2024 TJC Preliminary Exam H2 Mathematics Paper 1

1 A ball is rolling in a straight line such that its distance away from the starting point, s cm, can be modelled using the equation

$$s = at + \frac{b}{\sqrt{t+4}} + c,$$

where t is the time taken in seconds, and a, b and c are real constants.

The ball is at the starting point when $t = 0$, and moved 10 cm in the first 5 seconds. It moved another 9 cm in the next 16 seconds. Find the ball's distance away from the starting point when $t = 50$. [4]

2 On a single diagram, sketch the graphs of $y = |2x - p|$ and $y = qx$ where the following conditions are satisfied, indicating the axial intercepts.

- p and q are constants, $p > 1$ and $q > 0$, and
- the graphs have only one point of intersection. [2]

(a) State the least value of q. [1]

(b) Solve the inequality $|2x - p| > qx$, leaving your answer in terms of p and q. [2]

3 Find

(a) $\int \tan^2(x-1)\, dx$, [2]

(b) $\int \sin^{-1} 2x \; dx$. [3]

4 **Do not use a calculator in answering this question.**

(a) It is given $w = -\sqrt{3} + i$.

(i) Find arg w. [1]

(ii) Express iw^8 in the form $re^{i\theta}$ where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(b) **(i)** It is given that $(1 + ai)^2 = -3 - 4i$. Find the value of the real constant a. [2]

(ii) Hence solve the equation $2z^2 + (-3 + 2i)z + (1 - i) = 0$. [3]

5 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. C is the point on line OB such that AC is perpendicular to OB.

(a) By using a suitable scalar product, or otherwise, show that $\overrightarrow{OC} = \frac{(\mathbf{a}\cdot\mathbf{b})}{|\mathbf{b}|^2}\mathbf{b}$. [3]

(b) Give a geometrical interpretation of $\frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|}$. [1]

(c) It is given that $\mathbf{a} = \begin{pmatrix} -3 \\ 3 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ h \\ 0 \end{pmatrix}$. Given also that the length of the line segment AB is 5 units and angle AOB is an obtuse angle, find the exact value of h. [4]

6 The curve C is defined by the parametric equations

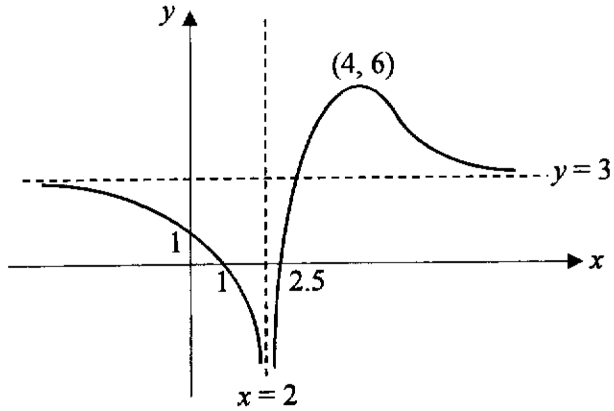
$$x = 1 - \cos t,\ \ y = \sin 2t,\ \text{ where } 0 \le t \le \frac{\pi}{2}.$$

(a) Sketch C, giving the exact coordinates of the points where C meets the x-axis. [1]

(b) The normal to C at the point where $t = \frac{\pi}{2}$ cuts the y-axis at D. Show that the y-coordinate of D is $-\frac{1}{2}$. [4]

(c) Find the exact area of the region bounded by C, the normal in part **(b)** and the y-axis. [5]

7 **(a)** The diagram shows the curve with equation $y = \mathrm{f}(x)$. The curve crosses the x-axis at $x = 1$ and $x = 2.5$, crosses the y-axis at $y = 1$ and has a maximum point at $(4,\ 6)$. The equations of the asymptotes are $x = 2$ and $y = 3$. Sketch the graph of $y = \mathrm{f}'(x)$, giving the equations of asymptotes, coordinates of turning points and axial intercepts, where possible. [2]



(b) The curve C has equation $y = \dfrac{x^2 + kx - 1}{x+1}$, where k is a non-zero constant.

(i) Find the range of values of k for which C has no stationary points. [4]

(ii) Given that $y = x + 3$ is an asymptote of C, show that $k = 4$. [2]

(iii) State a sequence of transformations which transform the graph of $y = \dfrac{x}{4} - \dfrac{1}{x}$ onto the graph of $y = \dfrac{x^2 + 4x - 1}{x+1}$. [3]

8 The function f and g are defined by

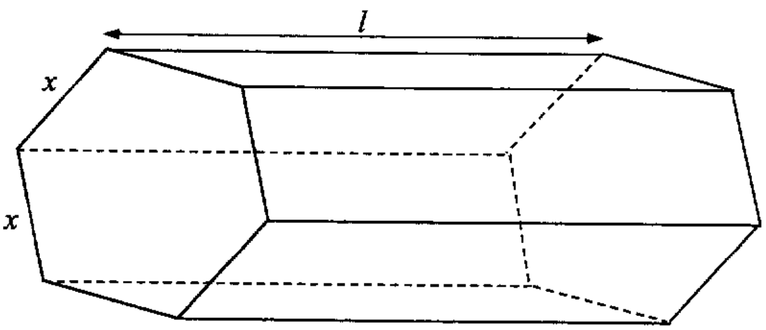
$$\mathrm{f} : x \mapsto -(x-4)^2 + 5, \quad x \in \mathbb{R},\ x \le 3,$$
$$\mathrm{g} : x \mapsto \mathrm{e}^{|x-3|}, \quad x \in \mathbb{R},\ x \le 10.$$

(a) Show that the composite function gf exists. [2]

(b) Find an expression for gf(x) and state the domain of gf. Hence find the value of x such that $x = (\mathrm{gf})^{-1}(1)$. [5]

(c) The function g has an inverse if its domain is restricted to $\alpha \le x \le 10$. State the smallest possible value of α and find $\mathrm{g}^{-1}(x)$, stating its domain. [4]

9



The figure above shows a metal rod with length l cm. The cross-section of the rod is a regular hexagon with sides of length x cm.

(a) A regular hexagon is made up of six identical triangles. Show that the area of the cross-section of the rod is $\dfrac{3\sqrt{3}x^2}{2}$ cm^2. [2]

(b) Suppose the rod has a fixed volume of C cm^3, show that the total surface area, S cm^2, of the rod may be expressed as $S = 3\sqrt{3}x^2 + \dfrac{4C}{\sqrt{3}x}$. [3]

(c) By using differentiation, find the value of x, in terms of C, which minimises S. [4]

Lucas heats up one of these metal rods. When heated, the metal rod expands uniformly such that it always retains its shape. At time t seconds, the length of each side of the hexagon is x cm, the length of the rod is l cm and the volume of the rod is V cm^3.

(d) Given that x and l are both increasing at a constant rate of 0.0025 cms^{-1}, find the rate of increase of V at the instant when $x = 2$ and $l = 5$. [2]

10 Anand writes a computer programme to simulate a population of organisms in a controlled environment. It is assumed that none of the organisms die or leave the environment within the duration of a simulation.

(a) In Simulation A, 200 organisms are introduced to the environment on Day 1. At the start of each subsequent day, 48 more organisms are introduced to the environment. Find the first day when the number of organisms in the environment exceeds 2025 at the end of that day. [2]

(b) In Simulation B, 15 organisms are introduced to the environment on Day 1. At the start of each subsequent day, each organism in the environment spawns two more organisms of the same type, i.e there are 45 organisms at the end of Day 2. Find the number of organisms in the environment at the end of Day 20. [2]

(c) In Simulation C, 5 organisms are introduced to the environment on Day 1. At the start of each subsequent day, the organisms in the environment will spawn in either one of the following ways.

I: Each organism will spawn three more organisms of the same type.

II: Each organism will spawn five more organisms of the same type.

On Day 2 to Day 9, the organisms undergo process **I** on m days and process **II** on the other days. Given that there are 1,105,920 organisms at the end of Day 9, find the value of m. [2]

Anand then adjusts the programme such that the simulation would allow for organisms to die at certain junctures.

(d) In Simulation D, 100 organisms are introduced to the environment at the start of Day 1. At the end of each day, 10% of the total population in the environment would die. At the start of Day 2 and each subsequent day, 20 organisms are introduced to the environment.

(i) Find an expression for the population size, P, in the environment at the start of Day n, after the organisms have been introduced. Leave your answer in the form $s - t\left(r^{n-1}\right)$, where s and t are positive integers and r is a real number. [4]

(ii) Describe what happens to the population size in the environment in the long term. [1]

(iii) Explain why the conclusion in **(ii)** does not depend on the population size in the environment on Day 1. [1]

11 A metal ball is released from the surface of the liquid in a tall cylinder. The ball falls vertically through the liquid and the distance, x cm, that the ball has fallen in time t seconds is measured. The speed of the ball at time t seconds is v cms^{-1}. The ball is released in a manner such that $x=0$ and $v=0$ when $t=0$.

(a) The motion of the ball is modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\frac{1}{2}\frac{\mathrm{d}x}{\mathrm{d}t}-10=0.$$

It is given that $v=\frac{\mathrm{d}x}{\mathrm{d}t}$.

(i) Show that the differential equation can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}t}=10-\frac{1}{2}v.$$ [1]

(ii) Using the differential equation in **(a)(i)**, find v in terms of t. **Hence** find x in terms of t. [6]

(b) The metal ball is now released in another tall cylinder filled with a different liquid. However, for this liquid, the motion of the ball is modelled by the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t}=10-k^2v^2, \text{ where } k \text{ is a positive constant.}$$

It is given that $x=0$ and $v=0$ when $t=0$.

(i) Find v in terms of t and k. [5]

(ii) When the ball falls through this liquid, its speed will approach its "terminal speed" which is the speed it will attain after a long time. Find the ball's terminal speed in terms of k. You must show sufficient working to justify your answer. [2]

2024 TJC Preliminary Exam H2 Mathematics Paper 2

Section A: Pure Mathematics [40 marks]

1 A curve C has equation $y=\dfrac{3-x}{x-1}$.

(a) Sketch C, stating the equations of the asymptotes. [2]

(b) Find the exact volume of the solid generated when the region bounded by C, the x-axis and the line $x=9$ is rotated through 2π radians about the x-axis. Leave your answer in the form $\pi(a+b\ln 2)$, where a and b are constants to be determined. [5]

2 **(a)** Find the series expansion of $\frac{(8+x)^{\frac{1}{3}}}{\cos 2x}$ in ascending powers of x up to and including the term in x^2. [5]

(b) Find the range of validity of x for the expansion to be valid. [2]

3 A sequence $\{a_n\}$ is defined by $a_0 = 2$ and $a_n = a_{n-1} - \frac{2}{3}\left(\frac{1}{3}\right)^{n-2}$ where $n \geq 1$.

