YISHUN INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
Higher 2

CANDIDATE NAME	
CG	

MATHEMATICS

Paper 1

9758/01

28 AUGUST 2024

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your CG, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiners' Use

Question	Marks
1	
2	
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4	
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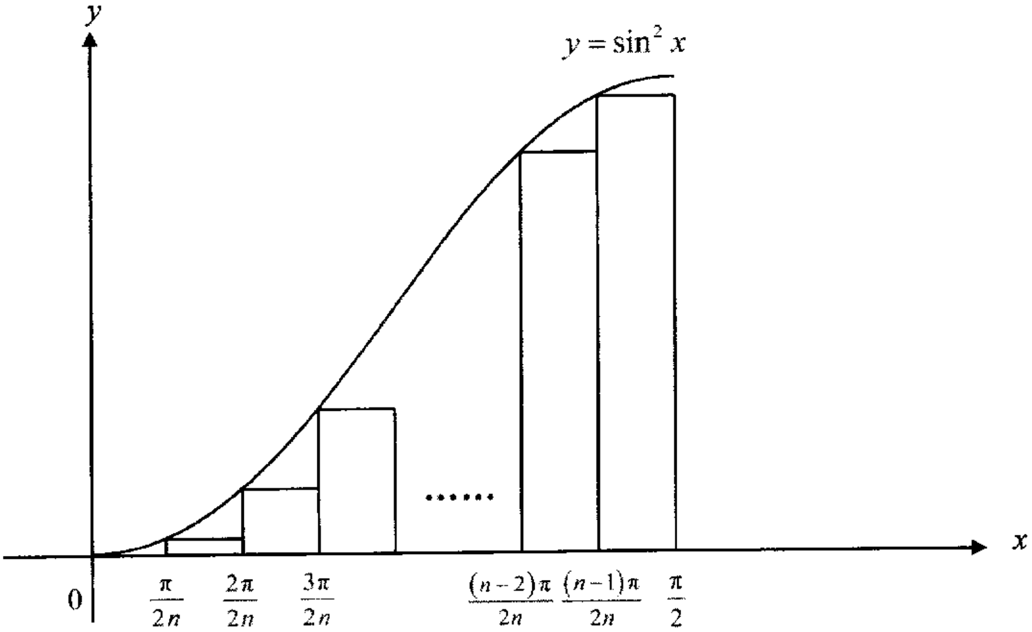
9	
10	
11	
Presentation	
Total	/ 100

1 **(a)** Without using a calculator, solve the inequality $\frac{x}{x-1} \geq \frac{6}{4+x}$. [4]

(b) Hence, solve $\frac{|x|}{|x|+1} \geq \frac{6}{4-|x|}$. [2]

2 The diagram below shows a sketch of the graph of $y = \sin^2 x$ for $0 \leq x \leq \frac{\pi}{2}$.

Rectangles each of width $\frac{\pi}{2n}$ are drawn under the curve for $0 \leq x \leq \frac{\pi}{2}$.



(a) Show that A, the total area of all the rectangles, is given by $a\sum_{k=1}^{n-1} \sin^2 \frac{k\pi}{2n}$, where a is to be determined. [2]

(b) Find the exact value of $\lim_{n \to \infty} a\sum_{k=1}^{n-1} \sin^2 \frac{k\pi}{2n}$. [2]

(c) Hence, find the value of $\int_0^1 \sin^{-1}\sqrt{y} \, dy$. [2]

3 **Do not use a calculator in answering this question.**

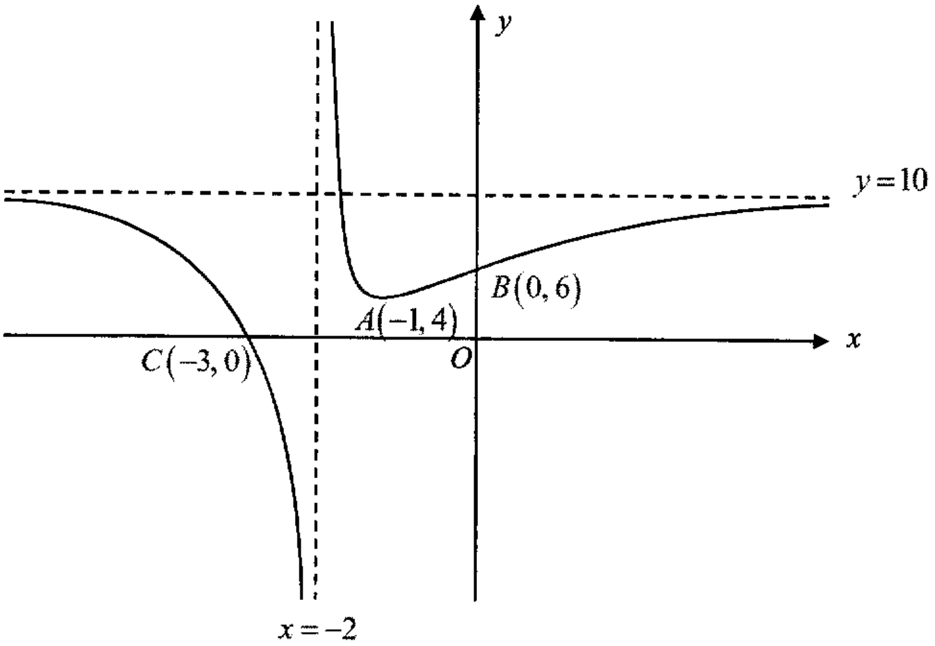
(a) Given that $f(x)$ is a polynomial of degree 4 with real coefficients, explain whether it is possible for $f(x) = 0$ to have 3 non-real roots and 1 real root. [1]

(b) One of the roots of the equation $2x^4 - 15x^3 + ax^2 - 63x + b = 0$, where a and b are real, is $3 - 2i$. Find the other roots of the equation and the values of a and b. [6]

4 **(a)** It is given that $x\frac{\mathrm{d}y}{\mathrm{d}x} = xy(\ln x + \ln y) - y$. Using the substitution $w = xy$, show that the differential equation can be transformed to $\frac{\mathrm{d}w}{\mathrm{d}x} = \mathrm{f}(w)$, where the function $\mathrm{f}(w)$ is to be found. [3]

(b) Hence, given that $y = \frac{1}{2}\mathrm{e}^3$ when $x = 2$, solve the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} = xy(\ln x + \ln y) - y$, to find y in terms of x. [5]

5 **(a)** The graph of $y = \mathrm{f}(x)$ is shown below. The graph has a turning point at $A(-1, 4)$, and axial intercepts at $B(0, 6)$ and $C(-3, 0)$. The lines $x = -2$ and $y = 10$ are the asymptotes.



On separate diagrams, and showing clearly the coordinates of the turning points and any points of intersection with the axes and the equations of the asymptotes where possible, sketch the graphs of

(i) $y = \frac{1}{\mathrm{f}(x)}$, and [3]

(ii) $y = \mathrm{f}'(x)$ [2]

(b) State a sequence of transformations that will transform the ellipse $4x^2 + y^2 - 4y = 0$ to the unit circle $x^2 + y^2 = 1$. [3]

6 A curve is defined by the parametric equations

$$x=\frac{1-t^2}{1+t^2}\ ,\ y=\frac{2t}{1+t^2}\ ,\ \ -1\le t\le 0.$$

(a) Using differentiation, find the equation of the tangent to the curve at the point where $t=-\frac{1}{2}$. [4]

(b) Sketch the curve and the tangent in part **(a)** on the same diagram, labelling the coordinates of the points of intersection with the axes. [2]

(c) Show that the area bounded by the curve, the tangent and the x-axis can be expressed in the form $c-\int_a^b \frac{8t^2}{\left(1+t^2\right)^3}\,\mathrm{d}t$, where a, b and c are constants to be determined. Hence evaluate this area. [3]

7 **(a)** Show that $\frac{4r-6}{(2r+1)(2r+3)(2r+5)}$ can be expressed in the form $\frac{A}{2r+1}+\frac{B}{2r+3}+\frac{C}{2r+5}$, where A, B and C are constants to be determined. [2]

(b) Hence, find an expression for $\sum_{r=1}^{N}\frac{4r-6}{(2r+1)(2r+3)(2r+5)}$ in terms of N. You do not need to give your answer as a single fraction. [4]

(c) Using your answer in part **(b)**, find the exact value of $\sum_{r=7}^{\infty}\frac{4r-10}{(2r-1)(2r+1)(2r+3)}$. [3]

8 The function f is defined by

$$\mathrm{f}:x\mapsto x(5-x),\qquad x\in\mathbb{R},x\ge 3.$$

(a) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

It is given that

$$\mathrm{g}(x)=\begin{cases}4+\dfrac{4}{x-10} & \text{for } x\le 8,\\[2mm] \dfrac{12-x}{2} & \text{for } 8<x\le 12.\end{cases}$$

(b) Sketch the graph of $y=\mathrm{g}(x)$. [2]

(c) Find the value of x such that $\mathrm{g}^{-1}(3.5)=x$. [2]

(d) Explain why the composite function gf exists and find gf (x). [2]

(e) Find the range of gf. [1]

9 **(a)** Find $\int x^2\sqrt{1+2x^3}\ \mathrm{d}x$. [2]

(b) Find $\int \sin 3x \sin 4x\ \mathrm{d}x$. Hence, find the exact value of $\int_0^{\frac{\pi}{3}} \sin 3x\left|\sin 4x\right|\ \mathrm{d}x$. [5]

(c) Find $\int \mathrm{e}^{-x}\cos 3x\ \mathrm{d}x$. Hence, find the exact value of $\int_0^{\pi} \mathrm{e}^{-x}\cos 3x\ \mathrm{d}x$. [5]

10 Two aeroplanes are observed flying in straight lines, with respect to an airport control tower located at $(0, 0, 0)$. The flight paths of aeroplanes A and B can be modelled by

$$\mathbf{r}=\begin{pmatrix}10\\4\\3\end{pmatrix}+\lambda\begin{pmatrix}-2\\-5\\6\end{pmatrix} \quad\text{and}\quad \mathbf{r}=\begin{pmatrix}-8\\3\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-3\\4\end{pmatrix}$$

respectively, where λ and μ is the time elapsed in minutes since the start of the observation for each aeroplane. The x, y and z-directions are due east, due north and vertically upwards respectively, with all distances in kilometres.

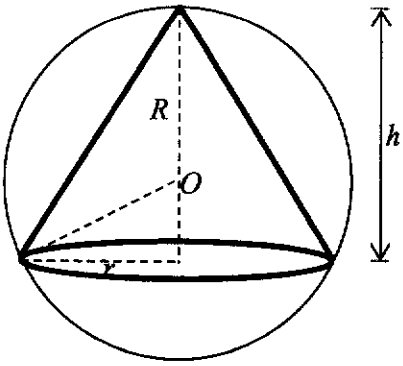
(a) The flight paths intersect at point P. Find the coordinates of P and explain why the two aeroplanes will not collide. [4]

(b) Find the acute angle between the flight path of aeroplane A and the horizontal ground. [2]

(c) Find a cartesian equation of the plane Π which contains both flight paths. [3]

(d) The airport building has a slanted wall which is parallel to the flight path of aeroplane B, and the wall is inclined at an angle of 60° with the horizontal ground. The cartesian equation of the wall is given by $ax+by+z=1$. Given that $b>1$, find the values of a and b. [4]

11 [It is given that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and that the volume of a circular cone with base radius r and height h is $\frac{1}{3}\pi r^2 h$.]



A hollow crystal sphere with centre O has a fixed radius of R cm and it is made of material with negligible thickness. A golden right circular cone with base radius r cm and height h cm is inscribe
such that its vertex and the circumference of the circular base are both in contact with the
of the sphere. It is also given that $h>R$ (see diagram).

(a) Show that $h = R + \sqrt{R^2 - r^2}$. [1]

(b) Show that the maximum possible volume of the cone is $k\pi R^3$ cm^3, where k is a constant whose exact value is to be found. You do not need to show that this volume is a maximum. [6]

It is now assumed that the volume of the inscribed cone is maximum for the rest of this question.

The space between the bottom of the sphere and the circular base of the cone is fully filled with fluorescent liquid. Unfortunately, the liquid is leaking at a constant rate of 2 cm^{3}s^{-1} at the bottom of the sphere. The volume of the liquid, L cm^3, at the instant when the depth of the liquid is x cm is given by $L = \frac{1}{3}\pi x^2 (3R - x)$.

It is now given that $R = 10$.

