn	Solution
5	**Sampling Methods**
(i)	Systematic sampling is a sampling method in which the **entire population is listed in some order**. The population is **divided into sampling intervals of *k* members.** After obtaining **a random starting point from the first *k* members, every k^{th} member is chosen** from the list until the required number is achieved.
(ii)	Possible Advantages: • It is easy to conduct the survey as the members of the sample are easily accessible. • It is easy to conduct as the surveyor does not need the list of all the residents in the neighbourhood. Possible Disadvantages: • It is a biased sample as only residents who visit the bakery during the evening rush hour is surveyed. Hence the sample may not be representative. • It is a biased sample as some people may visit the bakery multiple times during the evening rush hours increasing their chances to be selected. • It may not be easy to get residents to visit the bakery in sequence so selection of every k^{th} resident in this case may be difficult.

Qn	Solution
6	**Binomial Distribution**

Let X be the number of red balls drawn, out of n balls.

$X \sim \mathrm{B}(n,p)$



$$\begin{aligned}
P &= \mathrm{P}(\text{player wins}) \\
&= \mathrm{P}(X \le 1) + \mathrm{P}(X=2)\mathrm{P}(X=0) \\
&= \mathrm{P}(X=0) + \mathrm{P}(X=1) + \mathrm{P}(X=2)\mathrm{P}(X=0) \\
&= \binom{n}{0}(0.2)^0(0.8)^n + \binom{n}{1}(0.2)^1(0.8)^{n-1} + \left[\binom{n}{2}(0.2)^2(0.8)^{n-2}\right]\left[\binom{n}{0}(0.2)^0(0.8)^n\right] \\
&= (0.8)^n + n(0.2)(0.8)^{n-1} + \left[\binom{n}{2}(0.2)^2(0.8)^{n-2}\right](0.8)^n \\
&= (0.8+0.2n)(0.8)^{n-1} + \binom{n}{2}(0.2)^2(0.8)^{2n-2}
\end{aligned}$$

(Shown)

(i)

$$P = (0.8+0.2n)(0.8)^{n-1} + \binom{n}{2}(0.2)^2(0.8)^{2n-2} < 0.1$$

Using G.C.,

When $n=18$, $P=0.10218>0.1$
When $n=19$, $P=0.08509<0.1$

$\therefore$ least $n=19$

(ii)

Let Y be the number of games won, out of 100 games played.
$Y \sim \mathrm{B}(100, 0.3)$.

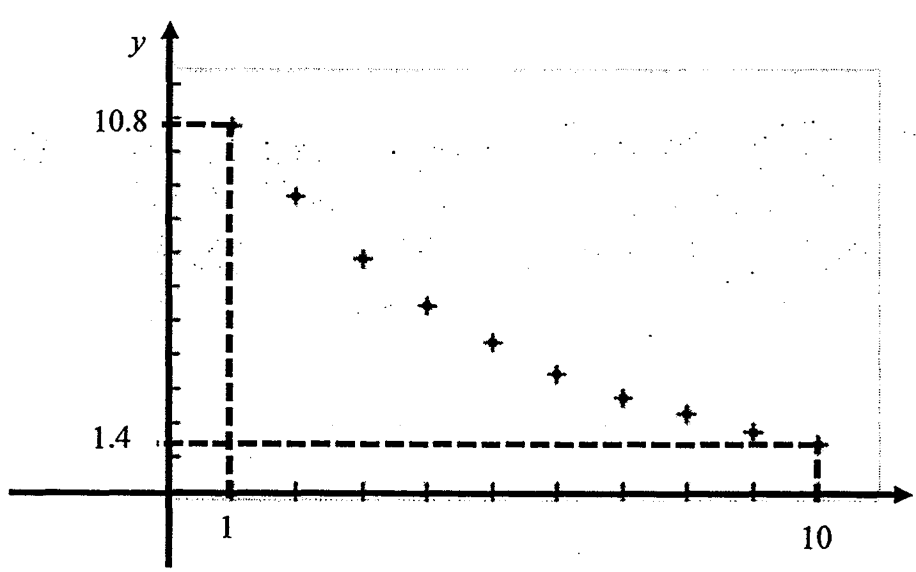
	Required probability $= \mathrm{P}(Y \geq 40)$ $= 1 - \mathrm{P}(Y \leq 39)$ $= 0.020988$ ≈ 0.0210 (3 s.f.)

Qn	Solution
7	**Hypothesis Testing**
	Let X denote the number of hours of sleep each child gets at night. Let μ denote the population mean hours of sleep each child gets at night. Assumption: $X \sim \mathrm{N}(\mu, \sigma^2)$ $\therefore \overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$. $\mathrm{H_0}$: $\mu = 6.5$ $\mathrm{H_1}$: $\mu < 6.5$ Test statistic : $T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$ Level of Significance: 8% Reject $\mathrm{H_0}$ if p-value < 0.08 Under $\mathrm{H_0}$, using GC, p-value $= 0.0998$ Since p-value $= 0.0998 > 0.08$, we do not reject $\mathrm{H_0}$ and conclude that there is insufficient evidence at 8% level of significance, that supports Ms Patricia's claim. $s^2 = \frac{15}{14}(0.849) = 0.90964$ (5 s.f) Level of Significance: 8% Reject $\mathrm{H_0}$ if t-value < -1.48389 $\frac{\overline{x} - 6.5}{\sqrt{0.90964}/\sqrt{15}} < -1.48389$ $\overline{x} < 6.13458$ $\therefore$ set of values of $\overline{x} = \{\overline{x} \in \mathbb{R} : 0 < \overline{x} < 6.13\}$

