Name	( )	Class	

2016 Year 6 Preliminary Examination II
Higher 1

MATHEMATICS

8864/01

Paper 1

20 September 2016

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)
Cover Page

READ THESE INSTRUCTIONS FIRST

Do not open this booklet until you are told to do so.
Write your name, class and index number in the space at the top of this page.

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, place the cover page on top of your answer paper and fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [35 marks]

1 Find the range of values of m for which $2x^2 - m(6x-5m) - 2$ is always positive. [3]

2 **(a)** Given $y = \ln\left[(x+1)^2(1-2x^2)^{-5}\right]$, find $\frac{dy}{dx}$. [2]

(b) By considering $\frac{1}{x+1} + \frac{1}{(x+1)^2}$, evaluate the integral $\int_0^2 \frac{x+2}{(x+1)^2}\,dx$ exactly. [3]

3 The equation of a curve is given as $y = a + e^{bx}$ where a and b are constants. It is given that the equation of the tangent to the curve at the point $M(0, 2)$ is $y = 3x + 2$.

(i) Find the values of a and b. [4]

(ii) Find the equation of the normal at M. [2]

4 The equation of a curve C is given by $y = 3 + \frac{2}{x-2}$.

(i) Sketch C, stating clearly the equations of the asymptotes and the coordinates of any points of intersection with the axes. [3]

(ii) Find the exact area of the region enclosed by the curve C, the y-axis and the lines $y = 5$ and $y = 8$. [4]

(iii) By adding a suitable curve to the sketch in part **(i)**, solve $\frac{2}{x-2} \le x - 2$ exactly.

[3]

5 **(a)** A fish tank that is a cuboid with square base has a fixed volume of 2000 cm^3. The side walls are made of acrylic and the base is made of marble. The cost per cm^2 of marble is twice the cost per cm^2 of acrylic. Let the side of the base of the tank be x cm. Assume that acrylic costs \$1 per cm^2 and thickness of the fish tank is negligible. Using differentiation, find the value of x which the cost of constructing the fish tank is a minimum. [6]

(b) Another fish tank of different shape has a height of 30 cm. Water is being pumped into the fish tank at a rate of 2 cm^3/s. The fish tank was initially empty and when the depth of the water is h cm, the volume of the water V cm^3 is given by

$$V = \left(h^2 + h\right)^{\frac{2}{3}}$$

(i) Find the rate of change of the depth of the water when $h = 8$. [3]

(ii) Sketch a graph of $\frac{dh}{dt}$ against h for $0 < h < 30$. Describe how the rate of change in the depth of the water varies as the fish tank is filled up. [2]

Section B: Statistics [60 marks]

6 A school has 200 teachers of whom 5% are in the 21 – 30 age group, 60% are in the 31 – 40 age group and the rest are in the age group of 41 and above. During a meeting held in a Lecture Theatre, the principal intends to obtain a sample of 20 teachers for a survey. She decides to select 20 teachers from the last occupied row for the survey.

(i) Name the sampling method described. State a reason, in the context of the question, why this sampling method is not desirable. [2]

(ii) Suggest a method of obtaining a representative sample and describe how it may be carried out. [3]

7 The total distance travelled by each car in a random sample of 500 cars is measured. The random variable Y with mean 50724 km and standard deviation 13112 km denotes the distance travelled by a car.

(i) Find the value of the constant c such that the probability that the sample mean differs from the population mean by at most c is 0.95. [4]

(ii) State, giving a reason, whether it is necessary to make any assumption about the distribution of the distances travelled by cars in order to carry out the calculation in part (i). [1]

8 The random variable X represents the time taken in minutes for a haircut at a barber's shop. X is normally distributed with mean 11 and standard deviation 3.