(c) Find the rate of decrease of x at the instant when the depth of the liquid is 4 cm. [3]

(d) How long does it take for the liquid to be completely drained? [2]

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YISHUN INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
Higher 2

CANDIDATE NAME

CG

MATHEMATICS

Paper 2

9758/02

10 SEPTEMBER 2024

3 hours

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Presentation				
Total	**/ 100**			

Section A: Pure Mathematics [40 marks]

1 **Do not use a calculator in answering this question.**

Two complex numbers are such that $z_1 = \dfrac{1-\mathrm{i}}{\cos\frac{1}{8}\pi - \mathrm{i}\sin\frac{1}{8}\pi}$ and $z_2 = -1+\sqrt{3}\,\mathrm{i}$.

(a) Find $\dfrac{-2z_1}{z_2{}^*}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(b) Find the three smallest positive integer values of n such that $\left(\dfrac{-2z_1}{z_2{}^*}\right)^n$ is a negative real number. [3]

2 Vectors $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors such that $\sqrt{3}\,|\mathbf{u}\times\mathbf{v}| = \mathbf{u}\bullet\mathbf{v}$ and the angle between the direction of $\mathbf{u}$ and the direction of $\mathbf{v}$ is θ.

(a) Explain why θ is an acute angle and show that $\theta = 30°$. [3]

It is now given that $\mathbf{u}$ is a unit vector and $|\mathbf{v}| = 3$.

(b) **(i)** Give the geometrical interpretation of $\mathbf{u}\bullet\mathbf{v}$. [1]

(ii) Find the value of $(\mathbf{u}\times\mathbf{v}+\mathbf{u})\bullet(\mathbf{v}\times\mathbf{u})$. [3]

3 **(a)** The region R is bounded by the curve $y = \dfrac{1}{1+\sqrt{x}}$, the x-axis and the lines $x = 1$ and $x = 4$.

R is rotated about the x-axis through 2π radians. Using the substitution $u = \sqrt{x}$, find the exact volume of the solid generated. [6]

(b) The region L is bounded by the curve $y = \dfrac{1}{1+\sqrt{x+a}}$, the x-axis and the lines $x = 3$ and $x = 6$.

Given that the volume of the solid generated when L is rotated about the x-axis through 2π radians is the same as the volume calculated in part (a), state the value of a. [1]

4 **(a)** A geometric series has first term a and common ratio r, where $r \neq 1$. The first, second and third terms of this series are the first, seventh and tenth terms of an arithmetic series with common difference d, where $d < 0$.

(i) Find a in terms of d. [3]

(ii) Given that the sum to infinity of the geometric series is 1, find the exact value of d. [2]

(b) The first term of another arithmetic series is a negative integer. The sum of the first six terms of the series is 15 and the product of the first four terms of the series is 0. Find the first positive term of the series. [4]

5 **(a)** **(i)** Using the expansions from the List of Formulae (MF26), find the Maclaurin series for $\ln(1+\sin 2x)$ in ascending powers of x, up to and including the term in x^4, where $0 \leq x \leq \frac{\pi}{4}$. [3]

(ii) State the equation of tangent to the curve $y = \ln(1+\sin 2x)$ at the point when $x = 0$. [1]

(b) In the triangle ABC, $AB = 1$, angle $BAC = \theta$ radians and angle $ABC = \frac{5\pi}{6}$ radians.

(i) Show that $AC = \dfrac{1}{\cos\theta - \sqrt{3}\sin\theta}$. [3]

(ii) Given that θ is a sufficiently small angle, show that

$$AC \approx 1 + a\theta + b\theta^2$$

for constants a and b to be determined. [3]

Section B: Probability and Statistics [60 marks]

6 Carl plays a game with only 2 outcomes, win or lose, and the probability of winning in the first round is 0.3. For any subsequent rounds, the probability of winning is 0.4 if he wins the previous round and 0.2 otherwise. Carl plays the game for three rounds.

(a) Draw a tree diagram to illustrate this information. [2]

(b) Find the probability that Carl wins exactly once, given that he wins at least once. [3]

On another day, Carl decides to play the game until he loses 1 round.

(c) Find the probability that Carl plays the game for an even number of rounds. [2]

7 A company has 12 employees consisting of 3 men and 9 women.

(a) The director wishes to gather opinions on female office dress code on Fridays from the female employees, so he sends a questionnaire to the 9 women. Explain whether these 9 women form a sample or a population. [1]

(b) How many ways are there to divide the 12 employees into three groups of 4 each such that each group consists of exactly one man? [2]

(c) The 12 employees are now seated at a round table. How many ways are there to seat them such that no two men are seated next to each other? [2]

8 A manufacturer produces a large number of tumblers of three colours: black, pink and white in the respective ratios 3 : 2 : 1. The tumblers are packed into boxes of 20 each.
Assume that the number of white tumblers in each box follows a binomial distribution.

(a) Find the probability that the number of white tumblers in a randomly chosen box is between 4 and 9 inclusive. [2]

(b) Find the probability that the 19th tumbler chosen from the box is the fourth white tumbler. [2]

It is known that, on average, p % of the tumblers are defective. The number of defective tumblers in a box also follows a binomial distribution. The probability that a box contains at most one defective tumbler is 0.95.

(c) Write down an equation satisfied by p. Hence find the value of p. [3]

9 A box contains two balls numbered 2 and k balls numbered 3, where $k \geq 3$. In a game, three balls are drawn from the box at random and the score is obtained by multiplying together the numbers indicated on the balls that were drawn. Let the random variable X denote the score obtained.

(a) Show that $\mathrm{E}(X) = \dfrac{9(3k^2+3k+2)}{(k+1)(k+2)}$. [4]

(b) Given that $\mathrm{E}(X) = \dfrac{144}{7}$, find the value of k. [1]

(c) Assuming that each game is independent of one another, find the probability that the average score of 30 games is more than 21. [3]

10 In an experiment, a sensor is released from a height of 12,000 m to record the atmospheric pressure at different heights above sea-level. The data from the sensor is recorded in the table below.

Height above sea-level, h (m)	0	2880	5470	7735	9650	12000
Atmospheric pressure, p (kPa)	101.3	76.8	49.2	35.0	29.5	21.5

It is thought that the atmospheric pressures at different heights above the sea-level can be modelled by one of the formulae

$$p = ah + b, \qquad \ln p = ch + d,$$

where a, b, c and d are constants.

(a) Find, correct to 4 decimal places, the value of the product moment correlation coefficient

- **(i)** between h and p, [1]
- **(ii)** between h and $\ln p$. [1]

(b) Explain which of $p = ah + b$ and $\ln p = ch + d$ is the better model and find the equation of a suitable regression line for this model. [3]

(c) It is known that an oxygen supply should be used when the atmospheric pressure is less than 57.2 kPa. A skydiver attempts to perform a skydive by jumping from a height of 4800 m without an oxygen supply. Use the equation of your regression line to estimate the atmospheric pressure and predict if the performance could be attempted safely. [2]

(d) Give two reasons why you would expect the estimate in part **(c)** to be reliable. [2]

11 A school shares that, on average, a student takes 90 minutes to complete an online learning assignment. Teacher Ian wishes to test whether this mean time taken has been understated. The time required to complete an online learning assignment, t minutes, is measured for a random sample of 45 students. The results are summarised as follows.

$$n = 45 \qquad \sum(t-90) = 15.39 \qquad \sum(t-90)^2 = 89.05$$

(a) Calculate unbiased estimates of the population mean and variance of the time taken for a student to complete an online learning assignment. [2]

(b) Test, at the 2% significance level, whether the mean time taken to complete an online learning assignment has been understated. You should state your hypotheses and define any symbols you use. [5]

(c) State the meaning of the p-value obtained in part **(b)**. [1]

After conducting some focus group discussions, the school claims that the population mean time taken for a student to complete an online learning assignment is μ_0 minutes.

(d) Teacher Ian takes another random sample of 40 students, where its mean and standard deviation of the time taken are 90.7 minutes and 1.3 minutes respectively. Given that a test of this sample at the 2% significance level indicates that the school's claim is valid, find the range of possible values of μ_0. [4]

12 **In this question you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

A vegetable seller sells potatoes and tomatoes. The masses of potatoes, X grams, have the distribution $\mathrm{N}(175, 25^2)$.

(a) Given that $\mathrm{P}(X < 166) = \mathrm{P}(X > m)$, find the value of m. [1]

(b) The probability that the mass of a randomly chosen potato differs from the population mean mass by more than a grams is 0.3. Find the value of a. [2]

(c) Find the expected number of potatoes with mass less than 182 grams in a randomly chosen batch of 90 potatoes. [2]

The masses of tomatoes, Y grams, have the distribution $\mathrm{N}(125, 15^2)$.

(d) Find the probability that the mass of a randomly chosen tomato is more than 130 grams. [1]

(e) A sample of 50 tomatoes are randomly chosen. Find the probability that exactly 20 tomatoes in this sample each has a mass more than 130 grams. [2]

The selling price of the potatoes is \$0.32 per 100 g and the selling price of the tomatoes is \$0.22 per 100 g.

(f) Find the probability that the price of a randomly chosen potato is greater than the total price of 2 randomly chosen tomatoes. [4]

2024 YIJC H2MA Prelim Examination Paper 1

	Solution	Comments
1(a)	$\frac{x}{x-1} \geq \frac{6}{4+x}$ $\frac{x}{x-1} - \frac{6}{x+4} \geq 0$ $\frac{x(x+4)-6(x-1)}{(x-1)(x+4)} \geq 0$ $\frac{x^2+4x-6x+6}{(x-1)(x+4)} \geq 0$ $\frac{x^2-2x+6}{(x-1)(x+4)} \geq 0$ $\frac{x^2-2x+1^2-1^2+6}{(x-1)(x+4)} \geq 0$ $\frac{(x-1)^2+5}{(x-1)(x+4)} \geq 0$ Since $(x-1)^2+5>0$ for all real values of x, $\therefore (x-1)(x+4)>0$ $x<-4$ or $x>1$	Do not cross-multiply here because we do not know the signs of $x-1$ and $4+x$. Need to explain why $x^2-2x+6=0$ does not have any critical value. Be clear of the difference between "and" and "or". Do not use comma. "and" means BOTH inequalities have to be satisfied for $\frac{(x-1)^2+5}{(x-1)(x+4)} \geq 0$ to be true while "or" means either $x<-4$ or $x>1$ is satisfied for $\frac{(x-1)^2+5}{(x-1)(x+4)} \geq 0$ to be true.
(b)	Replace x with $-\lvert x\rvert$ $-\lvert x\rvert<-4$ or $-\lvert x\rvert>1$ $\lvert x\rvert>4$ or $\lvert x\rvert<-1$ (no solution) $x<-4$ or $x>4$	Do not assume that once you see $\lvert x\rvert$ in the new inequality, it means the replacement is that. Check the signs. Need to reject $\lvert x\rvert<-1$ as $\lvert x\rvert \geq 0$

2 (a)	Area of 1st rectangle $= \frac{\pi}{2n} \times \left(\sin^2(0)\right)$ Area of 2nd rectangle $= \frac{\pi}{2n} \times \left(\sin^2\left(\frac{\pi}{2n}\right)\right)$ Area of 3rd rectangle $= \frac{\pi}{2n} \times \left(\sin^2\left(\frac{2\pi}{2n}\right)\right)$ Area of 4th rectangle $= \frac{\pi}{2n} \times \left(\sin^2\left(\frac{3\pi}{2n}\right)\right)$ $\vdots$ Area of n^{th} rectangle $= \frac{\pi}{2n} \times \left(\sin^2\left(\frac{(n-1)\pi}{2n}\right)\right)$ Area of all rectangles $= \frac{\pi}{2n} \times \left(\sin^2 0\right) + \frac{\pi}{2n} \times \left(\sin^2\left(\frac{\pi}{2n}\right)\right) + \frac{\pi}{2n} \times \left(\sin^2\left(\frac{2\pi}{2n}\right)\right)$ $+ \ldots + \frac{\pi}{2n} \times \left(\sin^2\left(\frac{(n-1)\pi}{2n}\right)\right)$ $= \frac{\pi}{2n} \sum_{k=1}^{n-1} \left(\sin^2\left(\frac{k\pi}{2n}\right)\right)$ $= \frac{\pi}{2n} \sum_{k=1}^{n-1} \frac{1}{2}\left(1-\cos\left(\frac{k\pi}{n}\right)\right)$ $= \frac{\pi}{4n} \sum_{k=1}^{n-1} \left(1-\cos\left(\frac{k\pi}{n}\right)\right)$	Check that the height of the n^{th} rectangle is the <u>y coordinate of the corner of the final rectangle touching the curve</u> For this "show" question, show clear steps, and should not write working as $\frac{\pi}{2n} \times \left(\sin^2 0\right) + \frac{\pi}{2n} \times \left(\sin^2\left(\frac{\pi}{2n}\right)\right) + \frac{\pi}{2n} \times \left(\sin^2\left(\frac{2\pi}{2n}\right)\right) + \ldots$ as this would mean that the sum is an infinite sum which is incorrect and misleading. Some students did not use the double angle formula to complete their working. The double angle formula is <u>assumed knowledge</u>.
(b)	We have $\lim_{n \to \infty} \frac{\pi}{4n} \sum_{k=1}^{n-1} \left(1-\cos\left(\frac{k\pi}{n}\right)\right) = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$ $= \frac{1}{2} \int_0^{\frac{\pi}{2}} 1-\cos 2x \, dx$ $= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$ $= \frac{1}{2}\left[\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin 2\left(\frac{\pi}{2}\right) - 0\right]_0^{\frac{\pi}{2}}$ $= \frac{\pi}{4}$	Many students managed to recognise that they had to compute the definite integral from 0 to $\frac{\pi}{2}$. This is a <u>standard question</u> and a limit for a rectangles question generally hints at definite integrals.