(a) By considering $\sum_{n=1}^{N}(a_n - a_{n-1})$, find an expression for a_N. [4]

(b) Hence explain whether the sequence is convergent. [1]

4 The line l_1 passes through the point A with coordinates $(1,0,4)$ and is perpendicular to the plane π_1 with equation $2x-y+4z=-3$.

(a) Find the coordinates of the point B where l_1 meets π_1. [4]

(b) Verify that the point C with coordinates $(5,9,-1)$ lies on π_1. [1]

(c) Find a vector equation of the line l_2 which is a reflection of the line AC in π_1. [3]

(d) Find a vector equation, in scalar product form, of the plane π_2 which contains l_1 and l_2. [3]

5 **(a)** The complex number $z = x + iy$ is represented by the point $P(x, y)$ in an Argand diagram and satisfies the equation

$$zz^* = (2+3i)z^* + (2-3i)z + 12 .$$

(i) Show that P is a point on a circle, and state the centre and radius of the circle. [3]

(ii) The point Q represents the complex number $-4-5i$. Find the smallest possible length PQ. [2]

(b) **(i)** Show that $\dfrac{1+\cos\theta+i\sin\theta}{1-\cos\theta-i\sin\theta} = i\cot\dfrac{\theta}{2}$. [3]

(ii) It is given that $z = e^{i\theta}$. Find the set of integer values of n such that $\left(\dfrac{1+z}{1-z}\right)^n$ is always real. [2]

Section B: Probability and Statistics [60 marks]

6 **(a)** The 11 letters of the word REFRESHMENT are arranged in a row.

(i) Find the number of different arrangements that can be made. [2]

(ii) Find the number of different arrangements that can be made such that all the E's are together and all the R's are together but the E's and the R's are not together. [2]

(b) A 4-letter codeword is formed using the letters in the word REFRESHMENT. Find the number of different codewords that can be formed. [4]

7 Two boys, Joseph and Elliot, play a game by each tossing a coin. Joseph tosses a 20-cent coin and Elliot tosses a 50-cent coin. The probability that the 20-cent coin and the 50-cent coin shows a head are $\frac{3}{5}$ and p respectively.

If both coins show heads, Joseph gets to keep Elliot's coin.

If both coins show tails, Joseph gives his coin to Elliot.

If one coin shows a head and the other coin shows a tail, both get to keep their own coins.

Let W, in cents, be the amount of money Joseph wins in a game.

(a) Find, in terms of p, the probability distribution of W and $E(W)$. [4]

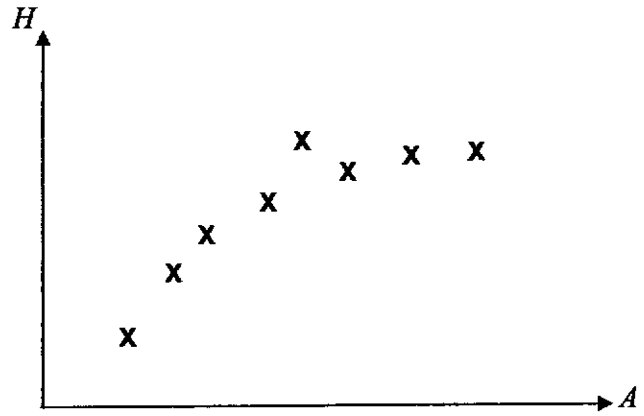
(b) Find the value of p for the game to be fair. [1]

(c) Suppose Elliot's coin is fair i.e. $p = \frac{1}{2}$ and the boys played 40 games. Find the probability that Joseph wins an average of more than 15 cents per game. [3]

8 A trainee nurse Angeline is investigating how the head circumferences of young children vary with age. The age, A months, and the head circumference, H cm, of a random sample of 8 young children are given in the table.

A	2	5	7	11	13	16	20	24
H	34.5	38.5	41	43	47	45	46.3	46.5

(a) The value of the product moment correlation coefficient between H and A is 0.880, correct to 3 decimal places, and a scatter diagram for the data is shown below.



(i) Explain whether a linear model is a good model for the relationship between H and A. [1]

(ii) Identify one of the data that Angeline may have recorded wrongly and justify your answer. [1]

(b) **For the rest of this question, you should not include the wrongly recorded data.**

Based on the scatter diagram in **(a)**, Angeline thinks that a model with equation $H = a + b\ln A$ is an appropriate model.

(i) Sketch a scatter diagram for H against $\ln A$. [1]

(ii) Use your calculator to find the equation of the least square regression line of H on $\ln A$ and the value of the corresponding product moment correlation coefficient. [2]

(iii) Use your equation to estimate the head circumference of a 13-month-old child. Give two reasons why you would expect this estimate to be reliable. [3]

9 In this question, you should state the parameters of any normal distributions you use.

In golf, a player's driving distance refers to the distance a ball travels when it is hit from the tee using a golf club known as a driver.

Records from past competitions show the following statistics.

- The proportion of male players with driving distance greater than 261 metres is equal to the proportion of male players with driving distance less than 170 metres.
- The proportion of male players with driving distance greater than 261 metres is equal to the proportion of female players with driving distance greater than 181 metres.
- The mean driving distance of a female player is 147.5 metres.

It may be assumed that the driving distances of male and female players follow normal distributions. The standard deviations of the driving distances of male and female players are denoted by σ_m and σ_f respectively.

(a) State the mean driving distance of a male player. [1]

(b) Show that $\sigma_m = k\sigma_f$, where k is a constant to be determined. [2]

It is given that $\sigma_m = 24.14$ and $\sigma_f = 17.77$.

(c) Find the probability that the driving distance of a randomly chosen male player is more than 1.5 times the driving distance of a randomly chosen female player. [3]

(d) Find the probability that the difference in driving distances of two randomly chosen female players is less than 15 metres. [3]

10 A ministry spokesman reported that students spend an average of 6.5 hours per week on co-curricular activities (CCA) in school. Mr Gru believes that the average time spend on CCA per week in his school is less than this average. To test his belief, he tasks his student Kevin to take a random sample of 50 students in his school. The times, x hours, spent on CCA per week are summarised below.

$$\sum x = 306.68 \qquad \sum x^2 = 1916.22$$

(a) State what it means for a sample to be random in this context. [1]

(b) Calculate unbiased estimates of the population mean and variance of the times spent on CCA per week. [2]

(c) Carry out a test and determine whether the p-value provides strong evidence to support Mr Gru's belief. [4]

(d) Kevin suggests to Mr Gru that it is necessary to assume that the times spent on CCA per week is normally distributed in order to carry out the test. Explain whether this assumption is necessary. [1]

(e) Student Bob takes another random sample of 50 students and finds that the mean and standard deviation of their times spent on CCA per week are m hours and 0.9 hours respectively. The result of a test at the 1% significance level is that the average time spent on CCA per week by students in his school differs from the average time reported by the ministry spokesman. Find the range of values of m. [5]

11 On average, Alex sleeps less than 6 hours on 75% of nights. The probability that he wakes up late on a school day is 0.625. On days where he wakes up late for school, there is a 96% chance that he has slept less than 6 hours the night before.

(a) Find the probability that he wakes up late when he has slept less than 6 hours the night before. [3]

(b) Determine, with justification, whether the event that he wakes up late for school is independent of the event that he has slept less than 6 hours. [1]

A school week has 5 school days. The number of days he wakes up late for school in a school week is denoted by X.

(c) State, in context, 2 assumptions needed for X to be well-modelled by a binomial distribution. [2]

Assume now that X can be modelled by a binomial distribution.

(d) Find the probability that, in a randomly chosen week, Alex wakes up late for school on at most 3 days. [1]

A school term has 10 weeks.

(e) Find the probability that Alex wakes up late for school on at most 3 days in a week for more than 4 weeks in a randomly chosen school term. State the distribution that you use. [3]

(f) Find the probability that Alex wakes up late for school on 32 days in a randomly chosen school term. State the distribution that you use. [2]

(g) Find the probability that Alex wakes up late for school on at most 3 days in a week for 4 weeks and wakes up late for school on 4 days in a week for the other weeks in a randomly chosen school term. [2]

BP-771

2024 TJC Preliminary Exam H2 Mathematics Paper 1 (Suggested solutions)

1 A ball is rolling in a straight line such that its distance away from the starting point, s cm, can be modelled using the equation

$$s = at + \frac{b}{\sqrt{t+4}} + c,$$

where t is the time taken in seconds, and a, b and c are real constants.

The ball is at the starting point when $t = 0$, and moved 10 cm in the first 5 seconds. It moved another 9 cm in the next 16 seconds. Find the ball's distance away from the starting point when $t = 50$. [4]

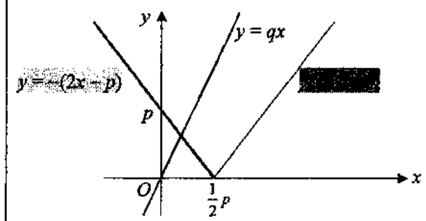
	[Solutions]	Remarks
	When $t=0, s=0$, $\quad 0=\frac{1}{2}b+c$ …………(1) When $t=5, s=10$, $\quad 10=5a+\frac{1}{3}b+c$ ……(2) When $t=21, s=19$, $\quad 19=21a+\frac{1}{5}b+c$ ……(3) Solving (1), (2) & (3) using GC, $a=0.08333$ or $\frac{1}{12}$ $b=-57.5$ or $-\frac{115}{2}$ $c=28.75$ or $\frac{115}{4}$ $\therefore s=\frac{1}{12}t-\frac{115}{2\sqrt{t+4}}+\frac{115}{4}$ When $t=50$, $s=\frac{1}{12}(50)-\frac{115}{2\sqrt{50+4}}+\frac{115}{4}=25.092$ Thus the distance away from starting point is 25.1 cm (3 s.f.)	- **3 unknowns** need 3 **equations** to solve. To simplify equations to as shown. - To read key word such as "another", "next" and "starting point". - To watch presentation of labeling "when…." and to number all equations - Use GC to solve! - Always read back to see objective in this case find s when t=50.

2 On a single diagram, sketch the graphs of $y=\lvert 2x-p\rvert$ and $y=qx$ where the following conditions are satisfied, indicating the axial intercepts.