(i) Find the value of a such that $P(X < a) = 0.369441$. [1]

(ii) Find the least value of n such that the probability of at most 3 out of n randomly selected haircuts which take less than 10 minutes is less than 0.25. [3]

(iii) Using a suitable approximation, find the probability that at least 50 out of 100 randomly selected haircuts take more than 10 minutes. [4]

9 (i) In a Mathematics test for 12-year-old children, the raw scores, X, are normally distributed with mean μ and standard deviation σ. Given that $P(X<45)=0.1$ and $P(X<65)=0.9$, find the value of μ and σ. [3]

(ii) In an English language test for the same group of 12-year-old children, the raw scores, Y, are normally distributed with mean 60 and standard deviation 11. Find the least raw score which would be obtained by the highest scoring 10% of the children, giving your answer correct to the nearest integer. [2]

(iii) For a randomly chosen 12-year-old child, find the probability that his English score is at least 10 marks higher than his Mathematics score. [3]

(iv) Two 12-year-old children are randomly selected. Find the probability that exactly one of them scored more than 65 marks for the Mathematics test. [2]

10 (a) A bag contains r red balls and g green balls. One ball is randomly selected without replacement from the bag and the colour is noted. A second ball is then randomly selected from the bag.

(i) Represent the above information using a tree diagram. [2]

(ii) Find the probability that two balls selected are of different colours. [2]

(b) Let A and B be events such that $P(B)=\frac{1}{4}$, $P(B \mid A)=\frac{1}{5}$ and $P(A \mid B')=\frac{7}{10}$.

(i) State, with reason, whether A and B are independent. [1]

(ii) Find $P(A \cap B')$. [3]

(iii) Find $P(A \cup B)$. [2]

11 The table below shows the time t, in minutes, taken by Andy to complete a particular stage of a computer game x weeks after he started playing the game.

x	1	2	3	4	5	6
t	35.2	25.8	22.7	30.1	15.4	12.3

(i) Draw a scatter diagram for the data, labelling the axes clearly. [2]

(ii) Calculate the product moment correlation coefficient and comment on why its value does not necessarily mean that the best model for the relationship between x and t is $t = ax + b$ or $x = ct + d$. [2]

(iii) Identify the pair of data which should be regarded as an outlier. [1]

For the rest of the question, the **outlier is removed**.

(iv) Calculate the new product moment correlation coefficient. [1]

(v) Andy would like to estimate the time for him to complete the stage of the game 8 weeks after he has started playing the game. Find the equation of a suitable regression line and use it to obtain the estimate. Comment on the reliability of this estimate. [3]

(vi) Explain whether you will use regression line t on x or regression line x on t to estimate the number of weeks after Andy has started playing the computer game when the time taken by him to complete the particular stage is 20 min. [1]

12 A company packs and supplies salt in small packets. The mass of salt in one packet is denoted by x grams. The company claims that the mean mass of salt is at least 10 grams. To test this claim, a random sample of 100 packets of salt is weighed and the masses are summarised by

$$\sum x = 980 \quad \sum x^2 = 9700$$

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) Carry out a test at the 5% significance level whether the company's claim is valid. [4]

(iii) The company wishes to test if the mass of the packets of salt it packs differs from 10 grams. Without further calculation, carry out the test at the 5% significance level. [2]

The company introduces a new packaging system and the new population variance is known to be 0.9^2 grams^2. A new random sample of 60 packets of salt is chosen and the mean of this sample is m grams. A test at the 1% significance level indicates that the company's initial claim is valid for this improved process.

(iv) Find the least possible value of m, giving your answer correct to 2 decimal places. [4]

– End of Paper –

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COVER PAGE

HIGH SCHOOL EXAMINATIONS

CANDIDATE NAME	:	____________________
CLASS / INDEX NO	:	☐☐ / ☐☐
SUBJECT TITLE / PAPER NO	:	____________________
DATE	:	☐☐ / ☐☐ / ☐☐

Instructions to Candidates

1 Write the information required in the box above.

2 Write the number of the questions attempted in the left hand margin of the writing paper and also in the left hand column of the box on the cover.

3 * Place this cover page on top of your answer sheets.

* Use the string provided to tie this cover page to your answer sheets securely.

* Answer sheets should not be tied so tightly that the examiners cannot easily turn the pages.

* Do not staple your answer sheets.