(c)	$\int_0^1 \sin^{-1}\sqrt{y}\,dy = \left(\frac{\pi}{2}\right)(1) - \int_0^{\frac{\pi}{2}} \sin^2 x\,dx$ $= \frac{\pi}{2} - \left(\frac{\pi}{4}\right)$ $= \frac{\pi}{4}$	This question was poorly done. Students need to recognise that the integrand is the inverse of the function given in the original question, which would mean the required area is the **area of the region bounded by the curve, the y-axis, and the line $y=1$.**
3(a)	It is not possible as by conjugate root theorem, since all the coefficients of $f(x)$ are real, **the non-real roots must occur in conjugate pairs**. Therefore, it is not possible to have an odd number of non-real roots.	Students need to write the keywords "coefficients of $f(x)$ are real" or "conjugate root theorem". By applying the conjugate root theorem, they would know that the non-real roots occur in conjugate pairs so there should not be odd number of non-real roots.
3(b)	Since all the coefficients of the equation are real, $x=3+2i$ is also a root. $[x-(3-2i)][x-(3+2i)]$ $=(x-3+2i)(x-3-2i)$ $=(x-3)^2-(2i)^2$ $=x^2-6x+9-(-4)$ $=x^2-6x+13$ $2x^4-15x^3+ax^2-63x+b=\left(x^2-6x+13\right)\left(2x^2+cx+\frac{b}{13}\right)$ Comparing coefficients of x^3: $-15=-6(2)+(1)(c)$ $c=-3$ Comparing coefficients of x: $-63=-6\left(\frac{b}{13}\right)+13c$ $-63=-\frac{6b}{13}+13(-3)$ $-6b=-312$ $b=52$ Comparing coefficient of x^2:	Students need to write the phrase "all coefficients of the equation are real" OR "by conjugate root theorem", AND state "3+2i" (the conjugate root) is also a root of the equation. Students need to learn how to compare coefficients WITHOUT expanding $\left(x^2-6x+13\right)\left(2x^2+cx+\frac{b}{13}\right)$. Students who are still unsure of how to compare coefficients without expanding should approach their tutors to clarify.

	$a=\left(\frac{b}{13}\right)-6c+2(13)$ $a=\left(\frac{52}{13}\right)-6(-3)+26$ $a=48$ $\therefore a=48, b=52$ $2x^2-3x+4=0$ $x=\frac{-(-3)\pm\sqrt{(-3)^2-4(2)(4)}}{2(2)}$ $x=\frac{3\pm\sqrt{-23}}{4}$ $x=\frac{3}{4}+\frac{\sqrt{23}}{4}i$ or $x=\frac{3}{4}-\frac{\sqrt{23}}{4}i$ $\therefore$ The other roots are $x=3+2i$, $x=\frac{3}{4}+\frac{\sqrt{23}}{4}i$ and $x=\frac{3}{4}-\frac{\sqrt{23}}{4}i$. **Method 2 (tedious):** $x^2=(3-2i)^2=9-12i-4=5-12i$ $x^3=(3-2i)(5-12i)=15-36i-10i+24i^2=-9-46i$ $x^4=(3-2i)(-9-46i)=-27-138i+18i+92i^2=-119-120i$ $2x^4-15x^3+ax^2-63x+b=0$ $2(-119-120i)-15(-9-46i)+a(5-12i)-63(3-2i)+b=0$ $(-292+5a+b)+(576-12a)i=0$ Comparing Imaginary part: $576-12a=0$ $a=48$ Comparing Real part: $-292+5a+b=0$ $-292+5(48)+b=0$ $b=52$ $\therefore a=48, b=52$ $2x^4-15x^3+48x^2-63x+52=0$ Since all the coefficients of the equation are real, $x=3+2i$ is also a root.	Many students forgot this part of the question to find the other roots of the equation. Students are advised to read the question carefully to check that they have answered the question fully and not just one part of the question. Students should also be reminded that roots are numbers or constants (which can be non-real numbers too), and roots are NOT algebraic expressions such as $2x^2-3x+4$. Students are <u>advised **NOT** to use method 2</u>. Some students who used method 2 did not read the first line in the question that calculators are not allowed in this question, and so they NEED to show <u>full workings</u> when calculating $(3-2i)^2, (3-2i)^3, (3-2i)^4$ as shown in the solutions on the left.

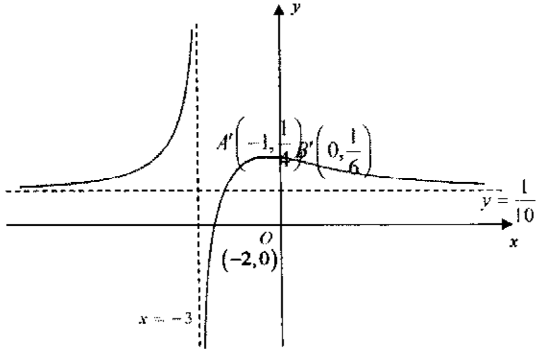
	$[x-(3-2\text{i})][x-(3+2\text{i})]$ $=(x-3+2\text{i})(x-3-2\text{i})$ $=(x-3)^2-(2\text{i})^2$ $=x^2-6x+9-(-4)$ $=x^2-6x+13$ $2x^4-15x^3+48x^2-63x+52=(x^2-6x+13)(2x^2+cx+4)$ Compare coefficient of x: $-63=-6(4)+13c$ $c=-3$ $2x^2-3x+4=0$ $x=\frac{-(-3)\pm\sqrt{(-3)^2-4(2)(4)}}{2(2)}$ $x=\frac{3\pm\sqrt{-23}}{4}$ $x=\frac{3}{4}+\frac{\sqrt{23}}{4}\text{i}$ or $x=\frac{3}{4}-\frac{\sqrt{23}}{4}\text{i}$ $\therefore$ The other roots are $x=3+2\text{i}$, $x=\frac{3}{4}+\frac{\sqrt{23}}{4}\text{i}$ and $x=\frac{3}{4}-\frac{\sqrt{23}}{4}\text{i}$.	
4(a)	Let $w=xy$. Differentiating with respect to x, we get $\frac{\mathrm{d}w}{\mathrm{d}x}=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$ ------ (1) $x\frac{\mathrm{d}y}{\mathrm{d}x}=xy(\ln x+\ln y)-y$ $x\frac{\mathrm{d}y}{\mathrm{d}x}+y=xy(\ln xy)$ Now, substituting (1) and $w=xy$, we get $\frac{\mathrm{d}w}{\mathrm{d}x}=w(\ln w)$ where $\mathrm{f}(w)=w\ln w$.	The original DE contains dy/dx so for substitution, we will need to replace all y in the original DE to w. The final answer in this question has only w term, $\mathrm{f}(w)$.

(b)	By (i), we have $\frac{\mathrm{d}w}{\mathrm{d}x}=w\ln w$. Then $\left(\frac{1}{w\ln w}\right)\frac{\mathrm{d}w}{\mathrm{d}x}=1$ $\int\frac{1}{w\ln w}\mathrm{d}w=\int 1\mathrm{d}x$ $\int\frac{\left(\frac{1}{w}\right)}{\ln w}\mathrm{d}w=\int 1\mathrm{d}x$ $\ln\lvert\ln w\rvert=x+C$ $\ln w=\pm\mathrm{e}^{x+C}$ $w=\mathrm{e}^{\pm\mathrm{e}^{x+C}}$ $w=\mathrm{e}^{A\mathrm{e}^x}$ where $A=\pm\mathrm{e}^C$. Since $w=xy$, then $xy=\mathrm{e}^{A\mathrm{e}^x}$ $y=\frac{\mathrm{e}^{A\mathrm{e}^x}}{x}$ When $x=2, y=\frac{1}{2}\mathrm{e}^3$. Then $\frac{1}{2}\mathrm{e}^3=\frac{\mathrm{e}^{A\mathrm{e}^2}}{2}$ $\mathrm{e}^3=\mathrm{e}^{A\mathrm{e}^2}$ $3=A\mathrm{e}^2$ $A=3\mathrm{e}^{-2}$ Therefore, $y=\frac{\mathrm{e}^{3\mathrm{e}^{x-2}}}{x}$.	To solve this DE, we will have to use variable separable.

5(a)(i)

$y = \dfrac{1}{\mathrm{f}(x)}$

$y = \mathrm{f}(x)$	
$A(-1,4)$	Turning point $A'\left(-1,\dfrac{1}{4}\right)$.
$B(0,6)$	y-intercept $B'\left(0,\dfrac{1}{6}\right)$
Vertical asymptote $x=-2$	x-intercept $(-2,0)$
x-intercept $C(-3,0)$	Vertical asymptote $x=-3$
Horizontal asymptote $y=10$	Horizontal asymptote $y=\dfrac{1}{10}$

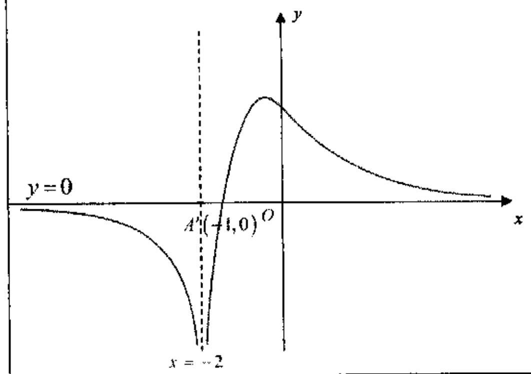


1. Use ruler to sketch curve.
2. Curves tend towards asymptotes.
3. Label all axial intercepts, turning points and asymptotes.
4. Curve must be smooth. Shouldn't have any kink.

5(a)(ii)

$y = \mathrm{f}(x)$	$y = \mathrm{f}'(x)$
$A(-1,4)$	$A'(-1,0)$
$B(0,6)$	-
Vertical asymptote $x=-2$	Vertical asymptote $x=-2$
x-intercept $C(-3,0)$	-
Horizontal asymptote $y=10$	Horizontal asymptote $y=0$

1. Not possible to label the turning point for graph of $y=\mathrm{f}'(x)$.
2. Asymptotes at x-axis must be label with $y=0$.



5(b)

$4x^2+y^2-4y=0$

$4x^2+y^2-4y+4=4$

$4x^2+(y-2)^2=4$

$x^2+\left(\dfrac{y-2}{2}\right)^2=1$

$x^2+\left(\dfrac{y-2}{2}\right)^2=1\xrightarrow[A]{}x^2+\left(\dfrac{y}{2}\right)^2=1\xrightarrow[B]{}x^2+y^2=1$

A: Translate the graph 2 units in the negative y-direction

B: Scale parallel to the y-axis by a scale factor of $\dfrac{1}{2}$

Method 2 (make y the subject)

$4x^2+y^2-4y=0$

$4x^2+y^2-4y+4=4$

$4x^2+(y-2)^2=4$

$(y-2)^2=4-4x^2$

$y-2=\pm\sqrt{4-4x^2}$

$y=2\pm\sqrt{4-4x^2}$

$y=2\pm2\sqrt{1-x^2}\xrightarrow[A]{}y=1\pm\sqrt{1-x^2}\xrightarrow[B]{}y=\pm\sqrt{1-x^2}$

$y^2=1-x^2$

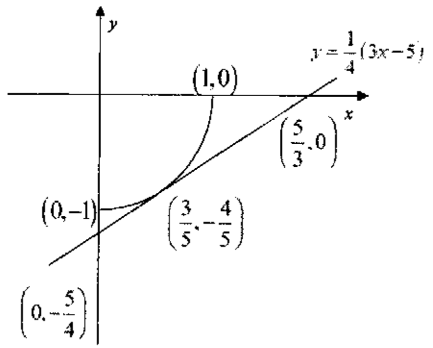
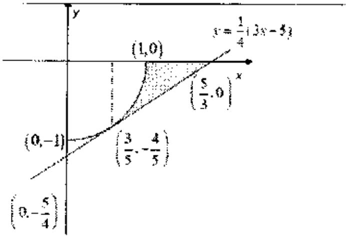
$x^2+y^2=1$

A: Scale parallel to the y-axis by a scale factor of $\dfrac{1}{2}$

B: Translate the graph 1 unit in the negative y-direction

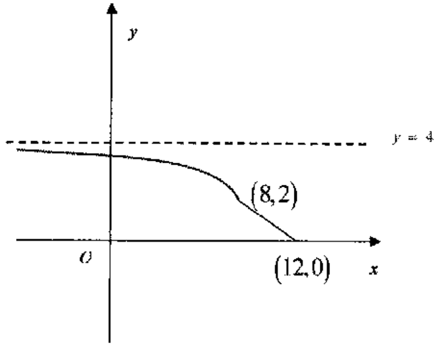
1. Have to complete the square as in conics transformation, you cannot have y and y^2 together.
2. Use proper terms. In your syllabus, only **Scale, translate or reflect**. NOT shift, move, stretch etc.

