SAINT ANDREW'S JUNIOR COLLEGE

Preliminary Examination

MATHEMATICS
Higher 1 **8865**

Thursday **24th August 2017** **3 hours**

Additional materials : Answer paper
List of Formulae(MF26)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks : **100**
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically state otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematic steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **6** printed pages including this page.
[Turn over]

Section A: Pure Mathematics [40 marks]

1 Find, algebraically, the set of values of k for which $kx^2+(k-1)x+\frac{9}{k}>0$ for all real values of x. [4]

2 Solve the simultaneous equations,

$$5^{x+2}=125\left(5^{y-5}\right),$$

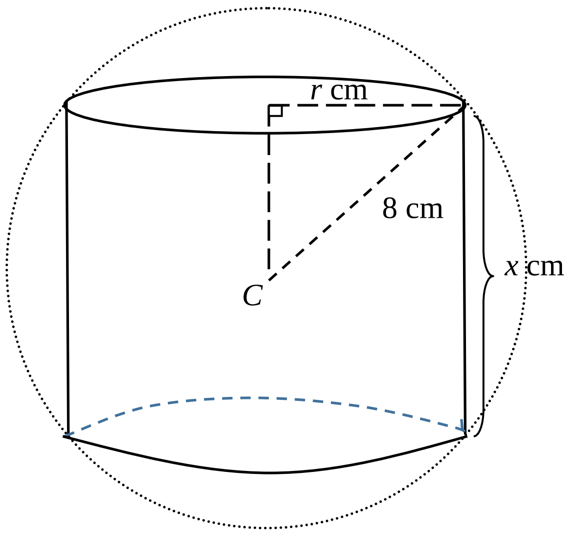
$$\log_{\sqrt{3}}(2x-y)=4+2\log_{\sqrt{3}}2.$$ [4]

3 (a) Differentiate with respect of x, simplifying your answers,

(i) $\dfrac{1}{\sqrt{x^2-3x+1}}$, [2]

(ii) $\ln\left(\dfrac{x^2+3x+2}{x^2+4x+3}\right)$. [3]

(b) In the diagram below, a solid right cylinder of height x cm and radius r cm is inscribed in a sphere of centre C with radius 8 cm. The circumference of the top and bottom circular surfaces of the cylinder are in contact with the sphere as shown in the figure below.



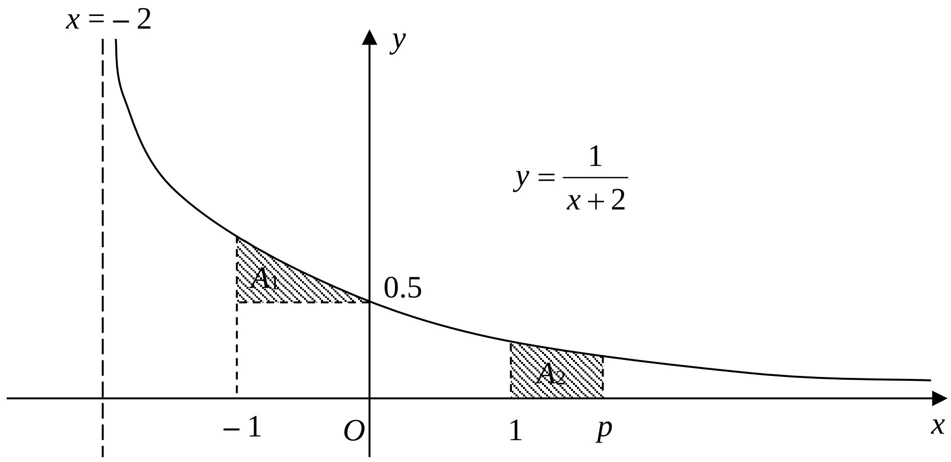
(i) Show that the volume of the cylinder is

$$V=\frac{\pi}{4}\left(256x-x^3\right)\ .$$ [2]

(ii) Hence, find the exact maximum value of V as x varies. [4]

4 (a) Find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\left(5\mathrm{e}^{2x}+1\right)^4\right)$. Hence, without the use of a calculator, find $\int_0^2 \mathrm{e}^{2x}\left(5\mathrm{e}^{2x}+1\right)^3 \mathrm{d}x$. [4]

(b) The diagram below, not drawn to scale, shows the curve C with equation $y=\frac{1}{x+2}$, $x>-2$. The region A_1 is bounded by C, the lines $x=-1$ and $y=0.5$. The region A_2 is bounded by C, the lines $x=1$, $x=p$ and the x-axis.