For Candidate's Use	For Examiner's Use
Question Number	Marks Obtained
Total Marks	

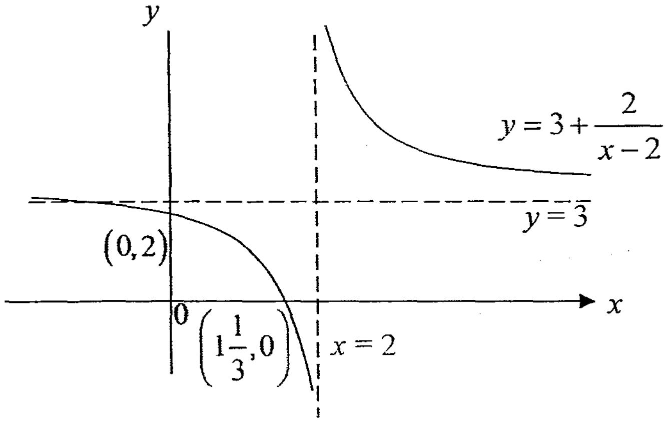
Calculator Model:
(if applicable)

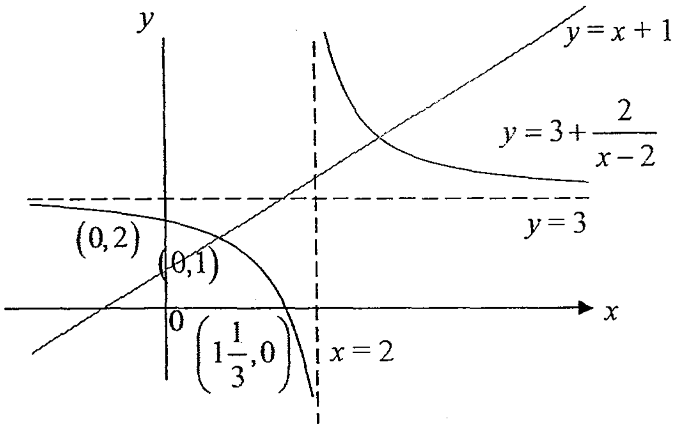
2016 Year 6 Prelim 2: 8864 H1 Mathematics Marking Scheme

Question 1 [3 marks]		
1	$2x^2 - m(6x-5m) - 2$ $= 2x^2 - 6mx + 5m^2 - 2$ For $2x^2 - m(6x-5m) - 2$ to be always positive, $(-6m)^2 - 4(2)(5m^2 - 2) < 0$ $36m^2 - 40m^2 + 16 < 0$ $-4m^2 + 16 < 0$ $m^2 - 4 > 0$ $(m-2)(m+2) > 0$ $m > 2$ or $m < -2$	

Question 2 [5 Marks]		
(a)	$y=\ln\left[(x+1)^2(1-2x^2)^{-5}\right]$ $y=\ln(x+1)^2+\ln\left(1-2x^2\right)^{-5}$ $y=2\ln(x+1)-5\ln\left(1-2x^2\right)$ $\frac{dy}{dx}=\frac{2}{x+1}-\frac{5(-4x)}{1-2x^2}$ $=\frac{2}{x+1}+\frac{20x}{1-2x^2}$	
(b)	$\frac{1}{x+1}+\frac{1}{(x+1)^2}=\frac{x+1+1}{(x+1)^2}=\frac{x+2}{(x+1)^2}$ $\int_0^2\frac{x+2}{(x+1)^2}\,dx=\int_0^2\frac{1}{x+1}+\frac{1}{(x+1)^2}\,dx$ $=\left[\ln(x+1)-\frac{1}{x+1}\right]_0^2$ $=\ln 3-\frac{1}{3}-(0-1)$ $=\ln 3+\frac{2}{3}$	

Question 3 [6 Marks]		
(i)	$y = a + e^{bx}$ $\frac{dy}{dx} = be^{bx}$ At $x = 0$, $\frac{dy}{dx} = b$ Gradient of tangent = 3 Thus $b = 3$ Sub (0, 2) into $y = a + e^{3x}$ $2 = a + 1$ $a = 1$	
(ii)	Gradient of Normal = $-\frac{1}{3}$ Equation of Normal: $y - 2 = -\frac{1}{3}(x - 0)$ $y = -\frac{1}{3}x + 2$	