6(a)	$\frac{dx}{dt}=\frac{(-2t)(1+t^2)-(2t)(1-t^2)}{(1+t^2)^2}$ $=\frac{-2t-2t^3-2t+2t^3}{(1+t^2)^2}$ $=\frac{-4t}{(1+t^2)^2}$ $\frac{dy}{dt}=\frac{2(1+t^2)-(2t)(2t)}{(1+t^2)^2}$ $=\frac{2-2t^2}{(1+t^2)^2}$ At $t=-\frac{1}{2}$, $x=\frac{1-\left(-\frac{1}{2}\right)^2}{1+\left(-\frac{1}{2}\right)^2}=\frac{3}{5}$, $y=\frac{2\left(-\frac{1}{2}\right)}{1+\left(-\frac{1}{2}\right)^2}=-\frac{4}{5}$ $\frac{dy}{dx}=\frac{\left(-\frac{1}{2}\right)^2-1}{2\left(-\frac{1}{2}\right)}=\frac{3}{4}$ Equation of the tangent at $t=-\frac{1}{2}$: $y-\left(-\frac{4}{5}\right)=\frac{3}{4}\left(x-\frac{3}{5}\right)$ $y=\frac{3}{4}x-\frac{5}{4}$ $y=\frac{1}{4}(3x-5)$ (shown)	Should simplify the answers for $\frac{dy}{dt}$ and $\frac{dy}{dt}$ so that it is easier to find $\frac{dy}{dx}$. Need to show clear working when finding values of x, y, $\frac{dy}{dx}$ and equation of tangent since a calculator is not allowed and the answer is given in the question.

(b)		Remember to sketch the parametric curve based on the given range of values of t Must draw the line such that it is a tangent to the parametric curve at $t=-\frac{1}{2}$
(c)	 Required area $=\frac{1}{2}\left(\frac{5}{3}-\frac{3}{5}\right)\left(\frac{4}{5}\right)-\left(-\int_{\frac{3}{5}}^{1}y\,dx\right)$ $=\frac{32}{75}+\int_{-\frac{1}{2}}^{0}\frac{2t}{(1+t^2)}\left[-\frac{4t}{(1+t^2)^2}\right]dt$ $=\frac{32}{75}-\int_{-\frac{1}{2}}^{0}\frac{8t^2}{(1+t^2)^3}dt$ $=0.203$ units^2 (Using GC)	As the region bounded by the curve, x-axis and the line $x=\frac{3}{5}$ is below x-axis, we need to use $-\int_{\frac{3}{5}}^{1}y\,dx$ to obtain the area

7(a)	$\frac{4r-6}{(2r+1)(2r+3)(2r+5)} \equiv \frac{A}{2r+1}+\frac{B}{2r+3}+\frac{C}{2r+5}$ $4r-6=A(2r+3)(2r+5)+B(2r+1)(2r+5)+C(2r+1)(2r+3)$ Let $2r=-1: -8=A(2)(4) \Rightarrow A=-1$ Let $2r=-3: -12=B(-2)(2) \Rightarrow B=3$ Let $2r=-5: -16=C(-4)(-2) \Rightarrow C=-2$ $\frac{4r-6}{(2r+1)(2r+3)(2r+5)}=\frac{-1}{2r+1}+\frac{3}{2r+3}+\frac{-2}{2r+5}$	Substitute $r=-\frac{1}{2}, r=-\frac{3}{2}, r=-\frac{5}{2}$ to make it easy to find A, B and C or use cover-up rule (should not be expanding and comparing coefficients)
	$\sum_{r=1}^{N}\frac{4r-6}{(2r+1)(2r+3)(2r+5)}=\sum_{r=1}^{N}\left(-\frac{1}{2r+1}+\frac{3}{2r+3}-\frac{2}{2r+5}\right)$ $=\left(-\frac{1}{3}+\frac{3}{5}-\cancel{\frac{2}{7}}\right)$ $+\left(-\frac{1}{5}+\cancel{\frac{3}{7}}-\cancel{\frac{2}{9}}\right)$ $+\left(-\cancel{\frac{1}{7}}+\cancel{\frac{3}{9}}-\cancel{\frac{2}{11}}\right)$ $+\left(-\cancel{\frac{1}{9}}+\cancel{\frac{3}{11}}-\cancel{\frac{2}{13}}\right)$ ... $+\left(-\cancel{\frac{1}{2N-3}}+\cancel{\frac{3}{2N-1}}-\cancel{\frac{2}{2N+1}}\right)$ $+\left(-\cancel{\frac{1}{2N-1}}+\cancel{\frac{3}{2N+1}}-\frac{2}{2N+3}\right)$ $+\left(-\cancel{\frac{1}{2N+1}}+\frac{3}{2N+3}-\frac{2}{2N+5}\right)$ $=-\frac{1}{3}+\frac{3}{5}-\frac{1}{5}-\frac{2}{2N+3}+\frac{3}{2N+3}-\frac{2}{2N+5}$ $=\frac{1}{15}+\frac{1}{2N+3}-\frac{2}{2N+5}$	Do not re-arrange the terms as the denominators in the fractions are already in ascending or descending order, otherwise it would be challenging to do the cancellations Must show 2 full cancellations at the start and 1 full cancellation at the end and all intermediate terms must be cancelled

(c)	**Method 1**: Replace r with $r+1$ in $\sum_{r=7}^{\infty}\frac{4r-10}{(2r-1)(2r+1)(2r+3)}$ $\sum_{r+1=7}^{\infty}\frac{4(r+1)-10}{(2(r+1)-1)(2(r+1)+1)(2(r+1)+3)}$ $=\sum_{r=6}^{\infty}\frac{4r-6}{(2r+1)(2r+3)(2r+5)}$ $=\sum_{r=1}^{\infty}\frac{4r-6}{(2r+1(2r+3)(2r+5)}-\sum_{r=1}^{5}\frac{4r-6}{(2r+1)(2r+3)(2r+5)}$ $=\lim_{N\to\infty}\left(\frac{1}{15}+\frac{1}{2N+3}-\frac{2}{2N+5}\right)-\left(\frac{1}{15}+\frac{1}{2(5)+3}-\frac{2}{2(5)+5}\right)$ $=\frac{1}{15}-\left(\frac{1}{15}+\frac{1}{13}-\frac{2}{15}\right)$ $=\frac{11}{195}$ **Method 2:** List the first few terms of the given series and then write sigma notation using general term in (b) $\sum_{r=7}^{\infty}\frac{4r-10}{(2r-1)(2r+1)(2r+3)}$ $=\frac{18}{(13)(15)(17)}+\frac{22}{(15)(17)(19)}+\frac{26}{(17)(19)(21)}+\ldots$ $=\sum_{r=6}^{\infty}\frac{4r-6}{(2r+1)(2r+3)(2r+5)}$ $=\sum_{r=1}^{\infty}\frac{4r-6}{(2r+1(2r+3)(2r+5)}-\sum_{r=1}^{5}\frac{4r-6}{(2r+1)(2r+3)(2r+5)}$ $=\lim_{N\to\infty}\left(\frac{1}{15}+\frac{1}{2N+3}-\frac{2}{2N+5}\right)-\left(\frac{1}{15}+\frac{1}{2(5)+3}-\frac{2}{2(5)+5}\right)$ $=\frac{1}{15}-\left(\frac{1}{15}+\frac{1}{13}-\frac{2}{15}\right)$ $=\frac{11}{195}$	Must use (b) result so not allowed to do partial fractions and then method of difference Start with the given sum to do the replacement and not start with the result in (b) Must use (b) result to find $\sum_{r=1}^{5}\frac{4r-6}{(2r+1)(2r+3)(2r+5)}$ and not use GC

8(a)	$y=x(5-x)$ $y=5x-x^2$ $x^2-5x+y=0$ $\left(x-\frac{5}{2}\right)^2-\frac{25}{4}+y=0$ $x=\frac{5}{2}\pm\sqrt{\frac{25}{4}-y}$ Since $x\geq 3$, $x=\frac{5}{2}+\sqrt{\frac{25}{4}-y}$ $f^{-1}(x)=\frac{5}{2}+\sqrt{\frac{25}{4}-x}$ $D_{f^{-1}}=(-\infty,6]$	Either complete the square or use quadratic formula to make *x* the subject. Need to write explanation for rejection of $x=\frac{5}{2}-\sqrt{\frac{25}{4}-y}$. Answer the question. Don't write it together with the rule. Note that for set notation, the smaller value is always on the left. (You should not be writing $[6,-\infty)$)
8(b)		Clear distinction between the curve and line needs to be seen Equation of asymptote, (8, 2) and (12, 0) must be labelled. The coordinates are important to show which expression is being drawn for the piecewise function. There should not be an empty circle at (8, 2) as the point is included in the piecewise function.
8(c)	$g^{-1}(3)=x$ $g(x)=3$ $4+\frac{4}{x-10}=3$ $\frac{4}{x-10}=-1$ $x=6$	$g^{-1}(3)=x$ $gg^{-1}(3)=g(x)$ $3=g(x)$ Reminder: $gg^{-1}(x)=x$. Need to choose the correct expression in the piecewise function to solve for *x* because the point is on the **curve part** when $g(x)=3$.

8(d)	$R_f=(-\infty,6]$ $D_g=(-\infty,12]$ Since $R_f\subseteq D_g$, gf exists $gf(x)=g(5x-x^2)$ $=4+\frac{4}{5x-x^2-10}$	You must state R_f and D_g before making the conclusion. From $R_f=(-\infty,6]$ which will become the "input" for g(*x*), we see that we are looking at the curve part of the function g since the curve part is defined for $(-\infty,8]$.
8(e)	$R_{gf}=[3,4)$	Draw the graph of $y=4+\frac{4}{5x-x^2-10}$ to determine the range. OR "Input" $(-\infty,6]$ into the curve part of the function g and determine the range.
9(a)	$\int x^2\sqrt{1+2x^3}\,dx=\frac{1}{6}\int 6x^2\left(1+2x^3\right)^{\frac{1}{2}}dx$ $=\frac{1}{6}\left(\frac{2}{3}\right)\left(1+2x^3\right)^{\frac{3}{2}}+C$ $=\frac{1}{9}\left(1+2x^3\right)^{\frac{3}{2}}+C$	Rewrite the integral to the form $\int f'(x)[f(x)]^n\,dx$
(b)	$\int\sin 3x\sin 4x\,dx=-\frac{1}{2}\int -2\sin 4x\sin 3x\,dx$ $=-\frac{1}{2}\int\cos 7x-\cos x\,dx$ $=-\frac{1}{2}\left(\frac{\sin 7x}{7}-\sin x\right)+C$ $=\frac{1}{2}\sin x-\frac{1}{14}\sin 7x+C$	