- p and q are constants, $p>1$ and $q>0$, and
- the graphs have only one point of intersection. [2]

(a) State the least value of q. [1]

(b) Solve the inequality $\lvert 2x-p\rvert > qx$, leaving your answer in terms of p and q. [2]

	[Solutions]	Remarks
		Useful Tips: When dealing with modulus curve, it is advisable to label the positive/negative equations on the diagram. Note that $\lvert 2x-p\rvert=\begin{cases}2x-p, & x\ge\frac{p}{2}\\ -(2x-p), & x<\frac{p}{2}\end{cases}$ When sketching the graphs, ensure that • **x and y intercepts** are clearly indicated • the graph of $y=\lvert 2x-p\rvert$ is **symmetrical** about $x=\frac{p}{2}$ • the line $y=qx$ should be steeper than $y=2x-p$ in order to have only one intersection point • the equation of each graph must be clearly labelled
(a)	For the graphs to have only one point of intersection, the line $y=qx$ has the **same or a greater gradient** than the line $y=2x-p$, i.e $q\ge 2$. Least value of $q=2$	
(b)	At the intersection point, Use graph sketched earlier! [redacted] $(2+q)x=p$ $x=\frac{p}{q+2}$ For $\lvert 2x-p\rvert>qx$, $x<\frac{p}{q+2}$	Note: When solving inequalities using graphical method, we should always attempt to first find intersection points (if any). From (a), intersection point occurs at $x<\frac{p}{2}$. Hence when finding the intersection point, we should equate $y=-(2x-p)$ with $y=qx$ instead of $y=\lvert 2x-p\rvert$ with $y=qx$.

	Alternative method for finding intersection point $\lvert 2x-p \rvert = qx$ $2x-p=qx$ or $2x-p=-qx$ $2x-qx=p$ $2x+qx=p$ $(2-q)x=p$ $(2+q)x=p$ $x=\frac{p}{2-q}$ $x=\frac{p}{2+q}$ (rejected $\because \frac{p}{2-q}<0$ since $q \geq 2$)	For students who used the alternative method, do pay attention to the correct reasons for rejecting the other answer.

3 Find

(a) $\int \tan^2(x-1)\,dx$, [2]

(b) $\int \sin^{-1}2x\,dx$. [3]

	[Solutions]		Remarks
(a)	$\int \tan^2(x-1)\,dx$ $=\int \sec^2(x-1)-1\,dx$ $=\tan(x-1)-x+c$	$\int \sin^2(ax+b)\,dx$ $\int \cos^2(ax+b)\,dx$ $\int \text{cosec}^2(ax+b)\,dx$ $\int \sec^2(ax+b)\,dx$ $\int \cot^2(ax+b)\,dx$	- Students to note how to handle trigo with power 2 and linear angle! 1) $\sec^2$, cosec^2 : **can integrate directly!** 2) $\tan^2$, $\cot^2$: use trigo identity to convert to (1)! 3) $\sin^2$, $\cos^2$: use double angle formula - Students must remember the trigo identity formula correctly!
(b)	$\int 1\cdot\sin^{-1}2x\,dx$ I C – ∫ I D $=(x)\sin^{-1}2x-\int (x)\cdot\frac{2}{\sqrt{1-(2x)^2}}\,dx$ $=x\sin^{-1}2x+\frac{1}{4}\int (-8x)(1-4x^2)^{-\frac{1}{2}}\,dx$ $=x\sin^{-1}2x+\frac{1}{4}\times\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}}+c$ $=x\sin^{-1}2x+\frac{1}{2}\sqrt{1-4x^2}+c$		- Students pls note that : "Inverse ≠ Reciprocal" - Do not mix up $\frac{d}{dx}\sin^{-1}x$ with $\int \sin^{-1}x\,dx$ where we need to use by part for integration! - Must remember chain rule for differentiation and note the angle for trigo when applying the formula "$\frac{d}{dx}\sin^{-1}(2x)=\frac{2}{\sqrt{1-(2x)^2}}$" - Students to remember the following formula *for* $n\neq -1$, identify carefully what is the f(x) that will affect the $f'(x)$ and to divide by the new power: $\int f'(x)[f(x)]^n = \frac{[f(x)]^{n+1}}{n+1}+c$

4 [redacted]

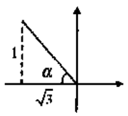
(a) It is given $w=-\sqrt{3}+\mathrm{i}$.

(i) Find arg w. [1]

(ii) Express $\mathrm{i}w^8$ in the form $re^{\mathrm{i}\theta}$ where $r>0$ and $-\pi<\theta\le\pi$. [3]

(b) **(i)** It is given that $(1+a\mathrm{i})^2=-3-4\mathrm{i}$. Find the value of the real constant a. [2]

(ii) [redacted] solve the equation $2z^2+(-3+2\mathrm{i})z+(1-\mathrm{i})=0$. [3]

	[Solutions]	Remarks
(a)(i)	$w=-\sqrt{3}+\mathrm{i}$ (2[nd] quadrant) $\tan\alpha=\left\lvert\frac{y}{x}\right\rvert=\left\lvert\frac{1}{-\sqrt{3}}\right\rvert$ $\Rightarrow \alpha=\frac{\pi}{6}$  $\arg w=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$	*As calculators are not allowed, detailed workings should be given.* Note that $\theta=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ gives $-\frac{\pi}{6}$, $-\frac{5\pi}{6}$, $\frac{5\pi}{6}$,....... The principal argument is the angle such that $-\pi<\theta\le\pi$.
(ii)	$\lvert w\rvert=\sqrt{(\sqrt{3})^2+1^2}=2$ $\boxed{\lvert \mathrm{i}w^8\rvert=\lvert\mathrm{i}\rvert\lvert w\rvert^8}=1\times2^8=256$ $\boxed{\arg(\mathrm{i}w^8)=\arg(\mathrm{i})+8\arg w}$ $=\frac{\pi}{2}+8\left(\frac{5}{6}\pi\right)$ $=\frac{43}{6}\pi=6\pi+\frac{7}{6}\pi\notin(-\pi,\pi]$ __Principal argument__ $=-\frac{5}{6}\pi$ Thus $\mathrm{i}w^8=256e^{-\frac{5}{6}\pi\mathrm{i}}$	$\lvert\mathrm{i}\rvert=1$ $\arg(\mathrm{i})=\frac{\pi}{2}$ Note that $\frac{43}{6}\pi\neq-\frac{5}{6}\pi$ r is the __modulus__ and must be a __positive real number.__ [redacted]
	Alternatively, from (i), $w=2e^{\frac{5}{6}\pi\mathrm{i}}$ $\therefore w^8=\left(2e^{\frac{5}{6}\pi\mathrm{i}}\right)^8=2^8e^{\frac{40}{6}\pi\mathrm{i}}=256\,e^{\frac{2}{3}\pi\mathrm{i}}$ $\mathrm{i}w^8=e^{\frac{\pi}{2}\mathrm{i}}\times256e^{\frac{2}{3}\pi\mathrm{i}}$ $=256e^{\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)\mathrm{i}}=256e^{\frac{7}{6}\pi\mathrm{i}}$ $=256e^{-\frac{5}{6}\pi\mathrm{i}}$	Note that $\mathrm{i}=(1)e^{\mathrm{i}\frac{\pi}{2}}$ [redacted]
(b)(i)	$(1+a\mathrm{i})^2=-3-4\mathrm{i}$ $1-a^2+2a\mathrm{i}=-3-4\mathrm{i}$ **Comparing imaginary parts,** $2a=-4$ $a=-2$ [Check: real parts $=1-a^2=1-2^2=-3$]	
(ii)	$2z^2+(-3+2\mathrm{i})z+(1-\mathrm{i})=0$ **Using the quadratic formula,** $z=\frac{3-2\mathrm{i}\pm\sqrt{(-3+2\mathrm{i})^2-4(2)(1-\mathrm{i})}}{2(2)}$ $=\frac{3-2\mathrm{i}\pm\sqrt{-3-4\mathrm{i}}}{4}$ $=\frac{3-2\mathrm{i}\pm(1-2\mathrm{i})}{4}$ Use earlier result in (i) $=1-\mathrm{i}$ or $\frac{1}{2}$	**Hence** means to use the result in part (i) $(1-2\mathrm{i})^2=-3-4\mathrm{i}$ $\Rightarrow\sqrt{-3-4\mathrm{i}}=\pm(1-2\mathrm{i})$ to obtain the roots of the equation.

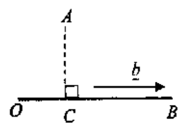
BP-773

5 The points A and B have position vectors **a** and **b** respectively. C is the point on line OB such that AC is perpendicular to OB.

(a) **By using a suitable scalar product**, or otherwise, show that $\overrightarrow{OC} = \dfrac{(\mathbf{a}\cdot\mathbf{b})}{\lvert\mathbf{b}\rvert^2}\mathbf{b}$. [3]

(b) Give a geometrical interpretation of $\dfrac{\lvert\mathbf{a}\cdot\mathbf{b}\rvert}{\lvert\mathbf{b}\rvert}$. [1]

(c) It is given that $\mathbf{a} = \begin{pmatrix}-3\\3\\-1\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}1\\h\\0\end{pmatrix}$. Given also that the length of the line segment AB is 5 units and angle AOB is an obtuse angle, find the exact value of h. [4]

	[Solutions]	Remarks
(a)	Since C is a point on line OB, $\overrightarrow{OC} = \lambda\underset{\sim}{b}$ for some $\lambda \in \mathbb{R}$  AC is perpendicular to OB $\overrightarrow{AC}\cdot\underset{\sim}{b} = 0$ $(\lambda\underset{\sim}{b} - \underset{\sim}{a})\cdot\underset{\sim}{b} = 0$ $\lambda\underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{b} = 0$ $\lambda\lvert\underset{\sim}{b}\rvert^2 = \underset{\sim}{a}\cdot\underset{\sim}{b}$ $\lambda = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert^2}$ Thus $\overrightarrow{OC} = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert^2}\underset{\sim}{b}$ (shown)	It is important to not just read the question but also to process the information provided. A good practice is to annotate on the question what each key piece of information translates to as you read the question e.g. …C is the point on line OB… (jot down $\overrightarrow{OC} = \lambda\underset{\sim}{b}$) …$AC$ is perpendicular to OB… (jot down $\overrightarrow{AC}\cdot\overrightarrow{OB} = 0$)
(b)	$\overrightarrow{OC} = \left(\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}\right)\dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$ $\lvert\overrightarrow{OC}\rvert = \left\lvert\dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}\right\rvert\left\lvert\dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}\right\rvert = \dfrac{\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert}{\lvert\underset{\sim}{b}\rvert}(1) = \dfrac{\lvert\underset{\sim}{a}\cdot\underset{\sim}{b}\rvert}{\lvert\underset{\sim}{b}\rvert}$ This is the length of projection of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$. OR this is the length OC.	Need to be careful of the term used. Length of projection ✓ not ~~line~~ of projection ✗