Find the exact value of p such that the regions A_1 and A_2 have equal areas. [5]

5 A group of environmentalists conducted a research project to study how the population of the monkeys in a forest could be affected by a newly set-up factory nearby. The population P, of monkeys in the forest, after t months of the opening of the factory, can be modelled by

$$P=500\,(3+\mathrm{e}^{-0.2t}).$$

(i) Find the initial population of the monkeys in the forest. [1]

(ii) Find, to the nearest whole number, the population of monkeys after two years. [1]

(iii) **Without the use of a calculator,** determine the number of complete months for the population of monkeys to first fall below 1515. [4]

(iv) Using this model, describe what will happen to the population of monkeys in the forest in the long run. [2]

(v) Sketch, in the context of this question, the graph of P against t. [3]

(vi) Suggest a possible limitation of this model to represent the population of monkeys in the real world. [1]

[Turn Over]

Section B: Statistics [60 marks]

6 Thomas has six tiles, each with a different letter of his name on it. Thomas randomly arranges these letters in a line. Find the probability that the six tiles are arranged

(a) in the correct order that spells his name. [2]

(b) such that the vowels are separated. [2]

(c) such that the vowels are at the two ends. [2]

7 (a) Eight pairs of values of variables x and y are measured. Draw a sketch of a possible scatter diagram of the data for each of the following cases:

(i) the product moment correlation coefficient is approximately -0.9, [1]

(ii) the product moment correlation coefficient is approximately zero. [1]

(b) A researcher recorded the water temperature T, in ^{o}C, and the depth D, in metres, at noon on a certain day at each of the eight locations in a lake. The results are summarized in the table below.

D (m)	10	50	80	120	200	250	340	400
T (^{o}C)	25.0	23.0	22.2	k	16.4	12.4	10.0	4.0

(i) It is known that the regression line of T on D is given by $T = -0.051424D + 25.908$. Show that the value of k is 19.7, correct to 1 decimal place. [1]

(ii) Give a sketch of the scatter diagram for the data. [2]

(iii) Calculate the product moment correlation coefficient for the revised data and comment on its value in the context of the question. [2]

(iv) Sketch the regression line T on D on your scatter diagram. [1]

(v) Hence, estimate the water temperature when the depth of the water is 550 metres. Comment on the reliability of this estimate. [1]

(vi) Given that 1 kilometre = 1000 metre, rewrite your equation from part **(i)** so that it can used to estimate the temperature of the water when the height is given in kilometres.
State the value of the regression coefficient. [2]

8 James plays a game by first throwing a biased four-sided die which has faces numbered 1, 2, 3 and 4. Let X denote the number obtained when the die is tossed once. The probability that x is shown on the die is given by $\mathrm{P}(X = x) = \dfrac{x}{10}$, where x = 1, 2, 3 and 4.

If the number obtained is 3 or 4, he records the number shown as his score.
If the number obtained is 1 or 2, he throws a fair six-sided die and records the sum of the two numbers from his two throws as his score.

(i) Draw a tree diagram to represent all possible outcomes. [2]

Events A and B are defined as follows:

Event A: James' score is at least a 4,
Event B: James' score is an odd number,

(ii) Show that $\mathrm{P}(A) = \dfrac{19}{30}$. [2]

(iii) Find $\mathrm{P}(A|B')$. [4]

(iv) State with a reason, whether the events A and B are mutually exclusive. [2]

9 A wholesaler sells two types of fruit, Type A and Type B. The masses of each type of fruit follow independent normal distributions with the following means and standard deviations:

Type	Mean / kg	Standard Deviation / kg
Type A	1.5	0.1
Type B	1.8	0.2

The wholesale price of Type A and Type B fruit are \$30 per kg and \$35 per kg respectively.