	Considering $0 \le x \le \frac{\pi}{3}$ where $\sin 4x \le 0 \Rightarrow \frac{\pi}{4} \le x \le \frac{\pi}{3}$ $\int_0^{\frac{\pi}{3}} \sin 3x \lvert\sin 4x\rvert \, dx$ $= \int_0^{\frac{\pi}{4}} \sin 3x \sin 4x \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin 3x \sin 4x \, dx$ $= \left[\frac{1}{2}\sin x - \frac{1}{14}\sin 7x\right]_0^{\frac{\pi}{4}} - \left[\frac{1}{2}\sin x - \frac{1}{14}\sin 7x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $= \left[\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{14}\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{2}(0) + \frac{1}{14}(0)\right]$ $- \left[\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{14}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{14}\left(-\frac{1}{\sqrt{2}}\right)\right]$ $= \frac{8}{7\sqrt{2}} - \frac{3\sqrt{3}}{14}$	
(c)	$\int e^{-x} \cos 3x \, dx$ $= -e^{-x} \cos 3x - \int -e^{-x}(-3\sin 3x) \, dx$ $= -e^{-x} \cos 3x - 3\int e^{-x} \sin 3x \, dx$ $= -e^{-x} \cos 3x - 3\left[-e^{-x} \sin 3x - \int -e^{-x}(3\cos 3x) \, dx\right]$ $= -e^{-x} \cos 3x + 3e^{-x} \sin 3x - 9\int e^{-x} \cos 3x \, dx$ $\int e^{-x} \cos 3x \, dx = -e^{-x} \cos 3x + 3e^{-x} \sin 3x - 9\int e^{-x} \cos 3x \, dx$ $10\int e^{-x} \cos 3x \, dx = -e^{-x} \cos 3x + 3e^{-x} \sin 3x + C$ $\int e^{-x} \cos 3x \, dx = \frac{1}{10} e^{-x}(-\cos 3x + 3\sin 3x) + D$ $\int_0^{\pi} e^{-x} \cos 3x \, dx = \left[\frac{1}{10} e^{-x}(-\cos 3x + 3\sin 3x)\right]_0^{\pi}$ $= \frac{1}{10} e^{-\pi}(-\cos 3\pi + 3\sin 3\pi)$ $- \frac{1}{10} e^{0}(-\cos 3(0) + 3\sin 3(0))$ $= \frac{1}{10} e^{-\pi} + \frac{1}{10}$	Remember to include the arbitrary constant once integration is completed

10 (a)	At P, $\begin{pmatrix}10\\4\\3\end{pmatrix} + \lambda\begin{pmatrix}-2\\-5\\6\end{pmatrix} = \begin{pmatrix}-8\\3\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-3\\4\end{pmatrix}$ $10 - 2\lambda = -8 + \mu \quad \Rightarrow \quad 2\lambda + \mu = 18$ $4 - 5\lambda = 3 - 3\mu \quad \Rightarrow \quad 5\lambda - 3\mu = 1$ $3 + 6\lambda = 1 + 4\mu \quad \Rightarrow \quad -6\lambda + 4\mu = 2$ Solving the simultaneous equations, $\lambda = 5, \ \ \mu = 8$ $\overrightarrow{OP} = \begin{pmatrix}10\\4\\3\end{pmatrix} + 5\begin{pmatrix}-2\\-5\\6\end{pmatrix} = \begin{pmatrix}0\\-21\\33\end{pmatrix}$ The coordinates of P are $(0, -21, 33)$. Since $\lambda \neq \mu$, the aeroplanes passed through P at different timings, so they do not collide.	• Idea: Similar to finding the point of intersection between two lines in functions and graphs, we find the coordinates of the point P by equating, in this case, the vector equations of the two flight paths. Recall that two vectors are equal if and only if the respective components are equal. Hence we equate the x-, y- and z-components to form three linear equations and solve them simultaneously to find the values of μ and λ. • Reminders: o Show all working by writing down the three linear equations explicitly, before rearranging them into a form suitable for solution. o You should use your GC to solve the simultaneous equations. The use of GC is not disallowed, and it would save you the time taken and reduce the possibility of arithmetic errors from solving them algebraically. • Read the question carefully: It asked for the **coordinates** of P. $\begin{pmatrix}0\\-21\\33\end{pmatrix}$ is the position vector of the point P, relative to the origin, in this case the control tower; specifically the position vector is $\overrightarrow{OP} = \begin{pmatrix}0\\-21\\33\end{pmatrix}$, which a point cannot be "equal" to. This must be written in the form of the tuple, (0, −21, 33), to be recognised as the coordinates of P. • Self-check: If you obtained equations which you could not solve using your GC, or this led (algebraically) to two different sets of values of μ and λ, then you should check your own working for arithmetic or sign slips. Since the question specified the flight paths intersect at a point, it is only reasonable that a solution must exist and it must be unique. • For the explanation why the aeroplanes will not collide

		consists of two interdependent parts: o The explanation must be in the context of the question i.e. stating that the values of μ and λ are different without explaining what this meant contextually was insufficient. o The explanation must be supported or justified by figures. In this case, claims that the flight paths can intersect but the aeroplanes do not collide since they do not pass through P at the same time must explicitly refer to the figures for μ and λ found.
(b)	Let the acute angle be θ. $\sin\theta=\frac{\left\|\begin{pmatrix}-2\\-5\\6\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{4+25+36}}=\frac{6}{\sqrt{65}}$ $\theta=48.1^\circ$	Idea: The standard procedure to find the angle between a line and a plane involves the dot product of the **direction** vector of the line and the **normal** vector of the plane, which is the horizontal ground. However, $\cos^{-1}\frac{\begin{pmatrix}-2\\-5\\6\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{2^2+5^2+6^2}}$ gives the angle with the normal vector of the plane, rather than with the plane itself. In this case, subtract this angle from 90°, or use $\sin\theta=\frac{\left\|\begin{pmatrix}-2\\-5\\6\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{2^2+5^2+6^2}}$ to find the required angle directly.
(c)	The normal vector of π is $\begin{pmatrix}-2\\-5\\6\end{pmatrix}\times\begin{pmatrix}1\\-3\\4\end{pmatrix}=\begin{pmatrix}-2\\14\\11\end{pmatrix}$ Since P lies in π, $\begin{pmatrix}0\\-21\\33\end{pmatrix}\cdot\begin{pmatrix}-2\\14\\11\end{pmatrix}=69$ or $\begin{pmatrix}10\\4\\3\end{pmatrix}\cdot\begin{pmatrix}-2\\14\\11\end{pmatrix}=69$ or $\begin{pmatrix}-8\\3\\1\end{pmatrix}\cdot\begin{pmatrix}-2\\14\\11\end{pmatrix}=69$	• Idea: The standard procedure is to find a normal vector of the plane by carrying out a cross product of two vectors parallel to the plane, followed by a dot product of this normal vector with the position vector of a point on the plane. o Since the plane Π contains both flight paths, the direction vectors of the two flight paths are parallel to the plane and can be used directly in a cross product to find the normal vector. o The plane Π containing both flight paths also means that the fixed point of each flight path

	$\pi:\underset{\sim}{r}\cdot\begin{pmatrix}-2\\14\\11\end{pmatrix}=69$ A cartesian equation of π is $-2x+14y+11z=69$	will lie in the plane. In this case, you can and should use either $\begin{pmatrix}10\\4\\3\end{pmatrix}$ or $\begin{pmatrix}-8\\3\\1\end{pmatrix}$ rather than the position vector of the point P from part **(a)**, such that a wrong answer from the earlier part does not lead to another wrong answer for this part. • Read the question carefully: It asked for the **cartesian equation** of the plane Π, whereas $\underset{\sim}{r}\cdot\begin{pmatrix}-2\\14\\11\end{pmatrix}=69$ is the scalar product form. In particular, part **(d)** should have helped you, since it explicitly stated that the given equation of the wall, $ax+by+z=1$, is the cartesian equation.
(d)	The wall is parallel to l_B implies that its normal vector is perpendicular to the direction vector of l_B. $\begin{pmatrix}a\\b\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-3\\4\end{pmatrix}=0 \quad\Rightarrow\quad a-3b+4=0$ The wall inclines at 60° with the horizontal ground implies that $\cos 60^\circ=\frac{\left\|\begin{pmatrix}a\\b\\1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|}{\sqrt{a^2+b^2+1}}$ $\frac{1}{2}=\frac{1}{\sqrt{a^2+b^2+1}}$ $\sqrt{a^2+b^2+1}=2$ $a^2+b^2+1=4$ Substituting $a=3b-4$, $(3b-4)^2+b^2=3$ $10b^2-24b+13=0$ $b=0.826$ (Reject since $b>1$) or $b=1.57$ (3 sf) $a=3(1.574165739)-4=0.722$ (3 sf)	• Idea: Sketching a diagram to represent the information and hence visualise the relationship between the respective lines and planes would have greatly facilitated the solution of this problem in three-dimensional geometry. o A simple sketch of the slanted wall with its normal vector followed by the flight path of B parallel to the wall would have shown that the flight path is perpendicular to the wall's normal vector. Hence we equate the dot product of the flight path's direction vector with the wall's normal vector to 0. o The angle 60° is the angle between two planes, which is a different situation from part **(b)** where the required angle is between a line and a plane. In this case, the technique for these two parts would not be the same. In particular, the angle between two planes is equal to the angle between

		their normal vectors, such that $\cos 60^\circ = \dfrac{\begin{pmatrix} a \\ b \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{a^2+b^2+1}}$.
11 (a)	Using Pythagoras' theorem, $(h-R)^2 = R^2 - r^2$ $h - R = \sqrt{R^2-r^2}$ since $h-R>0$ $h = R+\sqrt{R^2-r^2}$	Key points: • Clear use of Pythagoras' theorem to show required relationship
(b)	$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\left(R+\sqrt{R^2-r^2}\right)$ $\frac{\mathrm{d}V}{\mathrm{d}r} = \frac{2}{3}\pi r\left(R+\sqrt{R^2-r^2}\right)+\frac{1}{3}\pi r^2\left[\frac{-2r}{2\sqrt{R^2-r^2}}\right]$ $= \frac{1}{3}\pi r\left[2R+2\sqrt{R^2-r^2}-\frac{r^2}{\sqrt{R^2-r^2}}\right]$ Putting $\frac{\mathrm{d}V}{\mathrm{d}r}=0$, $2R+2\sqrt{R^2-r^2}-\frac{r^2}{\sqrt{R^2-r^2}}=0$ $2R\sqrt{R^2-r^2}+2\left(R^2-r^2\right)-r^2=0$ $2R\sqrt{R^2-r^2}=3r^2-2R^2$ Squaring both sides, $4R^2\left(R^2-r^2\right)=9r^4-12r^2R^2+4R^4$ $9r^4-8r^2R^2=0$ $r^2\left(9r^2-8R^2\right)=0$ $r^2=\frac{8}{9}R^2$ since $r \neq 0$ $r=\frac{2\sqrt{2}}{3}R$ since $r>0$	**Key points:** • expressing V in terms of r and attempt to find $\frac{\mathrm{d}V}{\mathrm{d}r}$ using product rule. • solving $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ with attempts in expressing r in terms of R. • squaring both sides to remove the sqrt sign **Common issues:** • Improper use of product rule • Confusion between r the variable and R the constant.