(c)	**Step 1: Determining possible h values** $\overrightarrow{AB} = \begin{pmatrix}1\\h\\0\end{pmatrix} - \begin{pmatrix}-3\\3\\-1\end{pmatrix} = \begin{pmatrix}4\\h-3\\1\end{pmatrix}$ Given: $\lvert\overrightarrow{AB}\rvert = \sqrt{16+(h-3)^2+1} = 5$ $(h-3)^2 = 25-17$ $h = 3\pm2\sqrt{2}$	Cambridge questions are very precise in their phrasing. When asked for exact value (singular), it literally means one value. This means further working is required to reject the non-applicable value.
	Step 2: Determining the correct h value **Approach 1A** Given: $\angle AOB$ is an obtuse angle $\overrightarrow{OA}\cdot\overrightarrow{OB} < 0$ $\begin{pmatrix}-3\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\h\\0\end{pmatrix} = -3+3h < 0 \Rightarrow h < 1$ Thus $h = 3-2\sqrt{2}$	For angle between two vectors, $\cos\theta = \dfrac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert}$ ✓ For approach 1A, it is sufficient to compare both h values and state the conclusion
	Approach 1B For $h = 3+2\sqrt{2}$ $\begin{pmatrix}-3\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\h\\0\end{pmatrix} = -3+3h = -3+3(3+2\sqrt{2}) = 6+6\sqrt{2} > 0$, while for $h = 3-2\sqrt{2}$ $\begin{pmatrix}-3\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\h\\0\end{pmatrix} = -3+3h = -3+3(3-2\sqrt{2}) = 6-6\sqrt{2} < 0$ $\therefore h = 3-2\sqrt{2}$	For approach 1B, as the substitution of h values require further evaluation to determine the sign of the dot product, it is insufficient to just prove either $h = 3+2\sqrt{2}$ leads to positive dot product or $h = 3-2\sqrt{2}$ leads to negative dot product. Both h values must be substituted to completely prove their validity or invalidity.

	Approach 2A $\angle AOB$ is an obtuse angle. $\Rightarrow \cos\theta < 0$ For $h = 3+2\sqrt{2}$ $\cos\theta = \frac{\underline{a}\cdot\underline{b}}{\lvert\underline{a}\rvert\lvert\underline{b}\rvert}$ $= \frac{\begin{pmatrix}-3\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\3+2\sqrt{2}\\0\end{pmatrix}}{\sqrt{(-3)^2+3^2+(-1)^2}\sqrt{1^2+(3+2\sqrt{2})^2+0^2}}$ $= 0.562 > 0$ For $h = 3-2\sqrt{2}$ $\cos\theta = \frac{\underline{a}\cdot\underline{b}}{\lvert\underline{a}\rvert\lvert\underline{b}\rvert}$ $= \frac{\begin{pmatrix}-3\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\3-2\sqrt{2}\\0\end{pmatrix}}{\sqrt{(-3)^2+3^2+(-1)^2}\sqrt{1^2+(3-2\sqrt{2})^2+0^2}}$ $= -0.562 < 0$ $\therefore h = 3-2\sqrt{2}$ **Approach 2B** Further process the $\cos\theta$ values to obtain the angle for direct comparison.	You may use the idea of the four quadrants to remember the signs of trigonometric ratios for different angles. For angle between two vectors, $\cos\theta = \frac{\underline{a}\cdot\underline{b}}{\lvert\underline{a}\rvert\lvert\underline{b}\rvert}$ ✔ It is not $\cos\theta = \frac{\lvert\underline{a}\cdot\underline{b}\rvert}{\lvert\underline{a}\rvert\lvert\underline{b}\rvert}$ ✘ $\cos\theta = \frac{\lvert\underline{a}\cdot\underline{b}\rvert}{\lvert\underline{a}\rvert\lvert\underline{b}\rvert}$ applies only to • acute angle between 2 lines, • acute angle between 2 planes • acute angle between a line and a plane.

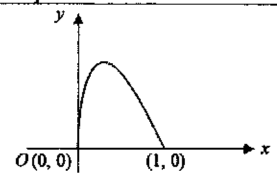
6 The curve C is defined by the parametric equations

$$x = 1-\cos t,\ y = \sin 2t,\ \text{ where } 0 \le t \le \frac{\pi}{2}.$$

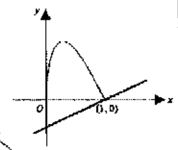
(a) Sketch C, giving the exact coordinates of the points where C meets the x-axis. [1]

(b) The normal to C at the point where $t = \frac{\pi}{2}$ cuts the y-axis at D. Show that the y-coordinate of D is $-\frac{1}{2}$. [4]

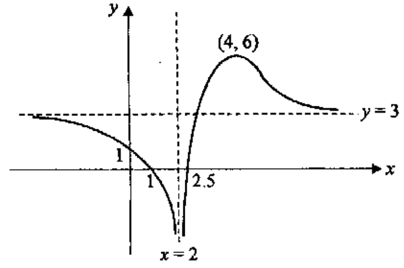
(c) Find the exact area of the region bounded by C, the normal in part **(b)** and the y-axis. [5]

	[Solutions]	Remarks
(a)		Always check (and double-check) that you have **set the correct range of the parameter t.** Remember that the default setting after resetting the GC is $0 \le t \le 2\pi$. Note the question requirement of giving **coordinates**.
(b)	$x = 1-\cos t \Rightarrow \frac{dx}{dt} = \sin t$ $y = \sin 2t \Rightarrow \frac{dy}{dt} = 2\cos 2t$ $\frac{dy}{dx} = \frac{2\cos 2t}{\sin t}$ When $t = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{2\cos\pi}{\sin\frac{\pi}{2}} = -2$, $x = 1, y = 0$ Hence gradient of normal $= \frac{-1}{-2} = \frac{1}{2}$ Equation of normal: $y = \frac{1}{2}(x-1)$ When $x = 0$, $y = -\frac{1}{2}$	Evaluate the value of $\frac{dy}{dx}$ first, instead of writing expressions in terms of $\frac{2\cos 2t}{\sin t}$

BP-775

(c)	Area $=\left(\frac{1}{2}\right)\frac{1}{2}(1)+\int_0^1 y\,dx$ $=\frac{1}{4}+\int_0^{\frac{\pi}{2}}\sin 2t\sin t\,dt$ $=\frac{1}{4}-\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos 3t-\cos t\,dt$ $=\frac{1}{4}-\frac{1}{2}\left[\frac{\sin 3t}{3}-\sin t\right]_0^{\frac{\pi}{2}}$ $=\frac{1}{4}-\frac{1}{2}\left[\frac{-1}{3}-1\right]=\frac{11}{12}$ units^2 	1. Notice that one part of the region is a triangle. Use the formula for area of triangle. 2. You should always start with the expression $\int_0^1 y\,dx$ before converting in terms of t. OR $\int \sin 2t\sin t\,dt$ $=\int (2\sin t\cos t)\sin t\,dt$ $=2\int \cos t(\sin t)^2\,dt$ $=2\left[\frac{(\sin t)^3}{3}\right]+C$

7 **(a)** The diagram shows the curve with equation $y=f(x)$. The curve crosses the x-axis at $x=1$ and $x=2.5$, crosses the y-axis at $y=1$ and has a maximum point at $(4,\,6)$. The equations of the asymptotes are $x=2$ and $y=3$. Sketch the graph of $y=f'(x)$, giving the equations of asymptotes, coordinates of turning points and axial intercepts, where possible. [2]

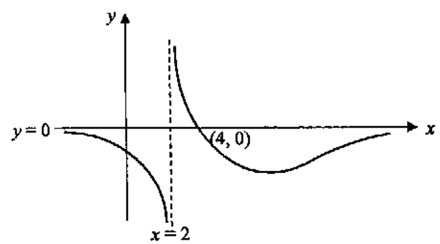


(b) The curve C has equation $y=\frac{x^2+kx-1}{x+1}$, where k is a non-zero constant.

(i) Find the range of values of k for which C has no stationary points. [4]

(ii) Given that $y=x+3$ is an asymptote of C, show that $k=4$. [2]

(iii) State a sequence of transformations which transform the graph of $y=\frac{x}{4}-\frac{1}{x}$ onto the graph of $y=\frac{x^2+4x-1}{x+1}$. [3]

	[Solutions]	Remarks
(a)	$y=f'(x)$ 	Note that $y=0$ is an asymptote that is not seen in the original graph. Note also that it is not possible to deduce the y-intercept of this graph.

(b)(i)	$y=\frac{x^2+kx-1}{x+1}=x+k-1-\frac{k}{x+1}$ $\frac{dy}{dx}=\frac{(x+1)(2x+k)-(x^2+kx-1)(1)}{(x+1)^2}=\frac{x^2+2x+k+1}{(x+1)^2}$ Let $\frac{dy}{dx}=0 \Rightarrow x^2+2x+k+1=0$ For no stationary points, the quadratic equation has no real roots Discriminant $=2^2-4(1)(k+1)<0$ $k+1>1 \Rightarrow k>0$	To approach this question, instead of thinking "$\frac{dy}{dx}<0$ or $\frac{dy}{dx}>0$", think "$\frac{dy}{dx}=0$ has no real solution" Note: $k=0$ is also a solution for this question but we do not mark students down for not mentioning it.
(ii)	Since $y=x+3$ is an asymptote of the hyperbola, $y=\frac{x^2+kx-1}{x+1}=x+3+\frac{c}{x+1}=\frac{x^2+4x+3+c}{x+1}$ Comparing coefficients, $k=4$ (and $c=-4$) (shown)	It is incorrect to state $x+3=\frac{x^2+kx-1}{x+1}$ (LHS is linear, RHS is not)
(iii)	$y=\frac{x^2+4x-1}{x+1}=x+3-\frac{4}{x+1}$ $y=\frac{x}{4}-\frac{1}{x}$ ↓ replace y by $\left(\frac{y}{4}\right)$ $y=4\left(\frac{x}{4}-\frac{1}{x}\right)=x-\frac{4}{x}$ ↓ replace x by $(x+1)$ $y=(x+1)-\frac{4}{(x+1)}$ ↓ replace y by $(y-2)$ $y=\left(x+1-\frac{4}{x+1}\right)+2=x+3-\frac{4}{x+1}$ The transformations are (in order): 1. A scaling parallel to the y-axis by factor 4. 2. A translation of 1 unit in the negative direction of x-axis. 3. A translation of 2 units in the positive direction of y-axis.	Use the idea of "replacement" to check your answers. Ensure that you use the correct terms and phrases for linear transformations. Cambridge has been particularly strict on this.