(i) Find the probability that the total mass of three Type A fruits exceeds 5 kg. [3]

(ii) Find the probability that the difference between the total mass of three Type A fruits and twice the mass of a Type B fruit is at least 0.5 kg. [3]

(iii) The total cost of buying two Type A fruits and one Type B fruit exceeds \$$k$ with a probability of 0.1. Determine the value of k. [3]

(iv) Two Type A fruits are selected at random. Determine the probability that only one of them weighs more than 1.3 kg. [2]

[Turn Over]

10 A pottery manufacturer makes teapots in batches of 30. On average, 8% of teapots are faulty.

(i) State, in context of this question, one assumption needed to model the number of teapots that are faulty by a binomial distribution. [1]

(ii) Find the probability that in a batch of 30 there is more than two faulty teapots. [2]

(iii) The manufacturer produces 240 batches of 30 teapots in one month. Find the expected number of batches which contain exactly one faulty teapot. [3]

(iv) 50 batches of 30 teapots each are randomly chosen. Find the probability that the mean number of faulty teapots in a batch is more than 2.5. [3]

1 1 A company packs and supplies salt in small packets. The mass of salt in one packet is denoted by x grams. The company claims that the mean mass of salt is at least 10 grams. To test this claim, a sample of 100 packets of salt is randomly chosen. Their masses are summarised by

$$\sum x = 970 \quad \sum x^2 = 9800$$

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) State, with a reason, whether it is necessary to assume a normal distribution for the test to be valid. [1]

(iii) Carry out a test at the 10% significance level whether the company's claim is valid. [4]

(iv) Explain what is meant by the phrase '10% level of significance' in the context of this question. [1]

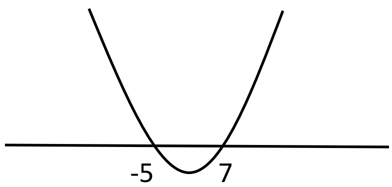
(v) Explain the meaning of p-value in this context. [1]

The company introduces a new packaging system and the new population variance is known to be 0.9^2 grams^2. A new random sample of 30 packets of salt is chosen and the mean of this sample is m grams. A test at the 5% significance level indicates that the company's initial claim is valid for this improved process.

(iv) Find the least possible value of m, giving your answer correct to 2 decimal places. [4]

End of Paper

2017 SAJC Preliminary Examination Solutions

1	$kx^2+(k-1)x+\dfrac{9}{k}>0$ Note that $k\neq 0$. For the quadratic expression to be positive, the following 3 conditions must be met: (1) Discriminant < 0 and (2) $k>0$ $(k-1)^2-4(k)\left(\dfrac{9}{k}\right)<0$ $k^2-2k+1-36<0$ $k^2-2k-35<0$ $(k+5)(k-7)<0$  $-5<k<7$ Combining (1) and (2), the set of values of k is $\left\{k : k\in\mathbb{R}, 0<k<7\right\}$
2	$5^{x+2}=5^3(5^{y-5})$ $x+2=3+y-5$ $x-y=-4$ $x=y-4$ ——————————————————(1)

	$\log_{\sqrt{3}}(2x-y)-2\log_{\sqrt{3}}2=4$ $\log_{\sqrt{3}}(2x-y)-\log_{\sqrt{3}}4=4$ $\log_{\sqrt{3}}\frac{(2x-y)}{4}=4$ $\frac{(2x-y)}{4}=3^{\frac{4}{2}}$ $2x-y=4\times 9$ $2x-y=36$ ------------ (2) Sub. (1) into (2) $2(y-4)-y=36$ $y=44$ $x=40$
3	(a) (i) Let $y=\frac{1}{\sqrt{x^2-3x+1}}=\left(x^2-3x+1\right)^{-\frac{1}{2}}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}\left(x^2-3x+1\right)^{-\frac{3}{2}}(2x-3)$ $=-\frac{2x-3}{2\sqrt{\left(x^2-3x+1\right)^3}}$ (ii) Let $y=\ln\left(\frac{x^2+3x+2}{x^2+4x+3}\right)=\ln\frac{(x+1)(x+2)}{(x+1)(x+3)}=\ln\frac{x+2}{x+3}$ $=\ln(x+2)-\ln(x+3)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x+2}-\frac{1}{x+3}$ $=\frac{1}{(x+2)(x+3)}$ Alternatively,

$$y=\ln\left(\frac{x^2+3x+2}{x^2+4x+3}\right)=\ln(x^2+3x+2)-\ln(x^2+4x+3)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x+3}{x^2+3x+2}-\frac{2x+4}{x^2+4x+3}$$