<table>
<tr><th colspan="3">Question 4 [10 marks]</th></tr>
<tr><td>(i)</td><td>
</td><td></td></tr>
<tr><td>(ii)</td><td>$y = 3 + \frac{2}{x-2}$
$y - 3 = \frac{2}{x-2}$
$x - 2 = \frac{2}{y-3}$
$x = 2 + \frac{2}{y-3}$
Area enclosed $= \int_5^8 \left(2 + \frac{2}{y-3}\right) \mathrm{d}y$
$= \left[2y + 2\ln\lvert y-3\rvert\right]_5^8$
$= [16 + 2\ln 5] - [10 + 2\ln 2]$
$= 6 + 2(\ln 5 - \ln 2)$
$= 6 + 2\ln\frac{5}{2}$</td><td></td></tr>
<tr><td>(iii)</td><td>$\frac{2}{x-2} \le x - 2$
$\frac{2}{x-2} \le x - 3 + 1$
$3 + \frac{2}{x-2} \le x + 1$
Add $y = x + 1$</td><td></td></tr>
</table>



$$3 + \frac{2}{x-2} = x + 1$$

$$\frac{2}{x-2} = x - 2$$

$$2 = x^2 - 4x + 4$$

$$x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 4(1)(2)}}{2}$$

$$= \frac{4 \pm \sqrt{8}}{2}$$

$$= 2 \pm \sqrt{2}$$

From the graph, $2 - \sqrt{2} \le x < 2$ or $x \ge 2 + \sqrt{2}$

Question 5 [11 Marks]

(a) Let V be the volume and h the height of the fish tank.

$$V = x^2 h$$

$$2000 = x^2 h$$

$$h = \frac{2000}{x^2}$$

Let A be the area of the four side walls and the base of the fish tank.

$A = x^2 + 4xh$

Hence the cost of fish tank is:

$$C = 2x^2 + (4xh)$$
$$= 2x^2 + \left(4x\frac{2000}{x^2}\right)$$
$$= 2x^2 + \frac{8000}{x}$$

$$\frac{dC}{dx} = 4x - \frac{8000}{x^2}$$

When $\frac{dC}{dx} = 0$,

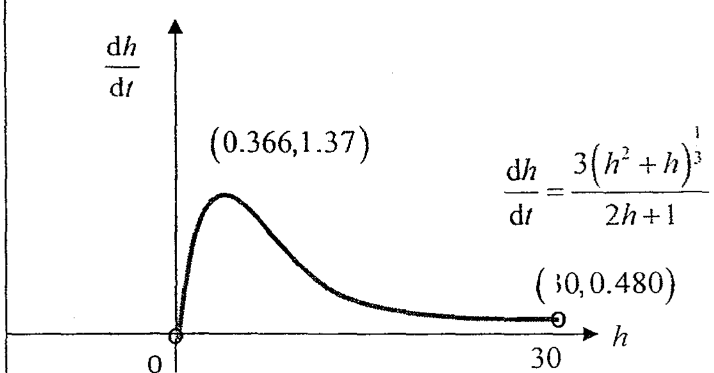
$$\frac{8000}{x^2} = 4x$$
$$x^3 = 2000$$
$$x = \sqrt[3]{2000}$$

$$\frac{d^2C}{dx^2} = 4 + \frac{16000}{x^3}$$

When $x = \sqrt[3]{2000}$, $\frac{d^2C}{dx^2} = 4 + \frac{16000}{2000} = 12 > 0.$

<u>Alt</u>

x	12.55	$\sqrt[3]{2000} \approx 12.6$	12.65
$\frac{dC}{dx}$	−0.5928	0	0.6070
	\	—	/

(bi)	$\frac{dV}{dt} = 2 \text{ cm}^3/s$ $V = \left(h^2 + h\right)^{\frac{2}{3}}$ $\frac{dV}{dh} = \frac{2}{3}\left(h^2 + h\right)^{-\frac{1}{3}}(2h+1)$ $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $2 = \frac{2}{3}\left(h^2 + h\right)^{-\frac{1}{3}}(2h+1) \times \frac{dh}{dt}$ $\frac{dh}{dt} = \frac{3\left(h^2 + h\right)^{\frac{1}{3}}}{2h+1}$ Substitute $h = 8$, $\frac{dh}{dt} = \frac{3\left(8^2 + 8\right)^{\frac{1}{3}}}{16+1} \approx 0.734$ $\therefore$ The rate of change of the depth of the water when $h = 8$ is 0.734 cm/s .	
(bii)	 The rate of change of depth of water increases from 0 to 1.37cm/s then it decreases as the height increases.	