	$V=\frac{1}{3}\pi r^2\left(R+\sqrt{R^2-r^2}\right)$ $=\frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left[R+\sqrt{R^2-\frac{8}{9}R^2}\right]$ $=\frac{8}{27}\pi R^2\left(R+\frac{1}{3}R\right)$ $=\frac{8}{27}\pi R^2\left(\frac{4}{3}R\right)$ $=\frac{32}{81}\pi R^3$ $k=\frac{32}{81}$ **Alternative solution (involving use of *h*):** $(h-R)^2=R^2-r^2$ $r^2=R^2-(h-R)^2$ $V=\frac{1}{3}\pi h\left(R^2-(h-R)^2\right)$ $=\frac{1}{3}\pi hR^2-\frac{1}{3}\pi h(h-R)^2$ $=\frac{1}{3}\pi hR^2-\frac{1}{3}\pi h\left(h^2-2Rh+R^2\right)$ $=\frac{2}{3}\pi Rh^2-\frac{1}{3}\pi h^3$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{4}{3}\pi Rh-\pi h^2$ Putting $\frac{\mathrm{d}V}{\mathrm{d}h}=0$, $\frac{4}{3}\pi Rh-\pi h^2=0$ $\pi h\left(\frac{4}{3}R-h\right)=0$ $h=0$ or $h=\frac{4}{3}R$	

	$V=\frac{2}{3}\pi Rh^2-\frac{1}{3}\pi h^3$ $=\frac{2}{3}\pi R\left(\frac{4}{3}R\right)^2-\frac{1}{3}\pi\left(\frac{4}{3}R\right)^3$ $=\frac{32}{27}\pi R^3-\frac{64}{81}\pi R^3$ $=\frac{32}{81}\pi R^3$	
(c)	$R=10$ $L=\frac{1}{3}\pi x^2(30-x)=10\pi x^2-\frac{1}{3}\pi x^3$ Differentiating both sides wrt t, $\frac{dL}{dt}=20\pi x\frac{dx}{dt}-\pi x^2\frac{dx}{dt}$ When $x=4$, $-2=\frac{dx}{dt}(80\pi-16\pi)$ $\frac{dx}{dt}=\frac{-2}{64\pi}=-0.00995$ (3 s.f.) The rate of decrease of x is 0.00995 cm s^{-1}.	**Key points:** • finding $\frac{dL}{dt}$ using either chain rule or implicit differentiation • identifying negative $\frac{dL}{dt}$ • expressing rate of change in numerical form.
(d)	When V is max, $r^2=\frac{8}{9}R^2$, $h=R+\sqrt{R^2-r^2}$ $=R+\sqrt{R^2-\frac{8}{9}R^2}$ $=\frac{4}{3}R$ $x=2R-h=\frac{2}{3}R$ Given that $R=10$. When $t=0$, $x=\frac{20}{3}$ $L=\frac{1}{3}\pi\left(\frac{20}{3}\right)^2\left(30-\frac{20}{3}\right)=1085.98$ Total time taken $=\frac{L}{2}\approx 543$ seconds (3 s.f.) or 9.05 minutes (3 s.f.)	**Key points:** • finding the value of x and L when V is max and t=0

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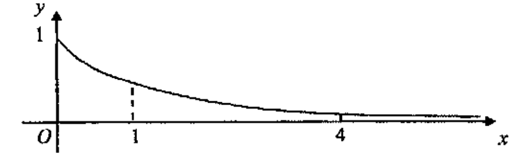
1a	$z_1=\dfrac{1-\mathrm{i}}{\cos\frac{1}{8}\pi-\mathrm{i}\sin\frac{1}{8}\pi}=\dfrac{\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}}=\sqrt{2}\mathrm{e}^{\mathrm{i}\theta\left(\frac{\pi}{8}-\frac{\pi}{4}\right)}=\sqrt{2}\mathrm{e}^{-\mathrm{i}\left(\frac{\pi}{8}\right)}$ $z_2{}^*=-1-\sqrt{3}\mathrm{i}=2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$ $-2=2\mathrm{e}^{\mathrm{i}\pi}$ $\dfrac{-2z_1}{z_2{}^*}=\dfrac{\left(2\mathrm{e}^{\mathrm{i}\pi}\right)\sqrt{2}\mathrm{e}^{-\mathrm{i}\left(\frac{\pi}{8}\right)}}{2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}}$ $=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\pi-\frac{\pi}{8}+\frac{2\pi}{3}\right)}$ $=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(\frac{37\pi}{24}-2\pi\right)}$ $=\sqrt{2}\mathrm{e}^{-\mathrm{i}\left(\frac{11\pi}{24}\right)}$ **Alternative method:** $z_1=\dfrac{1-\mathrm{i}}{\cos\frac{1}{8}\pi-\mathrm{i}\sin\frac{1}{8}\pi}=\dfrac{\sqrt{2}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}}{\mathrm{e}^{-\mathrm{i}\frac{\pi}{8}}}=\sqrt{2}\mathrm{e}^{\mathrm{i}\theta\left(\frac{\pi}{8}-\frac{\pi}{4}\right)}=\sqrt{2}\mathrm{e}^{-\mathrm{i}\left(\frac{\pi}{8}\right)}$ $z_2=-1+\sqrt{3}\mathrm{i}=2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ $\left\|\dfrac{-2z_1}{z_2{}^*}\right\|=\dfrac{\|-2\|\|z_1\|}{\|z_2{}^*\|}=\dfrac{2\sqrt{2}}{2}=\sqrt{2}$ $\arg(-2)+\arg\left(z_1\right)-\arg\left(z_2{}^*\right)=\pi+\left(-\dfrac{\pi}{8}\right)+\arg\left(z_2\right)$ $=\pi-\dfrac{\pi}{8}+\dfrac{2\pi}{3}$ $=\dfrac{37}{24}\pi$ $\arg\left(\dfrac{-2z_1}{z_2{}^*}\right)=\dfrac{37}{24}\pi-2\pi$ $=-\dfrac{11}{24}\pi$ $\dfrac{-2z_1}{z_2{}^*}=\sqrt{2}\mathrm{e}^{-\mathrm{i}\left(\frac{11\pi}{24}\right)}$	Key points: • Conversion to polar form OR product of complex numbers and using properties of modulus and argument.

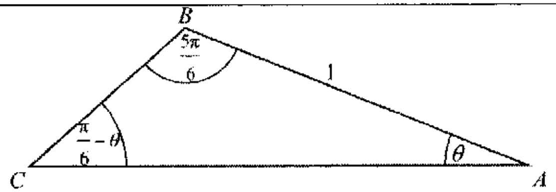
1b	Since $\left(\dfrac{-2z_1}{z_2{}^*}\right)^n<0$, $-\dfrac{11n\pi}{24}=(2k+1)\pi,\quad or\ -\dfrac{11n\pi}{24}=(2k-1)\pi,\ k\in\mathbb{Z}$ $n=-\dfrac{24(2k+1)}{11},\ or\ n=-\dfrac{24(2k-1)}{11}$ When $k=-6$, $n=24$ *or* When $k=-5$, $n=24$ When $k=-17$, $n=72$ *or* When $k=-16$, $n=72$ When $k=-28$, $n=120$ *or* When $k=-27$, $n=120$ The first three positive integer values of *n* are 24, 72 & 120 **Alternative Method**: $-\dfrac{11n\pi}{24}=-5\pi,-15\pi,\ -25\pi,\ \ldots$ $n=24,\ 72,\ 120$ The first three positive integer values of *n* are 24, 72 & 120.	Key points: • Identifying $\arg\left(\dfrac{-2z_1}{z_2{}^*}\right)=(2k+1)\pi$ from negative real number requirement. • Finding expression involving *n* and *k*. • Ensuring final answers are positive integers.
2a	Since $\sqrt{3}\|\mathbf{u}\times\mathbf{v}\|>0$, $\mathbf{u}\cdot\mathbf{v}>0$ which means $\cos\theta>0$ and hence θ is acute. $\sqrt{3}\|\mathbf{u}\times\mathbf{v}\|=\mathbf{u}\cdot\mathbf{v}$ $\sqrt{3}\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta=\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ $\dfrac{\|\mathbf{u}\|\|\mathbf{v}\|\sin\theta}{\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta}=\dfrac{1}{\sqrt{3}}$ $\tan\theta=\dfrac{1}{\sqrt{3}}$ $\theta=30°$	Key points: • Explaining how θ is acute starting from $\sqrt{3}\|\mathbf{u}\times\mathbf{v}\|$. Showing value of θ is not sufficient explanation.
bi	$\mathbf{u}\cdot\mathbf{v}$ is the length of projection of $\mathbf{v}$ onto $\mathbf{u}$	Key points: • Taking note that $\mathbf{u}$ is a unit vector and hence $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}=\mathbf{v}\cdot\hat{\mathbf{u}}$

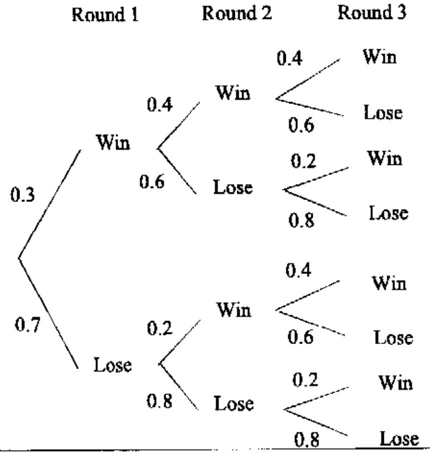
bii	$(\mathbf{u}\times\mathbf{v}+\mathbf{u})\cdot(\mathbf{v}\times\mathbf{u})$ $=(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{v}\times\mathbf{u})+[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{u})]$ $=(\mathbf{u}\times\mathbf{v})\cdot(-\mathbf{u}\times\mathbf{v})+0$ (since $\mathbf{u}\perp\mathbf{u}\times\mathbf{v}$) $=-\lvert\mathbf{u}\times\mathbf{v}\rvert^2$ $=-(\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\sin\theta)^2$ $=-((1)(3)\sin 30°)^2$ $=-\frac{9}{4}$	Key points: • Attaining $(\mathbf{u}\times\mathbf{v}+\mathbf{u})(\mathbf{v}\times\mathbf{u})$ $=(\mathbf{u}\times\mathbf{v})(\mathbf{v}\times\mathbf{u})+$ $[\mathbf{u}(\mathbf{v}\times\mathbf{u})]$ • Using $(\mathbf{u}\times\mathbf{v})(-\mathbf{u}\times\mathbf{v})$ $=-\lvert\mathbf{u}\times\mathbf{v}\rvert^2$ and $\mathbf{u}(\mathbf{v}\times\mathbf{u})=0$
3a	 $u=\sqrt{x}$ $\frac{du}{dx}=\frac{1}{2\sqrt{x}}=\frac{1}{2u}$ When $x=4, u=\sqrt{4}=2$. When $x=1, u=\sqrt{1}=1$. $\pi\int_1^4\left(\frac{1}{1+\sqrt{x}}\right)^2 dx$ $=\pi\int_1^2\left(\frac{1}{1+u}\right)^2 2u\ du$ $=\pi\int_1^2\frac{2u}{(1+u)^2}\,du$ $=\pi\int_1^2\frac{2(u+1)-2}{(1+u)^2}\,du$ $=2\pi\int_1^2\frac{1}{1+u}-\frac{1}{(1+u)^2}\,du$ $=2\pi\left[\ln(1+u)+\frac{1}{1+u}\right]_1^2$ $=2\pi\left[\ln 3+\frac{1}{3}-\ln 2-\frac{1}{2}\right]$ $=\left(2\ln\frac{3}{2}-\frac{1}{3}\right)\pi$	You should not be squaring the $2u$. i.e. It is wrong to do this: $\pi\int_1^2\left(\frac{1}{1+u}(2u)\right)^2 du$. This is because the formula for volume is $\pi\int_1^4 y^2\,dx$ so you need to substitute accordingly. You need to know what to substitute into dx. $dx=\frac{dx}{du}\times du$ There are a few ways to approach the integration here: - Juggling - Partial fraction (In this question, by parts also work but remember that it should always be the last resort)

b	Observe that both the upper and lower limits increased by 2 and in order to still have the same region, the graph will need to be translated by 2 units in the positive x-direction. Therefore, x is replaced by $x-2$. $a=-2$	Note that when the graph is translated by 2 units in the positive x-direction, x is replaced by $x-2$.
4ai	1st Term : a 2nd Term : $ar=a+6d$ 3rd Term : $ar^2=a+9d$ $\frac{a+6d}{a}=\frac{a+9d}{a+6d}$ $(a+6d)^2=a^2+9ad$ $a^2+12ad+36d^2=a^2+9ad$ $36d^2+3ad=0$ $3d(12d+a)=0$ $d=0$ or $12d+a=0$ Rejected as d < 0 $a=-12d$ **Alternative:** 1st Term : a 2nd Term : $ar=a+6d$ 3rd Term : $ar^2=a+9d$ 2nd Term – 1st Term: $a(r-1)=6d$ 3rd Term – 2nd Term: $ar(r-1)=3d$ $\frac{a(r-1)}{6}=\frac{ar(r-1)}{3}$ $r=\frac{1}{2}$ $a(\frac{1}{2})=a+6d$ $\frac{1}{2}a=-6d$ $a=-12d$	There are 3 variables a, r and d. We first eliminate r by utilising the knowledge of common ratio to form the equation $\frac{a+6d}{a}=\frac{a+9d}{a+6d}$. To solve this equation, we would need to expand out and eliminate certain terms. Since we are expressing a in terms of d, we should have a and d in our equation: $36d^2+3ad=0$. Since we see common term d, we can factorise and then solve the equation by equating the factors to 0.

aii	2nd Term : $ar = a + 6d$ $r = \frac{a+6d}{a} = 1 + 6\frac{d}{a}$ $r = 1 + 6(-\frac{1}{12}) = 1 - \frac{1}{2} = \frac{1}{2}$ $S_\infty = \frac{a}{1-r}$ $1 = \frac{-12d}{1-\frac{1}{2}}$ $\frac{1}{2} = -12d$ $d = -\frac{1}{24}$	Since we are given S_∞ and are trying to find value of d, we will think of $S_\infty = \frac{a}{1-r}$ and the objective is to ensure the equation only has d as the variable. So, since we have expressed a in terms of d in the previous part, we can use that to help us to find r. From there, we can then find the value of d using $S_\infty = \frac{a}{1-r}$.
b	$S_6 = \frac{6}{2}(2a + (6-1)d) = 15$ $2a + 5d = 5$ $a(a+d)(a+2d)(a+3d) = 0$ $a = 0$ (rejected, $a < 0$) or For $a = -d$, $3d = 5$, $d = \frac{5}{3}$, $a = -\frac{5}{3}$ (rejected, $a \in \mathbb{Z}$) or For $a = -2d$, $d = 5$, $a = -10$ or For $a = -3d$, $d = -5$, $a = 15$ (rejected, $a < 0$) Since $u_3 = -10 + 2(5) = 0$, $\therefore$ first positive term $= 0 + 5 = 5$ **Alternative for last part:** $u_n = a + (n-1)d$ $-10 + (n-1)(5) > 0$ $-10 + 5n - 5 > 0$ $5n > 15$ $n > 3$ When $n = 4$, $u_4 = -10 + (3)(5)$ $u_4 = 5$	Breakdown the question to form the 2 necessary equations: 1. $S_6 = \frac{6}{2}(2a + (6-1)d) = 15$ $2a + 5d = 5$ 2. $a(a+d)(a+2d)(a+3d) = 0$ Since there are 2 unknowns are 2 equations, we can solve them simultaneously. For equation 2, since it is expressed as product of factors equal to 0, we can equate each factor to 0. From there we can substitute equation 1 into each of the 4 equations and solve for the respective a and d. Since the question stated that the first term is a negative integer, we need to reject accordingly. With a and d, we can then find the first positive term. You should note just state that it is the 4th term. You need to state what the term is.