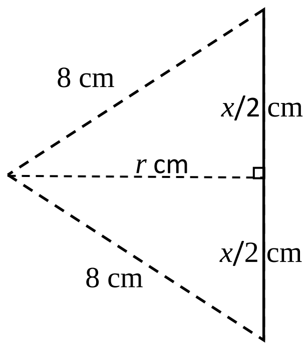
$$=\frac{2x+3}{(x+1)(x+2)}-\frac{2x+4}{(x+1)(x+3)}$$

$$=\frac{(2x+3)(x+3)-(2x+4)(x+2)}{(x+1)(x+2)(x+3)}$$

$$=\frac{x+1}{(x+1)(x+2)(x+3)}$$

$$=\frac{1}{(x+2)(x+3)}$$

(b)(i) By Pythagoras's Theorem,



the radius of the cylinder is $r=\sqrt{8^2-\left(\frac{x}{2}\right)^2}$

$$=\sqrt{64-\frac{x^2}{4}}=\sqrt{\frac{256-x^2}{4}}=\frac{\sqrt{256-x^2}}{2}$$

Volume of cylinder is

$$V=\pi\left(\sqrt{\frac{256-x^2}{4}}\right)^2(x)$$

$$=\frac{\pi x\left(256-x^2\right)}{4}$$

$$=\frac{\pi}{4}\left(256x-x^3\right)$$

(ii) $V=\frac{\pi}{4}\left(256x-x^3\right)$

$$\frac{dV}{dx}=\frac{\pi}{4}\left(256-3x^2\right)$$

When $\frac{dV}{dx} = 0$, $x^2 = \frac{256}{3}$

Since the height of the cylinder, x, is positive,

$$x = \sqrt{\frac{256}{3}} \quad (= 9.2376)$$

$$\frac{d^2V}{dx^2} = \frac{\pi}{4}(-6x) = -43.5 < 0$$ for positive value of x.

Alternative Method to Check for Maximum

x	9.2	$x = \sqrt{\frac{256}{3}}$	9.3
$\frac{dV}{dx}$	+1.63	0	–5.67
GI	/	—	\

Hence the volume V is maximum when $x = \sqrt{\frac{256}{3}}$ cm

$$\text{At Max } V = \frac{\pi}{4}\left(256 - x^2\right)x$$

$$= \frac{\pi}{4}\left(256 - \frac{256}{3}\right)\sqrt{\frac{256}{3}}$$

$$= \frac{\pi}{4}\left(\frac{512}{3}\right)\sqrt{\frac{256}{3}}$$

$$= \frac{\pi}{4}\left(\frac{512}{3}\right)\left(\frac{16\sqrt{3}}{3}\right)$$

$$= \frac{2048\sqrt{3}}{9}\pi \text{ cm}^3$$

4 (a)

	$\frac{\mathrm{d}}{\mathrm{d}x}\left(5e^{2x}+1\right)^4$ $=4\left(5e^{2x}+1\right)^3\left(10e^{2x}\right)$ $=40e^{2x}\left(5e^{2x}+1\right)^3$ $\int_0^2 e^{2x}\left(5e^{2x}+1\right)^3 dx=\frac{1}{40}\int_0^2 40e^{2x}\left(5e^{2x}+1\right)^3 dx$ From part (i) $\int_0^2 e^{2x}\left(5e^{2x}+1\right)^3 dx=\frac{1}{40}\left[\left(5e^{2x}+1\right)^4\right]_0^2$ $=\frac{1}{40}\left[\left(5e^4+1\right)^4-\left(5e^0+1\right)^4\right]$ $=\frac{1}{40}\left[\left(5e^4+1\right)^4-\left(5e^0+1\right)^4\right]$ $=\frac{1}{40}\left[\left(5e^4+1\right)^4-1296\right]$ (b) $\int_{-1}^{0}\frac{1}{x+2}\,\mathrm{d}x-(0.5)=\int_{1}^{p}\frac{1}{x+2}\,\mathrm{d}x$ $\left[\ln\lvert x+2\rvert\right]_{-1}^{0}-(0.5)=\left[\ln\lvert x+2\rvert\right]_{1}^{p}$ $(\ln 2-\ln 1)-0.5=\ln\lvert p+2\rvert-\ln 3$ $\ln(p+2)=\ln 2+\ln 3-0.5 \quad \text{since } p+2>0$ $(p+2)=e^{\ln 6-0.5}=6e^{-0.5}$ $p=6e^{-0.5}-2$
5	(i) $P=500\left(3+\mathrm{e}^{-0.2t}\right)$ When $t=0$, $P=500(3+1)$ $P=2000$ (ii) $P=500\left(3+\mathrm{e}^{-0.2(24)}\right)$ $P=1504$ monkeys