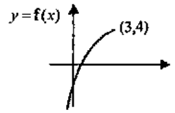
8 The function f and g are defined by

$$f: x \mapsto -(x-4)^2+5, \quad x\in\mathbb{R},\ x\le 3,$$
$$g: x \mapsto e^{|x-3|}, \quad x\in\mathbb{R},\ x\le 10.$$

(a) Show that the composite function gf exists. [2]

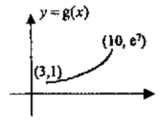
(b) Find an expression for gf(x) and state the domain of gf. Hence find the value of x such that $x=(\mathrm{gf})^{-1}(1)$. [5]

(c) The function g has an inverse if its domain is restricted to $\alpha \le x \le 10$. State the smallest possible value of α and find $g^{-1}(x)$, stating its domain. [4]

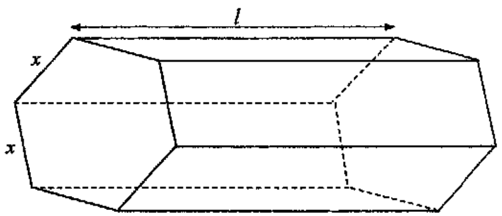
	[Solutions]	Remarks
(a)	$R_f=(-\infty,4]$ $D_g=(-\infty,10]$  Since $R_f \subset D_g$, gf exists.	- When finding range, please remember to draw graph according to DOMAIN! - Please **DO NOT OVERWRITE** the ") and]" when wrote wrongly as cannot tell which is the one! You are to **CANCEL and REWRITE**! - Please also remember the condition for checking composite function exit. - Check use of notation / keyword "subset" ! Note "subset, ⊂" is not the same as "element of, ∈"
(b)	$\mathrm{gf}(x)=e^{\left\|-(x-4)^2+5-3\right\|}=e^{\left\|-(x-4)^2+2\right\|},\ x\le 3$ $x=(\mathrm{gf})^{-1}(1)$, $(\mathrm{gf})(x)=1$ — Similar to how you do inverse trigo in secondary sch Eg $\sin^{-1}a=\theta \Rightarrow \sin\theta=a$ $e^{\left\|(x-4)^2-2\right\|}=1$ $\ln e^{\left\|(x-4)^2-2\right\|}=\ln 1$ $\left\|(x-4)^2-2\right\|=0$, $(x-4)^2=2$ — Note: in this case the modulus is not ignored but is because $(x-4)^2-2=\pm 0=0$ $x-4=\pm\sqrt{2}$ $x=4\pm\sqrt{2}$ Since $x\le 3$, $x=4-\sqrt{2}$ i.e. $x=(\mathrm{gf})^{-1}(1)=4-\sqrt{2}$	- Please remember domain of composite function is domain of 1st function! ie $D_{gf}=D_f$. Note: $D_{gf}\ne R_f$! - **Please do not ignore/ remove modulus without proper justification!** - Please remember the '±' when taking square root! - Whenever got 2 answers, please check to see if reject

BP-777

		any! In this the restriction of x is based on the domain of gf. - Students to be mindful to present answer clearly as there are too many x in the question, so pls ensure to write down the final line '$x=(gf)^{-1}(1)=4-\sqrt{2}$' to ensure this is answering the question
	Method 2: finding inverse function $y=e^{\left\|-(x-4)^2+2\right\|}$ $\ln y=\left\|-(x-4)^2+2\right\|$ $\pm\ln y=-(x-4)^2+2$ $(x-4)^2=2\mp\ln y$ $x-4=\pm\sqrt{2\mp\ln y}$ $x=4\pm\sqrt{2\mp\ln y}$ since $x\leq 3$, $x=4-\sqrt{2\mp\ln y}$ **IMPT: The definition of function in general: every input has only 1 output.** Hence CANNOT write $(gf)^{-1}(x)=4\pm\sqrt{2\mp\ln y}$ which will give 2 output because of the $\pm$ When $y=1$, $x=4-\sqrt{2}$ $\therefore x=(gf)^{-1}(1)=4-\sqrt{2}$	- Most students used this method but please be careful of how to remove modulus as well as the power 2! Will have 2 '$\pm$' in your working - Whenever have 2 answers, please check which to reject and **state reason clearly!** - Lastly, ensure to read question that they want it at $y=1$, hence need to sub in and write according to question to ensure you are answering the question!
(c)	Smallest value of $\alpha=3$ (for g to be one-one)	- To find smallest α, students can draw graph using GC and find the turning point.

	Since $3\leq x\leq 10$, $\|x-3\|=x-3$ $\therefore y=e^{x-3}$ $\ln y=x-3$ $x=\ln y+3$ $g^{-1}(x)=\ln x+3$ $D_{g^{-1}}=R_g=\left[1,e^7\right]$ 	- To find g^{-1}, do not ignore the modulus! - **And to reject with reason!** - Students to note that if they can reject, pls do so at the right juncture because if they proceed to reject at later part, they will have to provide more reasoning! - Most students know that $D_{g^{-1}}=R_g$ but please remember in finding R_g, you must draw graph **according to domain.**

9



The figure above shows a metal rod with length l cm. The cross-section of the rod is a regular hexagon with sides of length x cm.

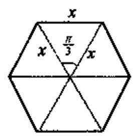
(a) A regular hexagon is made up of six identical triangles. Show that the area of the cross-section of the rod is $\dfrac{3\sqrt{3}x^2}{2}$ cm^2. [2]

(b) Suppose the rod has a fixed volume of C cm^3, show that the total surface area, S cm^2, of the rod may be expressed as $S = 3\sqrt{3}x^2 + \dfrac{4C}{\sqrt{3}x}$. [3]

(c) By using differentiation, find the value of x, in terms of C, which minimises S. [4]

Lucas heats up one of these metal rods. When heated, the metal rod expands uniformly such that it always retains its shape. At time t seconds, the length of each side of the hexagon is x cm, the length of the rod is l cm and the volume of the rod is V cm^3.

(d) Given that x and l are both increasing at a constant rate of 0.0025 cms^{-1}, find the rate of increase of V at the instant when $x = 2$ and $l = 5$. [2]

	[Solutions]	Remarks
(a)	Area $= \left(\frac{1}{2}(x^2)\sin\frac{\pi}{3}\right)\times 6$ $= \frac{3\sqrt{3}}{2}x^2$ 	Each of the 6 equilateral triangles **subtends** $\frac{360^\circ}{6} = 60^\circ$ **at the centre.** Write 60°, not 60. Use area of triangle = [redacted]
(b)	Volume of rod = cross sectional area × length $C = \frac{3\sqrt{3}}{2}x^2(l)$ $l = \frac{2C}{3\sqrt{3}x^2}$ $S = 2\left(\frac{3\sqrt{3}}{2}x^2\right) + 6xl$ $= 2\left(\frac{3\sqrt{3}}{2}x^2\right) + 6x\left(\frac{2C}{3\sqrt{3}x^2}\right)$ $= 3\sqrt{3}x^2 + \frac{4C}{\sqrt{3}x}$ (shown)	*As the answer is given, detailed steps are required.*
(c)	$\frac{dS}{dx} = 6\sqrt{3}x - \frac{4C}{\sqrt{3}x^2}$ [redacted] $6\sqrt{3}x = \frac{4C}{\sqrt{3}x^2}$ $18x^3 = 4C$ $x = \sqrt[3]{\frac{2C}{9}}$ **Using 2nd derivative test (preferred)** $\frac{d^2S}{dx^2} = 6\sqrt{3} + \frac{8C}{\sqrt{3}x^3}$ Since $x > 0$ and $C > 0$, $\frac{d^2S}{dx^2} > 0$ [or when $x = \sqrt[3]{\frac{2C}{9}}$, $\frac{d^2S}{dx^2} = 6\sqrt{3} + \frac{8C}{\sqrt{3}\left(\frac{2C}{9}\right)} = 18\sqrt{3}$ or $31.2 > 0$] Hence S is minimum when $x = \sqrt[3]{\frac{2C}{9}}$. **OR Using 1st derivative test** $\frac{dS}{dx} = 6\sqrt{3}x - \frac{4C}{\sqrt{3}x^2} = \frac{1}{\sqrt{3}}x\left(18 - \frac{4C}{x^3}\right)$	$\frac{d}{dx}\left(\frac{4C}{\sqrt{3}x}\right) = \left(\frac{4C}{\sqrt{3}}\right)\frac{d}{dx}(x^{-1})$ $= -\left(\frac{4C}{\sqrt{3}}\right)\frac{1}{x^2}$ $\frac{d}{dx}\left(\frac{4C}{\sqrt{3}x^2}\right) = \left(\frac{4C}{\sqrt{3}}\right)\frac{d}{dx}(x^{-2})$ $= -2\left(\frac{4C}{\sqrt{3}}\right)\left(\frac{1}{x^3}\right)$ Do not write $x = \sqrt[3]{\frac{2C}{9}}$ is a minimum point/value ✗

x	$\left(\sqrt[3]{\frac{2C}{9}}\right)^-$ eg 0.60C	$\left(\sqrt[3]{\frac{2C}{9}}\right)$	$\left(\sqrt[3]{\frac{2C}{9}}\right)^+$ eg 0.61C
$\frac{dS}{dx}$	$-0.180C < 0$	0	$0.133C > 0$
slope	╲	—	╱

Give values of $\frac{dS}{dx}$

Hence S is minimum when $x = \sqrt[3]{\frac{2C}{9}}$.

BP-779

(d)	$V=\frac{3\sqrt{3}}{2}x^2 l$ **Differentiate wrt t,** $\frac{dV}{dt}=\left(\frac{3\sqrt{3}}{2}\right)\left[l\left(2x\frac{dx}{dt}\right)+x^2\frac{dl}{dt}\right]$ When $x=2$ and $l=5$ and $\frac{dx}{dt}=\frac{dl}{dt}=0.0025$ $\frac{dV}{dt}=\left(\frac{3\sqrt{3}}{2}\right)\left[5(2(2)(0.0025))+2^2(0.0025)\right]$ $=0.159$ Thus the rate of increase of V is 0.159 cm^3s^{-1}.	**Since l is not a constant, we need to use product rule to find $\frac{dV}{dt}$**
	Alternatively, $V=\frac{3\sqrt{3}}{2}x^2 l$ **Differentiate wrt x,** $\frac{dV}{dx}=(3\sqrt{3}x)l+\left(\frac{3\sqrt{3}x^2}{2}\right)\frac{dl}{dx}$ $\frac{dl}{dx}=\left(\frac{dl}{dt}\right)/\left(\frac{dx}{dt}\right)=\frac{0.0025}{0.0025}=1$ $\frac{dV}{dt}=\frac{dV}{dx}\times\frac{dx}{dt}$ $=\left[(3\sqrt{3}(2))(4)+\left(\frac{3\sqrt{3}(2)^2}{2}\right)(1)\right]\times 0.0025$ $=0.156$ Thus the rate of increase of V is 0.159 cm^3s^{-1}.	**Since l is not a constant, we need to use product rule to find $\frac{dV}{dx}$**

10 Anand writes a computer programme to simulate a population of organisms in a controlled environment. It is assumed that none of the organisms die or leave the environment within the duration of a simulation.