5ai	$\ln(1+\sin 2x)$ $= \ln\left[1 + \left(2x - \frac{(2x)^3}{3!} + \ldots\right)\right]$ $\approx \ln\left[1 + \left(2x - \frac{4x^3}{3}\right)\right]$ $= \left(2x - \frac{4x^3}{3}\right) - \frac{1}{2}\left(2x - \frac{4x^3}{3}\right)^2 + \frac{1}{3}\left(2x - \frac{4x^3}{3}\right)^3 - \frac{1}{4}\left(2x - \frac{4x^3}{3}\right)^4 + \ldots$ $= \left(2x - \frac{4x^3}{3}\right) - \frac{1}{2}\left(4x^2 - \frac{16x^4}{3} + \ldots\right) + \frac{1}{3}\left(8x^3 - \ldots\right) - \frac{1}{4}\left(16x^4 - \ldots\right) + \ldots$ $\approx 2x - 2x^2 + \frac{4}{3}x^3 - \frac{4}{3}x^4$ (Up to the term in x^4) **Alternative for first part of expansion:** $\ln(1+\sin 2x)$ $= \sin 2x - \frac{1}{2}(\sin 2x)^2 + \frac{1}{3}(\sin 2x)^3 - \frac{1}{4}(\sin 2x)^4 + \ldots$ $= \left(2x - \frac{(2x)^3}{3!} + \ldots\right) - \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \ldots\right)^2 + \frac{1}{3}\left(2x - \frac{(2x)^3}{3!} + \ldots\right)^3 - \frac{1}{4}\left(2x - \frac{(2x)^3}{3!} + \ldots\right)^4 + \ldots$ $= \left(2x - \frac{4x^3}{3}\right) - \frac{1}{2}\left(4x^2 - \frac{16x^4}{3} + \ldots\right) + \frac{1}{3}\left(8x^3 - \ldots\right) - \frac{1}{4}\left(16x^4 - \ldots\right) + \ldots$ $\approx 2x - 2x^2 + \frac{4}{3}x^3 - \frac{4}{3}x^4$ (Up to the term in x^4)	Very few Students fail to read the question and attempted to find the Series using repeated differentiation. Many students did not manage to evaluate to the correct number of terms thus did not manage to get the correct answer.
aii	Equation of tangent is $y = 2x$	Some students did not understand the idea of reading from the Series for the equation of tangent and revaluate the equation from $\ln(1+\sin 2x)$

bi	 $\frac{AC}{\sin\frac{5\pi}{6}}=\frac{1}{\sin\left(\frac{\pi}{6}-\theta\right)}$ $\frac{AC}{\sin\frac{5\pi}{6}}=\frac{1}{\sin\frac{\pi}{6}\cos\theta-\cos\frac{\pi}{6}\sin\theta}$ $\frac{AC}{\frac{1}{2}}=\frac{1}{\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta}$ $AC=\frac{1}{\cos\theta-\sqrt{3}\sin\theta}$	Students mistakenly assume sum of interior angles of a triangle as 2π Some students forget to write sin despite showing understanding of sin rule.
bii	AC $=\frac{1}{\cos\theta-\sqrt{3}\sin\theta}$ $\approx\frac{1}{1-\frac{\theta^2}{2}-\sqrt{3}\theta}$ (since θ is sufficiently small) $=\left(1-\sqrt{3}\theta-\frac{\theta^2}{2}\right)^{-1}$ $=1-\left(-\sqrt{3}\theta-\frac{\theta^2}{2}\right)+\frac{-1(-2)}{2!}\left(-\sqrt{3}\theta-\frac{\theta^2}{2}\right)^2+\ldots$ $=1+\sqrt{3}\theta+\frac{\theta^2}{2}+3\theta^2+\ldots$ $=1+\sqrt{3}\theta+\frac{7}{2}\theta^2+\ldots$	Among students who manage to attempt this question, there are a few who use x in the evaluation instead of θ.

6a		• Note: For tree diagrams, probabilities must be written on the branches and outcomes must be written at the end of the branches. Idea: The probabilities represent conditional probabilities i.e. given the previous outcome has occurred what is the probability the next outcome will occur, hence the probabilities should lie on the branches from one outcome to the next. • Reminders: o Since the answer space is provided, make use of and occupy the space to draw your tree diagram. o The outcomes for the respective rounds should be aligned i.e. the W's and L's for the second round should lie along one straight vertical line, similarly for the third round.
b	$\mathrm{P}(\text{Wins exactly once} \mid \text{Wins at least once})$ $=\frac{\mathrm{P}(\text{WLL, LWL, LLW})}{1-\mathrm{P}(\text{LLL})}$ $=\frac{0.3\times0.6\times0.8+0.7\times0.2\times0.6+0.7\times0.8\times0.2}{1-0.7\times0.8\times0.8}$ $=\frac{85}{138}$ or 0.616 (3s.f.)	• Note the phrase "given that" in the question; this means that the required probability is a conditional probability, as such the use of $\mathrm{P}(A\mid B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ is required. o For the numerator, you are required to interpret and work out P(wins exactly once ∩ wins at least once). In particular, the events "wins exactly once" and "wins at least once" are **not** independent from simple reasoning, since one is actually a subset of the other. Hence, you **cannot** apply $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$. In this case, actually P(wins exactly once ∩ wins at least once) = P(wins exactly), which you would use your tree diagram to find. o For the denominator, P(wins at least once), the most efficient approach is to consider the complementary event i.e. P(wins at least once) = 1 – P(loses all rounds), rather than tediously obtaining and adding the 7 related probabilities from your tree diagram. • Reminder: Check the reasonableness of your final answer if it exceeds 1, since you are finding a probability.

c	P (Carl plays the game for an even number of rounds) $=0.3(0.6)+0.3(0.4)(0.4)(0.6)+0.3(0.4)^4(0.6)+\ldots$ $=\dfrac{0.3(0.6)}{1-(0.4)^2}$ $=\dfrac{3}{14}$	• Interpret the context of the question: o Since he is to play the game until he loses 1 round, this means that he will be winning consecutively until he loses. o Since this loss is still a round that he played, it must be counted in the number of rounds he played. o Hence, consider: P(plays even number of rounds) = P(plays 2 rounds) + P(plays 4 rounds) + P(plays 6 rounds) + … = P(win until loses 2nd round) + P(win until loses 4th round) + P(win until loses 6th round) + … = P(W,L) + P(W,W,W,L) + P(W,W,W,W,W,L) + … o This will be the sum to infinity of a geometric series. • As with any question involving a progression or series, write down the first three terms at least, so as to ascertain the common ratio in this case, and hence apply the correct formula for the sum to infinity of a convergent geometric series.
7a	The director is interested in the opinions of the female employes only. Since all female employees are chosen, these 9 women form a population.	There are students who did not give explanations. Many students gave unnecessary explanation such as all women have equal opportunity, responses being independent of each other, or even quoting Central Limit Theorem.
b	${}^9C_3\times{}^6C_3\times{}^3C_3=1680$ Note: Each group is uniquely identified by the man in the group, hence there is no need to account for double counting. **Alternative:** $\dfrac{\left({}^3C_1\times{}^9C_3\right)\times\left({}^2C_1\times{}^6C_3\right)\times\left({}^1C_1\times{}^3C_3\right)}{3!}=1680$	This question was not very well attempted. Commonly, students apply addition principle instead.
c	$(9-1)!\times{}^9C_3\times 3!=20321280$	This is the most well done part for this question.
8a	Let X be the number of white tumblers in a box of 20. $X\sim\mathrm{B}(20,\frac{1}{6})$ $\mathrm{P}(4\le X\le 9)=\mathrm{P}(X\le 9)-\mathrm{P}(X\le 3)$ ≈ 0.433 **Alternative**: $\mathrm{P}(X=4)+\mathrm{P}(X=5)+\ldots+\mathrm{P}(X=9)\approx 0.433$	• Read the question carefully: The number is "between 4 and 9 inclusive". This means that both 4 and 9 are included in the calculation of the required probability. o We can find P(4 ≤ X ≤ 9) as a difference of two cumulative probabilities. In particular, since X = 4 is included for the required probability, we subtract the cumulative probability for X up to and including X = 3 i.e. P(4 ≤ X ≤ 9) = P(X ≤ 9) – P(X ≤ 3), which we then use GC to calculate. o Alternatively, we can find P(4 ≤ X ≤ 9) as a sum of discrete probabilities. In this case, by listing we obtain P(4 ≤ X ≤ 9) = P(X = 4) + P(X = 5) + … + P(X = 9). However, the calculation by this approach is considerably more tedious. • Use of random variables: o Define any random variables you use in answering a question to find the required probability. For a binomial, the random variable will be a **count** i.e. the number of objects from a random sample that meets some criteria. In this case, your definition should include the size of the random sample and the probability of "success" i.e. meeting the criteria. o State the distribution for the random variable you use in the probability calculation. This is essential working; any probability calculation involving the use of a random variable without its distribution and parameters stated will not make sense. o Do not use the letter Z to represent your random variable since this denotes the standard normal. o State and use exact values of parameters of the distribution. For parts **(a)** and **(b)**, the probability of "success" i.e. that a randomly chosen tumbler is white is exactly $\frac{1}{6}$; you should not write 0.166… in your distribution. • Reminders: o Do **not** write GC commands as part of your working. This is not mathematical notation. Quoting from the front cover of your exam paper, "you are required to present the mathematical steps using mathematical notations and not calculator commands". o Check the reasonableness of your final answer if it exceeds 1, since you are finding a probability.