(iii)

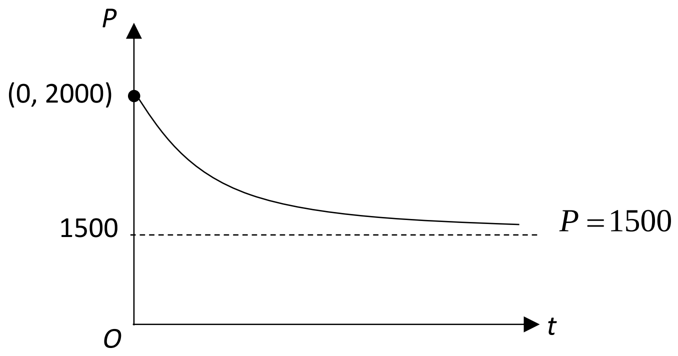
$$500\left(3+e^{-0.2t}\right)<1515$$
$$3+e^{-0.2t}<3.03$$
$$e^{-0.2t}<0.03$$
$$-0.2t<\ln(0.03)$$
$$t>\frac{\ln(0.03)}{-0.2}$$
$$t>17.533$$
$$t\approx 18$$

The population of the monkeys will first drop below 1515 after 18 complete months.

(iv) As $t\to\infty$, $P\to 500(3+0)=1500$.

The population of the monkeys will decrease and approach/stabilise at 1500.

(v)



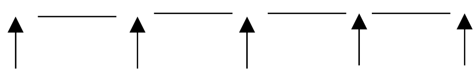
(vi) No. The model may not be accurate in the real world.

In the real world, population of the monkeys may not decrease due to the opening of the factory alone and the monkeys may not stabilise at 1500 due to human interference. (or any possible logical solutions)

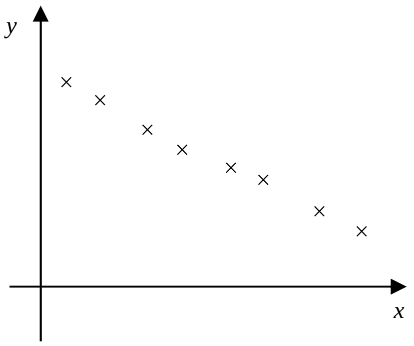
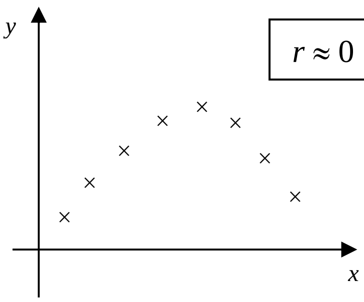
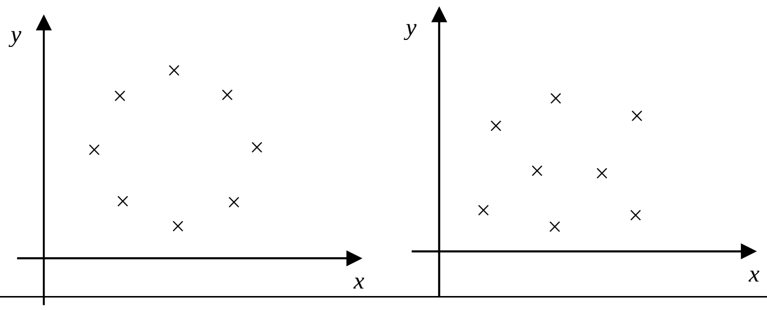
6

(i) Probability (tiles are arranged in correct order to spell his name) = $\dfrac{1}{6!}=\dfrac{1}{720}$

(ii)



4! ways to arrange consonants.