(a) In Simulation A, 200 organisms are introduced to the environment on Day 1. At the start of each subsequent day, 48 more organisms are introduced to the environment. Find the first day when the number of organisms in the environment exceeds 2025 at the end of that day. [2]

(b) In Simulation B, 15 organisms are introduced to the environment on Day 1. At the start of each subsequent day, each organism in the environment spawns two more organisms of the same type, i.e there are 45 organisms at the end of Day 2. Find the number of organisms in the environment at the end of Day 20. [2]

(c) In Simulation C, 5 organisms are introduced to the environment on Day 1. At the start of each subsequent day, the organisms in the environment will spawn in either one of the following ways.

I: Each organism will spawn three more organisms of the same type.

II: Each organism will spawn five more organisms of the same type.

On Day 2 to Day 9, the organisms undergo process **I** on m days and process **II** on the other days. Given that there are 1,105,920 organisms at the end of Day 9, find the value of m. [2]

Anand then adjusts the programme such that the simulation would allow for organisms to die at certain junctures.

(d) In Simulation D, 100 organisms are introduced to the environment at the start of Day 1. At the end of each day, 10% of the total population in the environment would die. At the start of Day 2 and each subsequent day, 20 organisms are introduced to the environment.

(i) Find an expression for the population size, P, in the environment at the start of Day n, after the organisms have been introduced. Leave your answer in the form $s-t\left(r^{n-1}\right)$, where s and t are positive integers and r is a real number.[4]

(ii) Describe what happens to the population size in the environment in the long term. [1]

(iii) Explain why the conclusion in **(ii)** does not depend on the population size in the environment on Day 1. [1]

	[Solutions]	Remarks
(a)	Let A_n be the existing number of organisms under Stimulation A on Day n. (see table below) $A_n = 200 + (n-1)(48) \geq 2025$ **(A.P.)** $n \geq 39.02$ Thus there are at least 2025 organisms in the environment on Day 40.	Tabulating the values to derive a trend for the information given is helpful in assessing whether the n[th] term or the sum to n[th] term is the logical value required. When tabulating values, the focus is on deriving the trend. Do not over evaluate. Evaluating the expression to obtain the final value may hinder the derivation of the correct trend.
(b)	Let B_n be the number of organisms under Stimulation B on Day n. (see table below) $B_{20} = 15(3)^{19} = 1.74 \times 10^{10}$ **(G.P.)**	It is important to read the question carefully. The phrase "there are 45 organisms at the end of Day 2" provides important information that the original 15 organisms whereby each spawned 2 others the next day yielded a total population of $15 \times 3 = 45$ organisms. This indicates the parent organism is part of the population i.e after the daily spawning process, the population is tripled.

Table for (a):

Day	Existing Number
1	200
2	$200 + 48 = 200 + 1(48)$
3	$(200 + 48) + 48 = 200 + 2(48)$
4	$((200 + 48) + 48) + 48 = 200 + 3(48)$
…	…
n	$200 + (n-1)(48)$

Table for (b):

Day	Current	Spawned	Total
1	15	0	15
2	15	15(2)	$15 + 15(2)$ $= 15(1+2)$ $= 15(3)^1$
3	$15(3)^1$	$15(3) \times 2$	$15(3) + 15(3) \times 2$ $= [15(3)](1+2)$ $= 15(3)^2$
4	$15(3)^2$	$15(3)^2$	$15(3)^2 + 15(3)^2 \times 2$ $= [15(3)^2](1+2)$ $= 15(3)^3$
…	…	…	…
n	$15(3)^{n-2}$	$15(3)^{n-2} \times 2$	$15(3)^{n-2} + 15(3)^{n-2} \times 2$ $= [15(3)^{n-2}](1+2)$ $= 15(3)^{n-1}$

(c) $5 \times 4^m \times 6^{8-m} = 1105920$

Using GC, $m = 5$

OR By factorisation, $1105920 = 5 \times 4^5 \times 6^3 \therefore m = 5$

Note:

Commutative Property of Multiplication

You would have probably learned it formally in the lower secondary levels.

Essentially, we can say that

$5 \times 4 \times 6 = 5 \times 6 \times 4$.

i.e. the order of multiplication does not matter, which we all know.

The implication here is that the order of spawning via process I or II does not matter and since it does not matter, all we need to know that in the 8 days running from days 2 to 9, m days involve spawning by process I and $8 - m$ days will involve spawning by process II, leading to the final population being computed by $5 \times 4^m \times 6^{8-m}$.

Remarks:

Extending from part **(b)**, the spawning 3 more organisms will means the population is multiplied by 4 times.

Likewise, spawning 4 more organisms will indicate the population is multiplied by 5 times.

Looking beyond the question, the important learning point to take away to scrutinize the information provided carefully as it may have a downstream impact towards the understanding of the different parts of the question.

Notice that once part (b) was incorrectly understood, part (c) would have likely used the wrong values of 3 and 5 instead of the required 4 and 6.

(d) (i)

n	Organisms at start of Day n (units)	Organisms at end of Day n (units)
1	100	$(0.9)100$
2	$(0.9)100 + 20$	$(0.9)^2(100) + (0.9)(20)$
3	$(0.9)^2(100) + (0.9)(20) + 20$ $= (0.9)^2(100) + (0.9)^1(20) + (0.9)^0(20)$	$(0.9)^3 100 + (0.9)^2(20) + (0.9)(20)$
4	$(0.9)^3(100) + (0.9)^2(20) + (0.9)^1(20) + (0.9)^0(20)$	:
:	:	:
n	$(0.9)^{n-1}(100) + (0.9)^{n-2}(20) + (0.9)^{n-3}(20) + \ldots + 20(0.9)^0$	:

At start of Day n, after the organisms are introduced, the population size

$= (0.9)^{n-1}(100) + (0.9)^{n-2}(20) + (0.9)^{n-3}(20) + \ldots + 20$

$= (0.9)^{n-1}(100) + 20\left(\frac{1-(0.9)^{n-1}}{1-0.9}\right)$

$= (0.9)^{n-1}(100) + 200(1-(0.9)^{n-1})$

$= 200 - 100(0.9)^{n-1}$

Comments

1. When tabulating a fairly complicated series, it is always important not to over evaluate. The focus is to identify a trend in the tabulated expression and not a number pattern arising from evaluated values.

2. Group like terms together to form a series. To this end, some rules of thumb which may be useful are:
   - Terms with the same constant value multiplied to the same ratio that changes exponentially likely forms a GP e.g. in this question we have:

     [redacted] $(0.9)^0$

     where 20 is the same constant value and 0.9 is the multiplier ratio that changes exponentially.

     Note that when the same constant value has no ratio multiplied to it, in this case 20, we can append the multiplier term $(\text{ratio})^0$, in this case 0.9^0 to ensure that the number of terms in the GP can be counted correctly, in this case from 0 to $n-2$, there will be $(n-2)-0+1 = n-1$ terms [the idea of no. of terms = upper limit – lower limit + 1]
   - A standalone constant raised to an exponent or a standalone constant multiplied to a ratio raised to an exponent is likely a power series e.g. in this question $100(0.9)^{n-1}$ or say 87^n for another unrelated instance.
   - Terms with the same constant being add progressively will likely form an AP.

3. It is good practice to put your working for identifying the trend in a table for proper organisation.

4. To determine the n^{th} expression correctly, the trick is to first observe the latest terms of each group to see what is the relation to n e.g. in this question

Day	Organisms at start of Day n (units)
4	$(0.9)^3(100) + (0.9)^2(20) + (0.9)^1(20) + (0.9)^0(20)$

For the power series $(0.9)^3(100)$, 0.9 is raised to the power $3 = 4 - 1$. So it follows that in the n^{th} term, the expected component will be $(0.9)^{n-1}(100)$

Likewise, for the terms comprising of $(0.9)^2(20) + (0.9)^1(20) + (0.9)^0(20)$, the highest power of 0.9 is 2. Hence for the n^{th} term, the expected expression will be $(0.9)^{n-2}(20) + \ldots + (0.9)^0(20)$.

(ii)	As $n \to \infty$, $(0.9)^{n-1} \to 0$ and thus $P \to 200$. In the long run, the population size approaches 200.	It is important not to provide the conclusion directly but instead provide a term-wise trend leading to the final conclusion to ensure clarity. Make sure that the conclusion is logical with reference to the context of the question e.g. in this case, the population cannot be a negative value.
(iii)	If the starting population was S instead of 100, the population at the end of Day n would be $(0.9)^{n-1}(S) + (0.9)^{n-2}(20) + (0.9)^{n-3}(20) + \ldots + 20$ $= (0.9)^{n-1}(S) + 20\left(\frac{1-(0.9)^{n-1}}{1-0.9}\right)$ $= (0.9)^{n-1}(S) + 200(1-(0.9)^{n-1})$ $= (0.9)^{n-1}(S) + 200 - 200(0.9)^{n-1}$ $= 200 + (S-200)(0.9)^{n-1}$ Since $(0.9)^{n-1} \to 0$ as $n \to \infty$ for all values of S, the population size would still approach 200.	The 100 in the final expression $200-100(0.9)^{n-1}$ does not represented the initial population. The misconception arises as the value of 100 coincided with the initial population of the question. If we trace the working, we will realise that 100 herein isn't the initial population.

11 A metal ball is released from the surface of the liquid in a tall cylinder. The ball falls vertically through the liquid and the distance, x cm, that the ball has fallen in time t seconds is measured. The speed of the ball at time t seconds is v cms^{-1}. The ball is released in a manner such that $x=0$ and $v=0$ when $t=0$.

(a) The motion of the ball is modelled by the differential equation

$$\frac{d^2x}{dt^2}+\frac{1}{2}\frac{dx}{dt}-10=0.$$

It is given that $v=\frac{dx}{dt}$.