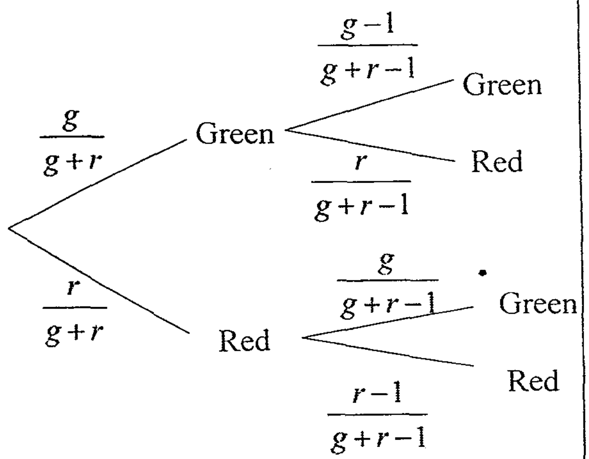
Question 6 [5 Marks]		
(i)	Quota Sampling. This method is non-random as not every teacher has an equal chance of being selected.	
(ii)	Stratified Sampling. Principal can draw **random** samples from each strata as follows:	

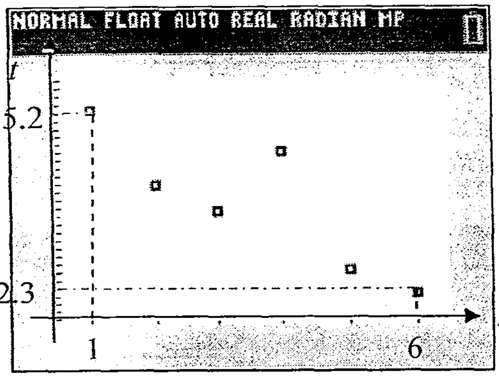
	Senior teachers	Beginning teachers	Classroom teachers
Number of teachers selected	$0.05 \times 20 = 1$	$0.6 \times 20 = 12$	$20 - 1 - 12 = 7$

Question 7 [5 Marks]		
(i)	Since $n = 500$ is large, by CLT, $\bar{Y} \sim \mathrm{N}\left(50724, \frac{13112^2}{500}\right)$ $\mathrm{P}\left(\lvert \bar{Y} - 50724 \rvert \le c\right) = 0.95$ $\mathrm{P}(-c \le \bar{Y} - 50724 \le c) = 0.95$ $\mathrm{P}(50724 - c \le \bar{Y} \le 50724 + c) = 0.95$ $\mathrm{P}(\bar{Y} \le 50724 - c) = 0.025$ $50724 - c \approx 49574.70364$ $c \approx 1150$ (3 sig fig)	
(ii)	It is not necessary to assume population normal as n = 500 is large, thus sample mean is normally distributed by the CLT.	

Question 8 [8 Marks]		
(i)	Using the GC, $a \approx 10$	
(ii)	Let Y be the r.v. for the number of haircut that takes less than 10 minutes, $Y \sim \text{B}(n, 0.369441)$. $\text{P}(Y \leq 3) < 0.25$ Using the GC, $n = 12,\ \text{P}(Y \leq 3) = 0.296 > 0.25$ $n = 13, \text{P}(Y \leq 3) = 0.231 < 0.25$ $n = 14, \text{P}(Y \leq 3) = 0.179 < 0.25$ The least value of $n = 13$.	
(iii)	Let W be the r.v. of the number of haircut more than 10 minutes out of 100, $W \sim \text{B}(100, 1 - 0.369441)$. $W \sim \text{B}(100, 0.630559)$ Since n =100 is large, np = 100(0.630559) = 63.1>5, $n(1-p)$ =36.9 >5, thus $W \sim \text{N}(63.0559, 23.295)$ approximately. $\text{P}(W \geq 50) \overset{cc}{=} \text{P}(W \geq 49.5) = 0.998$	