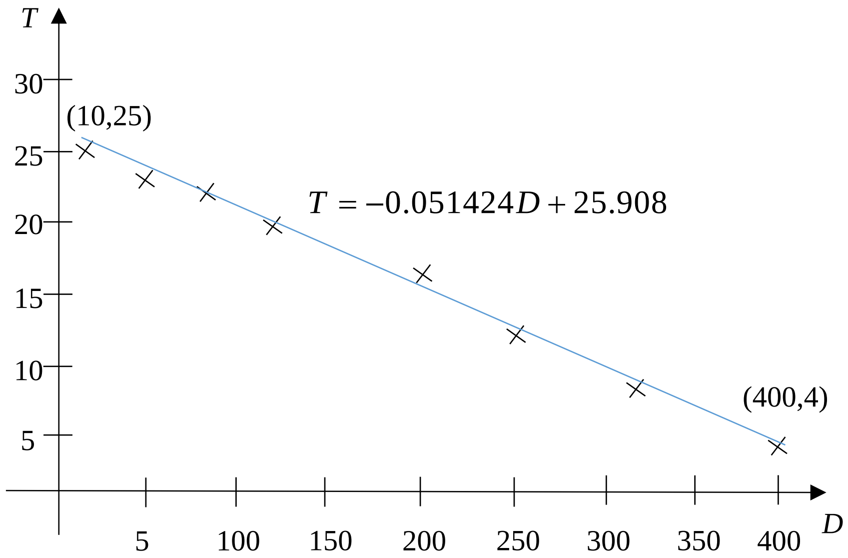
	5 possible slots to insert A & O P(the vowels are separated) = $\dfrac{4!\,{}^{5}P_{2}}{6!} = \dfrac{2}{3}$ (iii) 4! Ways to arrange consonants. 2! Ways to arrange A&O $\boxed{\text{A}}\ \boxed{\text{T, H, M, S}}\ \boxed{\text{O}}$ P(the vowels are at the two ends) = $\dfrac{4!2!}{6!} = \dfrac{1}{15}$
7	 $\boxed{r \approx -0.9}$  $\boxed{r \approx 0}$ 

Using $\overline{T} = -0.051424\overline{D} + 25.908$,

substitute $\overline{D} = \dfrac{1450}{8} = 181.25$, $\overline{T} = \dfrac{k+113}{8}$

$\Rightarrow \dfrac{k+113}{8} = -0.051424(181.25) + 25.908$

$\Rightarrow k = 19.6992 = 19.7$ (shown)



$r = -0.992$. It indicates a strong negative linear correlation between the depth of water and the temperature of water. As the depth of water increases, the temperature of water decreases.

Regression line T on D:

$T = -0.051424D + 25.908$

$T = -0.0514D + 25.9$ (3 s.f.)

When D = 550,

$T = -0.051424(550) + 25.908$

$T = -2.38$

Although $r = -0.992$ (3 s.f.) is close to -1, but the estimate may not be reliable as D = 550 falls outside the data range [10, 400]. Hence, the estimate is unreliable.

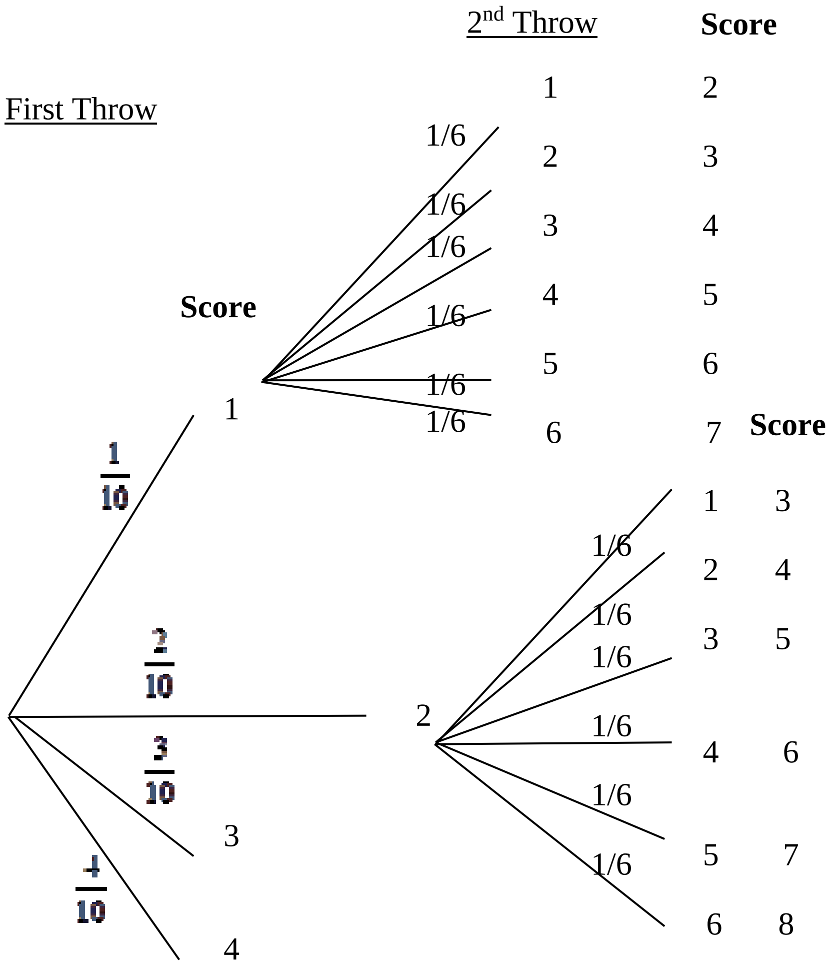
Regression line T on D where height is given in kilometres:

$T = -51.424D + 25.908$

$T = -51.4D + 25.9$ (3 s.f.)

The value of the regression coefficient is -0.000051424.

8



Let X be the score of the first throw and Y be the score of the second throw

$$\begin{aligned}
\mathrm{P}(A) &= 1-\mathrm{P}(A') = 1-\mathrm{P}(\text{James' score is at most a 3})\\
&= 1-\mathrm{P}(\text{score is 3 on the 1st throw})\\
&\quad -\mathrm{P}(X=1 \cap Y=1 \text{ or } 2)-\mathrm{P}(X=2 \cap Y=1)\\
&= 1-\frac{3}{10}-\left(\frac{1}{10}\times\frac{1}{6}\times 2\right)-\left(\frac{2}{10}\times\frac{1}{6}\right)=\frac{19}{30}
\end{aligned}$$

Alternative Method

$\mathrm{P}(A) = \mathrm{P}(X=4) + \mathrm{P}(X=1 \cap 3 \le Y \le 6)$

$$+\ \mathrm{P}(X=2 \cap 2 \le Y \le 6)$$

$$=\frac{4}{10}+\left(\frac{1}{10}\times\frac{1}{6}\times4\right)+\left(\frac{2}{10}\times\frac{1}{6}\times5\right)$$

$$=\frac{19}{30}$$

$$\mathrm{P}(A|B')=\frac{\mathrm{P}(A\cap B')}{\mathrm{P}(B')}=\frac{\text{P(score is at least 4 and even)}}{\text{P(score is even)}}$$

$$=\frac{\mathrm{P}(X=4)+\mathrm{P}(X=1\cap Y=3\text{ or }5)+\mathrm{P}(X=2\cap Y=4\text{ or }6)}{\mathrm{P}(X=4)+\mathrm{P}(X=1\cap Y=1\text{ or }3\text{ or }5)+\mathrm{P}(X=2\cap Y=2\text{ or }4\text{ or }6)}$$

$$=\frac{\frac{4}{10}+\left(\frac{1}{10}\times\frac{1}{6}\times2\right)+\left(\frac{2}{10}\times\frac{1}{6}\times3\right)}{\frac{4}{10}+\left(\frac{1}{10}\times\frac{1}{6}\times3\right)+\left(\frac{2}{10}\times\frac{1}{6}\times3\right)}$$

$$=\frac{8/15}{11/20}$$

$$=\frac{32}{33}$$

$$\mathrm{P}(A\cap B)$$

$$\mathrm{P}(X=1\cap Y=4\text{ or }6)+\mathrm{P}(X=2\cap Y=3\text{ or }5)$$

$$=\left(\frac{1}{10}\times\frac{1}{6}\times2\right)+\left(\frac{2}{10}\times\frac{1}{6}\times2\right)$$

$$=\frac{1}{10}\neq0$$

Events A and B are not mutually exclusive.