(i) Show that the differential equation can be written as

$$\frac{dv}{dt}=10-\frac{1}{2}v.$$ [1]

(ii) Using the differential equation in (a)(i), find v in terms of t. **Hence** find x in terms of t. [6]

(a)(i)	$\frac{d^2x}{dt^2}+\frac{1}{2}\frac{dx}{dt}-10=0$ --- (1) $v=\frac{dx}{dt}$ Differentiating w.r.t t: $\frac{dv}{dt}=\frac{d^2x}{dt^2}$ Substituting into (1), $\frac{dv}{dt}+\frac{1}{2}v-10=0$ $\frac{dv}{dt}=10-\frac{1}{2}v$ (shown)	Students are reminded to show all their workings clearly for shown question, in particular how $\frac{dv}{dt}$ is obtained.
(ii)	$\frac{dv}{dt}=10-\frac{1}{2}v=\frac{20-v}{2}$ Simplify into a single fraction first. $\int\frac{1}{20-v}\,dv=\int\frac{1}{2}\,dt$ $-\ln\lvert 20-v\rvert=\frac{1}{2}t+c$ $\lvert 20-v\rvert=e^{-\frac{1}{2}t-c}$ $20-v=\pm e^{-c}e^{-\frac{1}{2}t}$ $20-v=Ae^{-\frac{1}{2}t}$, where $A=\pm e^{-c}$ Alternatively, $\frac{dv}{dt}=10-\frac{1}{2}v$ $\int\frac{1}{10-\frac{1}{2}v}\,dv=\int 1\,dt$ $-2\ln\lvert 20-v\rvert=t+c$ $\ln\lvert 20-v\rvert=-\frac{1}{2}t-\frac{1}{2}c$ $\lvert 20-v\rvert=e^{-\frac{1}{2}t-\frac{1}{2}c}$ $20-v=\pm e^{-\frac{1}{2}t-\frac{1}{2}c}=\pm e^{-\frac{1}{2}t}e^{-\frac{1}{2}c}$ $20-v=Ae^{-\frac{1}{2}t}$, $A=\pm e^{-\frac{1}{2}c}$ Given: [redacted] $20=A$ Thus $v=20-20e^{-\frac{1}{2}t}$ Substituting $v=\frac{dx}{dt}$, $\frac{dx}{dt}=20-20e^{-\frac{1}{2}t}$ $x=\int 20-20e^{-\frac{1}{2}t}\,dt$ $x=20t-20\left(\frac{e^{-\frac{1}{2}t}}{-\frac{1}{2}}\right)+d$ $x=20t+40e^{-\frac{1}{2}t}+d$ [redacted] $0=40+d \Rightarrow d=-40$ Thus $x=20t+40e^{-\frac{1}{2}t}-40$	Students are reminded not to memorise solution but to seek an understanding on how each step is obtained. Remove modulus first before finding the value of A. Recall: $\int\frac{1}{ax+b}\,dx=\frac{1}{a}\ln\lvert ax+b\rvert+C$ Need to sub in given conditions to find value of A and express v in terms of t as stated in the question Note that the relation Distance = speed × time is used only when speed is a constant. Here, the speed is not a constant! Sub in given conditions to find d!

(b) The metal ball is now released in another tall cylinder filled with a different liquid. However, for this liquid, the motion of the ball is modelled by the differential equation

$$\frac{dv}{dt} = 10 - k^2v^2, \text{ where } k \text{ is a positive constant.}$$

It is given that $x = 0$ and $v = 0$ when $t = 0$.

(i) Find v in terms of t and k. [5]

(ii) When the ball falls through this liquid, its speed will approach its "terminal speed" which is the speed it will attain after a long time. Find the ball's terminal speed in terms of k. You must show sufficient working to justify your answer. [2]

(b)(i)	$\frac{dv}{dt} = 10 - k^2v^2$ $\int \frac{1}{(\sqrt{10})^2 - (kv)^2}\, dv = \int 1\, dt$ $\frac{1}{k}\int \frac{k}{(\sqrt{10})^2 - (kv)^2}\, dv = \int 1\, dt$ Put in modulus! $\frac{1}{2k\sqrt{10}} \ln\left\|\frac{\sqrt{10}+kv}{\sqrt{10}-kv}\right\| = t + f$ $\ln\left\|\frac{\sqrt{10}+kv}{\sqrt{10}-kv}\right\| = 2k\sqrt{10}t + 2k\sqrt{10}f$ $\frac{\sqrt{10}+kv}{\sqrt{10}-kv} = \pm e^{2k\sqrt{10}f} e^{2k\sqrt{10}t} = Be^{2k\sqrt{10}t}$, $B = \pm e^{2k\sqrt{10}f}$	Recall: $\int \frac{f'(x)}{a^2 - [f(x)]^2}\, dx = \frac{1}{2a}\ln\left\|\frac{a+f(x)}{a-f(x)}\right\| + C$ **Note:** There's no modulus in the formula in MF26 as it is stated that $\|x\| < a$ which makes $\frac{a+x}{a-x} > 0$. f(x): $\frac{1}{a^2-x^2}$; $\int$ f(x) dx: $\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)$ $(\|x\| < a)$ However, in this question, we do not know whether $\frac{\sqrt{10}+kv}{\sqrt{10}-kv}$ is positive or negative. Hence we need to put modulus for the ln function to be defined!
	When $t = 0$, $v = 0$: $\frac{\sqrt{10}}{\sqrt{10}} = 1 = B$ $\sqrt{10} + kv = e^{2k\sqrt{10}t}(\sqrt{10} - kv)$ $kv(1 + e^{2k\sqrt{10}t}) = \sqrt{10}(e^{2k\sqrt{10}t} - 1)$ $v = \frac{\sqrt{10}}{k}\left(\frac{e^{2k\sqrt{10}t} - 1}{e^{2k\sqrt{10}t} + 1}\right)$	Find v in terms of t and k!

(ii)	**Method 1** $v = \frac{\sqrt{10}}{k}\left(\frac{e^{2k\sqrt{10}t} - 1}{e^{2k\sqrt{10}t} + 1}\right)$ As $t \to \infty$, $e^{2k\sqrt{10}t} \to \infty$, $\frac{e^{2k\sqrt{10}t} - 1}{e^{2k\sqrt{10}t} + 1} \to 1$ and thus $v \to \frac{\sqrt{10}}{k}$ Hence the ball's terminal speed is $\frac{\sqrt{10}}{k}$ cms^{-1} **Method 2** $v = \frac{\sqrt{10}}{k}\left(\frac{e^{2k\sqrt{10}t} - 1}{e^{2k\sqrt{10}t} + 1}\right) \times \frac{e^{-2k\sqrt{10}t}}{e^{-2k\sqrt{10}t}}$ $= \frac{\sqrt{10}}{k}\left(\frac{1 - e^{-2k\sqrt{10}t}}{1 + e^{-2k\sqrt{10}t}}\right)$ As $t \to \infty$, $e^{-2k\sqrt{10}t} \to 0$ and thus $v \to \frac{\sqrt{10}}{k}\left(\frac{1-0}{1+0}\right) = \frac{\sqrt{10}}{k}$ Hence the ball's terminal speed is $\frac{\sqrt{10}}{k}$ cms^{-1} **Method 3** $v = \frac{\sqrt{10}}{k}\left(\frac{e^{2\sqrt{10}t} - 1}{e^{2\sqrt{10}t} + 1}\right)$ $= \frac{\sqrt{10}}{k}\left(1 - \frac{2}{e^{2\sqrt{10}t} + 1}\right)$ (by long division) As $t \to \infty$, $e^{2\sqrt{10}t} \to \infty$, $\frac{2}{e^{2\sqrt{10}t} + 1} \to 0$ and thus $v \to \frac{\sqrt{10}}{k}$ **Method 4** **At terminal speed,** $\frac{dv}{dt} = 10 - k^2v^2 = 0$	Do not write $\frac{\infty}{\infty}$! Essential workings Essential workings Essential workings

	$v^2 = \frac{10}{k^2}$ $v = \pm\frac{\sqrt{10}}{k}$ Since speed, $v > 0$, the ball's terminal speed is $\frac{\sqrt{10}}{k}$ cms^{-1}	

BP-785

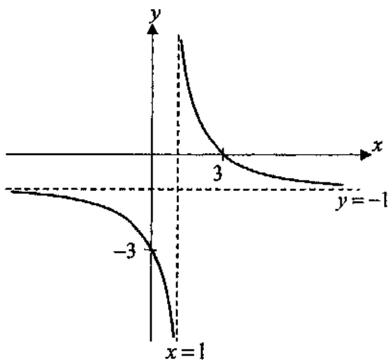
2024 TJC Preliminary Exam H2 Mathematics Paper 2 (Suggested solutions)

Section A: Pure Mathematics [40 marks]

1 A curve C has equation $y=\dfrac{3-x}{x-1}$.

(a) Sketch C, stating the equations of the asymptotes. [2]

(b) Find the exact volume of the solid generated when the region bounded by C, the x-axis and the line $x=9$ is rotated through 2π radians about the x-axis. Leave your answer in the form $\pi(a+b\ln 2)$, where a and b are constants to be determined. [5]

	[Solutions]	Remarks
1(a)		$y=\dfrac{3-x}{x-1}=-1+\dfrac{2}{x-1}$ Use dashed lines for asymptotes. When sketching additional lines to solve other parts of the problem, it is recommended that a side sketch be redrawn to avoid damaging the solution.
1(b)	Required volume $=\pi\int_3^9 \left(\dfrac{3-x}{x-1}\right)^2 \mathrm{d}x$ $=\pi\int_3^9\left(-1+\dfrac{2}{x-1}\right)^2 \mathrm{d}x$ $=\pi\int_3^9 1-\dfrac{4}{x-1}+\dfrac{4}{(x-1)^2}\mathrm{d}x$ $=\pi\left[x-4\ln\lvert x-1\rvert-4(x-1)^{-1}\right]_3^9$ $=\pi\left[\left(9-4\ln\lvert 9-1\rvert-4(9-1)^{-1}\right)-\left(3-4\ln\lvert 2\rvert-4(2)^{-1}\right)\right]$ $=\pi\left(\dfrac{15}{2}-4\ln(2^3)+4\ln 2\right)$ $=\pi\left(\dfrac{15}{2}-8\ln 2\right)$	An additional diagram drawn by the side is useful in helping identify the solid of revolution. It is more efficient to perform a long division first then squaring. Squaring first will result in a more complicated algebraic fraction that would still have to be dealt with by long division (or further splitting), which would increase the complexity of the working.