Qn	Solution
8	**Normal Distribution**
(i)	Let X be the mass of a randomly chosen bar of body soap in grams. Let $\overline{X}$ be the sample mean mass of 20 randomly chosen bars of body soaps in grams. $X \sim \mathrm{N}\left(110, 1.5^2\right)$ $\overline{X} \sim \mathrm{N}\left(110, \frac{1.5^2}{20}\right)$ $\overline{X}_1 - \overline{X}_2 \sim \mathrm{N}\left(0, 2\left(\frac{1.5^2}{20}\right)\right)$ $P\left(\left\|\overline{X}_1 - \overline{X}_2\right\| \le 0.5\right) = P\left(-0.5 \le \overline{X}_1 - \overline{X}_2 \le 0.5\right)$ $= 0.708 \text{ (3 s.f.)}$
(ii)	Let W be the mass of a portion of liquefied soap. $W = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{4}$ $W \sim \mathrm{N}\left(\frac{(5)(110)}{4}, \frac{(5)(1.5^2)}{4^2}\right)$ $W \sim \mathrm{N}\left(\frac{275}{2}, \frac{45}{64}\right)$ $\mathrm{P}(W > 140) = 0.00143 \text{ (3 s.f.)}$
(iii)	Unbiased estimate of population mean, $\bar{u} = \frac{\sum u}{n} = \frac{1590}{15} = 106$ Unbiased estimate of population variance, $s^2 = \frac{1}{n-1}\left(\sum u^2 - \frac{\left(\sum u\right)^2}{n}\right)$ $= \frac{1}{15-1}\left(169046 - \frac{(1590)^2}{15}\right)$ $= 36.1 \text{ (3 s.f.)}$ OR $\frac{253}{7}$

Qn	Solution
9	**Probability**
(ai)	Since $\mathrm{P}(A' \mid B)=\frac{3}{4}=\mathrm{P}(A')$, A' and B are independent events $\Rightarrow A$ and B are independent events $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$ $=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A)\,\mathrm{P}(B)$ (Since A and B are independent) $=\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\cdot\frac{1}{2}$ $=\frac{5}{8}$
(ii)	$\mathrm{P}(C \mid A)=\frac{\mathrm{P}(C\cap A)}{\mathrm{P}(A)}=\frac{2}{3}$ $\Rightarrow \mathrm{P}(A\cap C)=\frac{2}{3}\mathrm{P}(A)$ $=\frac{2}{3}\cdot\frac{1}{4}=\frac{1}{6}$
(bi)	Required number of ways $=4!\times\frac{3!}{2!}\times{}^{5}C_{2}$ $=720$ Choose 2 out of 5 slots to put 'Y's Arrange vowels. (AEE) V R D Group vowels together, and arrange these 4 groups.
(ii)	Required number of ways $={}^{4}C_{2}\times 2!\times 4!$ $=288$ 3. The 4 remaining letters can permute themselves within the 4 remaining blanks. _ E Y _ | _ Y E _ 1. Choose 2 out of the 1st 4 blanks to put E & Y. $\times 2!$ Because E & Y can switch positions. 2. Correspondingly, the remaining E & Y would take their respective positions in the next 4 blanks.

Qn	Solution
10	**Poisson Distribution**
(i)	The average number of tins sold per week is constant. The sale of one tin is independent of another throughout the week.
(ii)	Let X be the number of tins for chocolate cookies sold in a week. $X \sim \text{Po}(2.4)$ Let Y be the number of tins for raisin cookies sold in a week. $Y \sim \text{Po}(1.8)$ $X+Y \sim \text{Po}(4.2)$ $P(X+Y>9)=1-P(X+Y\le 9)=0.0111$ (3 s.f.)
(iii)	Let W be the total number of tins sold in 4 weeks. $W \sim \text{Po}(16.8)$ Since $\lambda=16.8>10$, $\therefore W$ □ $\text{N}(16.8,16.8)$ approximately. $\text{P}(15\le W\le 25)=\text{P}(14.5<W<25.5)$ after continuity correction $=0.69576=0.696$ (3 s.f.)
(iv)	The mean number of tins sold per week might not be constant from one week to another because of seasonal fluctuations such as sales and holidays.

Qn	Solution
11	**Correlation and Regression**
(i)	Using GC, $y=10.30667-0.99939x$ $y=10.307-0.999x$
(ii)	Using GC, $\sum(y-Y)^2=6.3689=6.37$ (to 3 s.f.)
(iii)	$\sum(y-Y')^2\ge 6.37$
	

(v)	For x and y, $r = -0.9635$ For x and $\ln y$, $r = -0.9999$
(vi)	Since the scatter diagram shows that the population decreases at a decreasing rate as the years pass and the r value for the model $\ln y = c + dx$ is closer to -1 than that of $y = a + bx$, $\ln y = c + dx$ is the better model.
(vii)	Using GC, when $x = 20, \ln y = -1.89587 \Rightarrow y = 0.150187$. The population in 20th year will be 150.