9 (i)

Let X and Y be the r.v. denoting the mass of the type A and type B fruits respectively.

$$X \sim N(1.5, 0.1^2)$$

	$Y \sim N(1.8, 0.2^2)$ $X_1 + X_2 + X_3 \sim N(4.5, 0.03)$ $P\left(X_1 + X_2 + X_3 > 5\right) = 0.00195$ (to 3 sf) (ii) Let $T = X_1 + X_2 + X_3 - 2Y$ $E(T) = 4.5 - 3.6 = 0.9$ $Var(T) = 0.03 + 4 \times 0.2^2 = 0.19$ $T \sim N(0.9, 0.19)$ $P\left(\lvert T \rvert \geq 0.5\right)$ $= P(T \leq -0.5) + P(T \geq 0.5)$ $= 0.000659548 + 0.8206023$ = 0.821 (3 sf) (iii) Let $C = 30(X_1 + X_2) + 35(Y)$ $E(C) = 30(1.5 + 1.5) + 35(1.8) = 153$ $Var(C) = 30^2(0.01 + 0.01) + 35^2(0.04) = 67$ $C \sim N(153, 67)$ $P(C > k) = 0.1$ $P(C \leq k) = 0.9$ $k = 163.49 \approx 163$ (3 sig. fig.) (iv) Required Probability $= P(X > 1.3)P(X \leq 1.3) \times 2!$ $= 0.97725 \times 0.02275 \times 2$ = 0.0445 (3 sf)

10	(i) The probability of making a faulty teapot is a constant. The event that a teapot is found faulty is independent of other another teapot. (ii) Let X be the number of faulty teapots in a batch of 30. $X \sim \mathrm{B}(30.0.08)$ $P(X > 2) = 1 - P(X \le 2) = 0.435$ (iii) Let Y be the number of batches which contain exactly one faulty teapot each. $P(X = 1) = 0.21382$ $Y \sim \mathrm{B}(240, 0.21382)$ E(Y) = $240 \times 0.21382 = 51.3$ (iv) Let $\overline{W}$ be the mean number of faulty teapots per batch. Since n = 50 is large, by Central Limit Theorem $\overline{W} \sim \mathrm{N}\left(30 \times 0.08, \dfrac{30 \times 0.08 \times 0.92}{50}\right)$ approximately $\overline{W} \sim \mathrm{N}(2.4, 0.04416)$ $P(\overline{W} > 2.5) = 0.317$

11	Unbiased estimate of the population mean,
	$\bar{x}=\frac{970}{100}=9.7$
	Unbiased estimate of the population variance,
	$s^2=\frac{1}{99}\left[9800-\frac{970^2}{100}\right]=\frac{391}{99}=3.9495$
	(ii) It is not necessary to assume a normal distribution for the test to be valid. Since n is large, by Central Limit Theorem, the mean mass of salt is normally distributed approximately.
	(iii) Let X be the mass of one packet of salt in grams and μ be the population mean of the mass of one packet of salt in grams.
	$H_o : \mu = 10$ $H_1 : \mu < 10$
	Under H_o,
	Since n = 100 is large, by Central Limit Theorem,
	$\overline{X} \sim N\left(10, \frac{391}{9900}\right)$ approximately,
	Use a left tailed z-test at 10 % level of significance,
	Test Statistic:
	$Z=\frac{\overline{X}-10}{s/10} \sim \mathrm{N}(0,1)$
	Using GC, the p-value = $0.0656 \le 0.10$
	Reject the null hypothesis and conclude that there is sufficient evidence at 10% significance level to reject the company's claim that the mass is at least 10g.
	(iv) There is a probability of 0.10 that we concluded that the mean mass of the packets of salt is less than 10 grams when it is actually at least 10 grams.
	The p-value is the lowest significance level at which the sample mean mass of salt is at least 10 grams.
	(v) Let Y be the random variable of the mass of each packet of salt in the new packaging system.

	$H_o : \mu = 10$ $H_1 : \mu < 10$ Since n = 30 is large, by Central Limit Theorem Under H_o, $Z = \dfrac{\overline{Y} - 10}{\frac{0.9}{\sqrt{30}}} \sim N(0,1)$ approximately Since company's claim is valid, H_o is not rejected at 5% level of significance, $\dfrac{m-10}{\frac{0.9}{\sqrt{30}}} > -1.6449$ $m > 9.7297$ Least possible value of m is 9.73.