Question 9 [10 marks]		
(i)	By symmetry, $\mu = \frac{45+65}{2} = 55$ $P(X < 45) = 0.1$ $P\left(Z < \frac{45-55}{\sigma}\right) = 0.1$ $\frac{-10}{\sigma} = -1.281551567$ $\sigma = 7.80304 \approx 7.80$	
(ii)	$Y \sim N(60, 11^2)$ $P(Y \geq a) \leq 0.1$ $P(Y < a) \geq 0.9$ $a \geq 74.1$ $\therefore$ Least raw score is 75	
(iii)	$Y - X \sim N(5, 11^2 + 7.80304^2)$ $P(Y \geq X + 10)$ $= P(Y - X \geq 10)$ $= 0.355$	
(iv)	The required probability $= 2P(X < 65)P(X > 65)$ $= 0.18$	

Question 10 [10 marks]		
(a)(i)	 	
(a)(ii)	P(2 balls of different colours) $=\left(\frac{g}{g+r}\right)\left(\frac{r}{g+r-1}\right)+\left(\frac{r}{g+r}\right)\left(\frac{g}{g+r-1}\right)$ $=\frac{2rg}{(r+g)(r+g-1)}$	
(b)(i)	Since $P(B\mid A)=\frac{1}{5}\neq P(B)=\frac{1}{4}$, A and B are not independent.	
(b)(ii)	Given $P(A\mid B')=\frac{7}{10}$ $\frac{P(A\cap B')}{P(B')}=\frac{7}{10}$ $P(A\cap B')=\frac{7}{10}P(B')$ $P(A\cap B')=\frac{7}{10}\left(1-\frac{1}{4}\right)=\frac{21}{40}$	
(b)(iii)	$P(A\cup B)=P(A\cap B')+P(B)$ $P(A\cup B)=\frac{21}{40}+\frac{1}{4}=\frac{31}{40}$	

Question 11 [10 Marks]		
(i)	 	
(ii)	$r \approx -0.851$ Although $\|r\|$ is rather close to 1, suggesting a linear relationship but looking at the scatter diagram, you will notice that there is a outlier.	
(iii)	(4, 30.1)	
(iv)	$r = -0.9750384 \approx -0.975$	
(v)	A suitable regression line: $t = 36.72209 - 4.24767x$ i.e. $t = 36.7 - 4.25x$ When $x = 8$, $t = 36.72209 - 4.24767(8)$ $= 2.7406977 \approx 2.74$ mins Although $\|r\|$ is close to 1, the estimate is not reliable because $x = 8$ is outside of data range, $[1, 6]$. Therefore it is a extrapolation.	
(vi)	regression line t on x (i.e $t = ax + b$)should be use since x is the controlled or independent variable.	

Question 12 [12 Marks]		
(i)	$\bar{x}=\frac{980}{100}=9.8$ $s^2=\frac{1}{99}\left[9700-\frac{980^2}{100}\right]=\frac{32}{33}$	
(ii)	Let μ be the population mean of the mass of one packet of salt. $H_o:\mu=10$ $H_1:\mu<10$ Since $n=100$ is large, Under H_o, $Z=\frac{\bar{x}-10}{s/10}\sim N(0,1)$ Using GC, the p-value = 0.0211<0.05 Reject the null hypothesis and there is sufficient evidence at 5% significance level to reject the company's claim that the mass is at least 10g.	
(iii)	For a 2-tailed test, the p-value = 2(0.0211) = 0.0422 < 0.05. Thus reject the null hypothesis and there is sufficient evidence at 5% significance level that the mass of the packets of salt it packs differs from 10 grams.	
(iv)	Let Y be the r.v. of the mass of each packet of salt in the new packaging system. $H_o:\mu=10$ $H_1:\mu<10$ Since $n=60$ is large, Under H_o, $Z=\frac{\bar{Y}-10}{\frac{0.9}{\sqrt{60}}}\sim N(0.1)$ Since company's claim is valid, H_o is not rejected. $\frac{m-10}{\frac{0.9}{\sqrt{60}}}\geq -2.3263$ $m\geq 9.7297\approx 9.73$ Least possible value of m is 9.73.	