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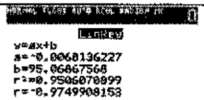
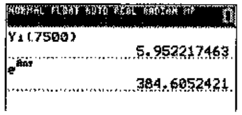
b	Let W be the number of white tumblers from 18 tumblers. $W \sim \mathrm{B}(18, \frac{1}{6})$ Required probability $= \mathrm{P}(W=3) \times \frac{1}{6}$ ≈ 0.0409	• Interpret the context of the question: o The question involves up to the 19th tumbler only i.e. the sampling does not involve all 20 tumblers in the box. In this case, the random variable and hence **the distribution will not be** the same as for part **(a)**. o Consider the number of white tumblers from the 19 tumblers chosen, denoted by Y, such that $Y \sim \mathrm{B}(19, \frac{1}{6})$. In this case, $\mathrm{P}(Y=4)$ is the probability that 4 white tumblers are chosen, but this does **not** mean that the 19th tumbler chosen will be a white one as required. (For instance, the 4 white tumblers could be the 1st, 2nd, 3rd and 4th tumblers chosen.) o Hence, P(19th tumbler chosen is 4th white one) = P(3 white tumblers chosen from first 18 tumblers) × P(19th tumbler is white), where P(19th tumbler is white) $= \frac{1}{6}$. In this case, for P(3 white tumblers chosen from first 18 tumblers), we define *another* random variable, represented by *another* letter, since the sample has changed and the distribution, which must be stated, is different from part **(a)**.
c	Let D be the number of defective tumblers in a box of 20. $D \sim \mathrm{B}(20, 0.01p)$ $\mathrm{P}(D \le 1) = 0.95$ $\mathrm{P}(D=0) + \mathrm{P}(D=1) = 0.95$ ${}^{20}C_0 (0.01p)^0 (1-0.01p)^{20} + {}^{20}C_1 (0.01p)^1 (1-0.01p)^{19} = 0.95$ $(1-0.01p)^{20} + 0.2p(1-0.01p)^{19} = 0.95$ Using G.C., $p \approx 1.81$	• Since p% of tumblers are defective, this means the probability that a randomly selected tumbler is defective is p%, which should be written in decimal form as $0.01p$ or in fraction form as $\frac{p}{100}$. In particular, p% is **not** the same as p; the distribution, which must be stated, should be $\mathrm{B}(20, 0.01p)$ or $\mathrm{B}(20, \frac{p}{100})$, **not** $\mathrm{B}(20, p)$. • After defining the random variable and stating the distribution, the required probability will be $\mathrm{P}(D \le 1) = 0.95$. This does not answer the question as it is not an equation in p; nowhere is the variable p explicitly seen in this equation. In this case, refer to MF26, where

		formulae for standard discrete distributions are included. Note that the formula is for the discrete probability $\mathrm{P}(X=x)$, not the cumulative probability $\mathrm{P}(X \le x)$. Hence, express $\mathrm{P}(D \le 1)$ as a sum of discrete probabilities i.e. $\mathrm{P}(D \le 1) = \mathrm{P}(D=0) + \mathrm{P}(D=1)$, in other words P(at most 1 defective) = P(none defective) + P(1 defective), then apply the formula respectively. • Note that **p is not an integer**, as such you should not be using the **table function** of your GC to find the value of p. In particular, since the value of p is non-exact, it should be found to 3 significant figures at least.
9a	**Method 1:** $\mathrm{P}(X=12) = \mathrm{P}(2,2,3 \text{ in any order})$ $= \frac{2}{k+2}\left(\frac{1}{k+1}\right)\left(\frac{k}{k}\right) \times 3$ $= \frac{6}{(k+1)(k+2)}$ $\mathrm{P}(X=18) = \mathrm{P}(2,3,3 \text{ in any order})$ $= \frac{2}{k+2}\left(\frac{k}{k+1}\right)\left(\frac{k-1}{k}\right) \times 3$ $= \frac{6(k-1)}{(k+1)(k+2)}$ $\mathrm{P}(X=27) = \mathrm{P}(3,3,3)$ $= \frac{k}{k+2}\left(\frac{k-1}{k+1}\right)\left(\frac{k-2}{k}\right)$ $= \frac{(k-1)(k-2)}{(k+1)(k+2)}$	Students forgot to multiply by 3, which is equivalent to multiplying $\frac{3!}{2!}$ as the case when $X=12$ means drawing $2,2,3$ in ANY order. This means that the balls that are drawn is such that there are 3 balls, of which 2 are <u>identical</u>, and so there is a need to multiply by $\frac{3!}{2!} = 3$. Similarly for the case of drawing three balls of $2,3,3$, there are 3 balls, of which 2 are identical, and so there is a need to multiply by $3 = \frac{3!}{2!}$ for the case when $X=18$. Students are reminded that for all "SHOW" type questions such as Q9(a), they are required to show full workings and **should not skip any steps and write fully all the steps as shown in the solutions for calculating** $\mathrm{E}(X)$. They should consult their tutors to clarify the demands of such questions if they are still unsure.

<table>
<tr><td></td><td>$$\mathrm{E}(X)=\frac{12(6)+108(k-1)+27(k-1)(k-2)}{(k+1)(k+2)}$$ $$=\frac{72+108k-108+27(k^2-3k+2)}{(k+1)(k+2)}$$ $$=\frac{27k^2+27k+18}{(k+1)(k+2)}$$ $$=\frac{9(3k^2+3k+2)}{(k+1)(k+2)} \quad \text{(shown)}$$
Method 2: P&C (tedious)
For example,
$$\mathrm{P}(X=27)=\frac{{}^{k}C_3}{{}^{k+2}C_3}$$ $$=\frac{\dfrac{k!}{(k-3)!3!}}{\dfrac{(k+2)!}{(k-1)!3!}}$$ $$=\frac{k!}{(k+2)!}\times\frac{(k-1)!3!}{(k-3)!3!}$$ $$=\frac{1}{(k+1)(k+2)}\times(k-1)(k-2)$$ $$=\frac{(k-1)(k-2)}{(k+1)(k+2)}$$</td><td></td></tr>
<tr><td>**b**</td><td>Since $\mathrm{E}(X)=\frac{144}{7}$
$\frac{9(3k^2+3k+2)}{(k+1)(k+2)}=\frac{144}{7}$
By GC, $k=6$</td><td>Most students got this part correct. Students are allowed to use the answer from Q9(a) even though they did not get the question correct and use their GC to obtain the value of k .</td></tr>
<tr><td>**c**</td><td><table><tr><td>x</td><td>12</td><td>18</td><td>27</td></tr><tr><td>$\mathrm{P}(X=x)$</td><td>$\frac{3}{28}$</td><td>$\frac{15}{28}$</td><td>$\frac{5}{14}$</td></tr></table>From GC, $\mathrm{Var}(X)=5.116001^2$</td><td>Since the question did not state that calculators should not be used, students can and should use their GC to calculate the variance.</td></tr>
</table>

<table>
<tr><td></td><td>Since $n = 30$ is large, by Central Limit Theorem,
$\bar{X} \sim \mathrm{N}\left(\frac{144}{7}, \frac{(5.116001)^2}{30}\right)$ approx.
$\mathrm{P}(\bar{X} > 21) = 0.323$ (3 s.f.)</td><td>Students need to write the following for this question:
• “Since $n = 30$ is large”
• “Central Limit Theorem” (spelt in FULL)
• Write the distribution $\bar{X} \sim \mathrm{N}\left(\frac{144}{7}, \frac{(5.116001)^2}{30}\right)$
Take note that the distribution is $\bar{X}$ and NOT X.
There are some students who wrote the correct distribution for $\bar{X}$ but obtained the wrong probability. These students should check where their calculations went wrong and not make this mistake in the coming A Levels.</td></tr>
<tr><td>**10a**</td><td>$p=ah+b$
$r=-0.9750$ (4 d.p.)

LinReg
y=ax+b
a=-0.0068136227
b=95.06867568
r²=0.9506070899
r=-0.9749908153
$\ln p = ch+d$
$r=-0.9965$ (4 d.p.)

LinReg
y=ax+b
a=-1.328536427ᴇ-4
b=4.64603469
r²=0.9930774508
r=-0.9965327144</td><td>Some students forgot that the question required them to round off to 4 decimal places. Students should read the question carefully before answering the questions.</td></tr>
<tr><td>**b**</td><td>$\ln p = ch+d$ is the better model since its product moment correlation coefficient value -0.9965 is closer to -1 than -0.9750.
The equation of $\ln p$ on h is
$\ln p = -0.000133h + 4.65$ (3 s.f.)</td><td>This question asks students to compare between two models and decide on the more appropriate model. This implies that students need to make a comparison, hence the word choice should be “closer to -1 ” instead of “close to -1”, so that it brings out the idea of some comparison of two (or more) models.</td></tr>
<tr><td>**c**</td><td>$\ln p = -0.000132854(4800) + 4.64603$
$= 4.0083308$
$p = e^{4.0083308}$
$= 55.0549$
$= 55.1$

Y₁(7500)
5.952217463
e^Ans
384.6052421
Since $55.1 < 57.2$,
the performance could not be attempted safely.</td><td>Some students forgot to answer the question on whether the performance could be attempted safely or not. Read the question carefully and always answer the question.</td></tr>
<tr><td>**d**</td><td>The estimate is reliable since
(i) The product moment correlation coefficient/r value - 0.9965 is very close to -1
(ii) $h = 4800$ is within the data range of h (or the estimate is an interpolation)</td><td>Most students could get this part correct, except that some missed out the reason that the r value is very close to -1.
Some students also gave only one reason when the question required</td></tr>
</table>

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		them to give TWO reasons. Students should always read the question carefully before answering them.

11a	Unbiased estimate of population mean $=\frac{\sum(t-90)}{45}+90$ $=\frac{15.39}{45}+90$ $=90.342$ Unbiased estimate of population variance $=\frac{1}{44}\left[89.05-\frac{15.39^2}{45}\right]$ $=1.90424(6\text{ s.f.})$ $=1.90\ (3\text{ s.f.})$	You need to remember how to find the unbiased estimate of population mean as the formula is not given.
b	Let μ be the **population mean time taken** for a student to complete an online learning assignment, in minutes. Let T be the time taken for a randomly chosen student to complete an online learning assignment. $H_0:\mu=90$ $H_1:\mu>90$ Level of significance $=2\%=0.02$ Since $n=45$ is sufficiently large, by Central Limit Theorem, $\bar{T}\sim\mathrm{N}\left(90,\frac{1.90424}{45}\right)$ approximately. Using GC, p-value $=0.0482025\ (6\text{ s.f.})$. Since p-value $=0.0482025>0.02$, we do not reject H_0. Hence there is insufficient evidence at the 2% significance level to indicate that the population mean time taken for a student to complete an online learning assignment is more than 90 minutes.	If the question asks for definition of symbol used, just define μ. Note that it is **population** mean and in context. Please adhere to the proper presentation of workings. Conclude in context.
C	The p-value in **(b)** indicates that there is a probability of 0.0482205 of drawing a random sample of 45 students whose sample mean time to complete an online learning assignment is more than 90.342 minutes, assuming the population mean time taken to complete an online learning assignment is 90 minutes.	Please learn this definition, Basically its explaining $p=\mathrm{P}(\bar{X}<\bar{x})$, $p=\mathrm{P}(\bar{X}>\bar{x})$, or both depending on your H₁.
d	Let T be the time taken for a randomly chosen student to complete an online learning assignment. Unbiased estimate of population variance $\frac{40}{39}(1.3^2)=1.73333\ (6\text{ s.f.})$ $H_0:\mu=\mu_0$ $H_1:\mu\neq\mu_0$	Note that sample variance is given so need to change to the unbiased estimate of population variance.

	Level of significance $= 2\% = 0.02$ Since $n = 40$ is sufficiently large, by Central Limit Theorem, $\overline{T} \sim \mathrm{N}\left(\mu_0, \frac{1.73333^2}{40}\right)$ approximately. For H_0 to be not rejected, we must have $-2.32635 < z_{test} < 2.32635$ $-2.32635 < \frac{90.7-\mu_0}{\left(\sqrt{\frac{1.73333}{40}}\right)} < 2.32635$ $-0.484268 < 90.7 - \mu_0 < 0.484268$ $90.2157 < \mu_0 < 91.1843$ (6 s.f.) $90.2 < \mu_0 < 91.2$ (3 s.f.)	

12a	$175 = \frac{166+m}{2}$ $m = 184$	
b	$X \sim \mathrm{N}(175, 25^2)$ $\mathrm{P}(\lvert X-175\rvert > a) = 0.3$ $\mathrm{P}(X-175 < -a \text{ or } X-175 > a) = 0.3$ $2P(X-175 < -a) = 0.3$ $P(X < 175-a) = 0.15$ $175 - a = 149.089$ $a = 25.9$ (3s.f.)	
c	Expected number $= 90 \times P(X < 182)$ $= 90 \times 0.610261$ $= 54.9$ (3s.f.)	Expected number ≠ mode
d	$Y \sim \mathrm{N}(125, 15^2)$ $P(Y > 130) = 0.36944 = 0.369$ (3s.f.)	
e	Let A be the number of tomatoes with a mass more than 130 grams, out of 50. $A \sim \mathrm{B}(50, 0.36944)$ $P(A = 20) = 0.10374 = 0.104$ (3s.f.)	
f	Price of Potatoes: \$0.32/100g $\Rightarrow \$\frac{0.32}{100}/\mathrm{g}$ Price of Tomatoes: \$0.22/100g $\Rightarrow \$\frac{0.22}{100}/\mathrm{g}$ Let $C = \frac{0.32}{100}(X) - \frac{0.22}{100}(Y_1 + Y_2)$ $\mathrm{E}(C) = \left(\frac{0.32}{100}\right)(175) - \left(\frac{0.22}{100}\right)(2)(125)$ $= 0.01$ $\mathrm{Var}(Y) = \left(\frac{0.32}{100}\right)^2 (25^2) + \left(\frac{0.22}{100}\right)^2 (2)(15^2)$ $= 0.008578$ $C \sim \mathrm{N}(0.01, 0.008578)$ $P(C > 0) = 0.543$ (3s.f.)	