2 **(a)** Find the series expansion of $\dfrac{(8+x)^{\frac{1}{3}}}{\cos 2x}$ in ascending powers of x up to and including the term in x^2. [5]

(b) Find the range of validity of x for the expansion to be valid. [2]

	[Solutions]	Remarks
(a)	$\dfrac{(8+x)^{\frac{1}{3}}}{\cos 2x}=\dfrac{8^{\frac{1}{3}}\left(1+\frac{x}{8}\right)^{\frac{1}{3}}}{\cos 2x}$ $=\dfrac{2\left(1+\frac{1}{3}\left(\frac{x}{8}\right)+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}\left(\frac{x}{8}\right)^2+\cdots\right)}{1-\frac{(2x)^2}{2}+\cdots}$ $=2\left(1+\dfrac{x}{24}-\dfrac{x^2}{576}+\cdots\right)\left(1-2x^2+\cdots\right)^{-1}$ $=2\left(1+\dfrac{x}{24}-\dfrac{x^2}{576}+\cdots\right)\left(1+2x^2+\cdots\right)$ $=2\left(1+\dfrac{x}{24}+2x^2-\dfrac{x^2}{576}+\cdots\right)$ $=2+\dfrac{1}{12}x+\dfrac{1151}{288}x^2+\cdots$	• Do not differentiate and use Maclaurin's theorem. • Use binomial expansion/standard series in MF26: $(1+x)^n=1+nx+\dfrac{n(n-1)}{2!}x^2+\ldots, \lvert x\rvert<1$ $\cos x=1-\dfrac{1}{2}x^2+\ldots$, all x • Expand until x^2 term. Do not waste time expanding more than what is required. • Rewrite $\dfrac{1}{(1-2x^2)}$ as $(1-2x^2)^{-1}$ so that we can further expand it using binomial expansion again
(b)	For $(1-2x^2)^{-1}$: $\lvert 2x^2\rvert<1$ $\lvert x\rvert<\dfrac{1}{\sqrt{2}}$ $-\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{\sqrt{2}}$ For [redacted] : [redacted] $\Rightarrow -8<x<8$ Taking intersection, the range of validity is $-\dfrac{1}{\sqrt{2}}<x<\dfrac{1}{\sqrt{2}}$	Note that there are two main binomial expansions, $\left(1+\frac{x}{8}\right)^{\frac{1}{3}}$ and $(1-2x^2)^{-1}$. We need to find the validity range for each expansion and then find the range of values which satisfy both validity range by taking intersection.

BP-787

3 A sequence $\{a_n\}$ is defined by $a_0 = 2$ and $a_n = a_{n-1} - \frac{2}{3}\left(\frac{1}{3}\right)^{n-2}$ where $n \geq 1$.

(a) By considering $\sum_{n=1}^{N}(a_n - a_{n-1})$, find an expression for a_N. [4]

(b) Hence explain whether the sequence is convergent. [1]

	[Solutions]	Remarks
3(a)	$a_n - a_{n-1} = -\frac{2}{3}\left(\frac{1}{3}\right)^{n-2}$ where $n \geq 1$ $\sum_{n=1}^{N}(a_n - a_{n-1}) = \left(-\frac{2}{3}\right)\sum_{n=1}^{N}\left(\frac{1}{3}\right)^{n-2}$ LHS $= \sum_{n=1}^{N}(a_n - a_{n-1})$ $= a_1 - a_0$ $+ a_2 - a_1$ $+ a_3 - a_2$ ⋮ $+ a_{N-2} - a_{N-1}$ $+ a_{N-1} - a_{N-2}$ $+ a_N - a_{N-1}$ $= a_N - a_0$ $= a_N - 2$ RHS $= \left(-\frac{2}{3}\right)\sum_{n=1}^{N}\left(\frac{1}{3}\right)^{n-2}$ $\left(= \left(-\frac{2}{3}\right)\left(\frac{1}{3}^{-1} + \frac{1}{3}^{0} + \frac{1}{3}^{1} + \ldots + \frac{1}{3}^{N-2}\right)\right)$ $= \left(-\frac{2}{3}\right)\frac{\left(\frac{1}{3}\right)^{-1}\left[1-\left(\frac{1}{3}\right)^{N}\right]}{1-\frac{1}{3}}$ $= -3\left[1-\left(\frac{1}{3}\right)^{N}\right]$ Thus $a_N - 2 = 3\left(\frac{1}{3}\right)^{N} - 3$ $\Rightarrow \quad a_N = 3\left(\frac{1}{3}\right)^{N} - 1$	- This is an equation. So when we take sum, we take sum **ON BOTH SIDES** of the equation. - For summation, if you cannot observe what sum is that, it is always good to write out few terms to see. - LHS: observe is sum of **DIFFRENCE of 2 SIMILAR TERMS**, so use MOD. - In MOD presentation, it is a **MUST to show the cancellation pattern.** - Pay attention to the **SUBSCRIPT** use. - RHS: Can write out a few terms to see as well to observe is sum of GP to **FINITE term**. - Sum of GP, learn to count the number of terms correctly. For summation **number of terms** can be obtained by **"upper – lower limit + 1"** if you did not remove any of the terms. **Pay attention to the use of capital letter vs small letter again.** - Simplify all answers.

3(b)	As $N \to \infty$, $\left(\frac{1}{3}\right)^{N} \to 0$. Hence $a_N \to -1$, a **finite value**. Thus the sequence is converges to -1.	- Check the concept of converge and watch necessary presentation. - Note the **difference of SEQUENCE converge vs SERIES converge.**

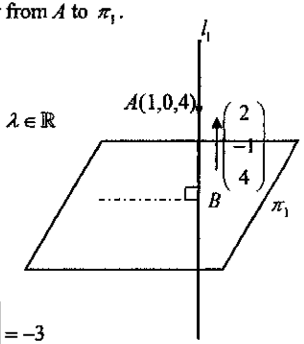
4 The line l_1 passes through the point A with coordinates $(1,0,4)$ and is perpendicular to the plane π_1 with equation $2x-y+4z=-3$.

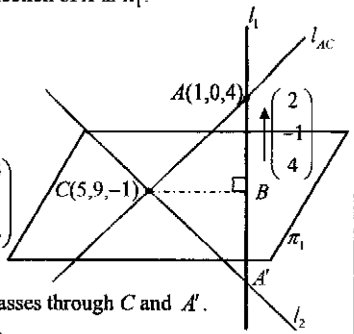
(a) Find the coordinates of the point B where l_1 meets π_1. [4]

(b) Verify that the point C with coordinates $(5,9,-1)$ lies on π_1. [1]

(c) Find a vector equation of the line l_2 which is a reflection of the line AC in π_1. [3]

(d) Find a vector equation, in scalar product form, of the plane π_2 which contains l_1 and l_2. [3]

	[Solutions]	Remarks
4(a)	B is the foot of perpendicular from A to π_1.  B lies on l_1: $\overrightarrow{OB}=\begin{pmatrix}1\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\4\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ B lies on π_1: $\overrightarrow{OB}\cdot\begin{pmatrix}2\\-1\\4\end{pmatrix}=-3$ Thus $\left[\begin{pmatrix}1\\0\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\4\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-1\\4\end{pmatrix}=-3$ $\Rightarrow (2+16)+\lambda(4+1+16)=-3$ $\Rightarrow 21\lambda=-21 \Rightarrow \lambda=-1$ Thus $\overrightarrow{OB}=\begin{pmatrix}1\\0\\4\end{pmatrix}-\begin{pmatrix}2\\-1\\4\end{pmatrix}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$ i.e. **coordinates** of B are $(-1,1,0)$	Take note the the direction of l_1 is indirectly given in the normal to π_1. **Notations**: Remember to mention that $\lambda\in\mathbb{R}$, and write "$\underset{\sim}{r}=$" when writing vector equations. Pay attention to details: **Coordinates** are required.
4(b)	$\overrightarrow{OC}\cdot\begin{pmatrix}2\\-1\\4\end{pmatrix}=\begin{pmatrix}5\\9\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\4\end{pmatrix}=10-9-4=-3$ Thus C lies on π_1. (Verified)	Do not just write the calculation; include some statements.

[redacted]	Let A' be the point of reflection of A in π_1. Using ratio theorem, $\overrightarrow{OB}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$ $\Rightarrow \overrightarrow{OA'}=2\overrightarrow{OB}-\overrightarrow{OA}$ $=2\begin{pmatrix}-1\\1\\0\end{pmatrix}-\begin{pmatrix}1\\0\\4\end{pmatrix}=\begin{pmatrix}-3\\2\\-4\end{pmatrix}$  The line of reflection l_2 passes through C and A'. $\overrightarrow{A'C}=\begin{pmatrix}-3\\2\\-4\end{pmatrix}-\begin{pmatrix}5\\9\\-1\end{pmatrix}=\begin{pmatrix}-8\\-7\\-3\end{pmatrix}$ Thus equation of l_2 is $\underset{\sim}{r}=\begin{pmatrix}5\\9\\-1\end{pmatrix}+\mu\begin{pmatrix}8\\7\\3\end{pmatrix}$, $\mu\in\mathbb{R}$.	**Draw a good diagram.** Think carefully about the information provided; ***B* is the midpoint of *A* and *A'*, *not C*.** Other variations include using $\overrightarrow{CB}=\dfrac{\overrightarrow{CA}+\overrightarrow{CA'}}{2}$ and so on, but these require more calculation.
4(d)	π_2 contains l_1 and l_2 A normal to π_2 is $\begin{pmatrix}8\\7\\3\end{pmatrix}\times\begin{pmatrix}2\\-1\\4\end{pmatrix}=\begin{pmatrix}31\\-26\\-22\end{pmatrix}$ Equation of π_2 is $\underset{\sim}{r}\cdot\begin{pmatrix}31\\-26\\-22\end{pmatrix}=\overrightarrow{OA}\cdot\begin{pmatrix}31\\-26\\-22\end{pmatrix}=\begin{pmatrix}1\\0\\4\end{pmatrix}\cdot\begin{pmatrix}31\\-26\\-22\end{pmatrix}$ i.e., $\underset{\sim}{r}\cdot\begin{pmatrix}31\\-26\\-22\end{pmatrix}=-57$ **<u>Alternative solution</u>** π_2 contains l_1 and $l_2 \Rightarrow \pi_2$ also contains line AC A normal to π_2 is $\overrightarrow{AC}\times\begin{pmatrix}2\\-1\\4\end{pmatrix}=\begin{pmatrix}4\\9\\-5\end{pmatrix}\times\begin{pmatrix}2\\-1\\4\end{pmatrix}=\begin{pmatrix}31\\-26\\-22\end{pmatrix}$ Equation of π_2 is $\underset{\sim}{r}\cdot\begin{pmatrix}31\\-26\\-22\end{pmatrix}=\overrightarrow{OC}\cdot\begin{pmatrix}31\\-26\\-22\end{pmatrix}=\begin{pmatrix}5\\9\\-1\end{pmatrix}\cdot\begin{pmatrix}31\\-26\\-22\end{pmatrix}$ i.e., $\underset{\sim}{r}\cdot\begin{pmatrix}31\\-26\\-22\end{pmatrix}=-57$	**Refer to your diagram.** There are multiple ways to obtain this answer, even without solving part (c), as $\overrightarrow{CA},\overrightarrow{CB}$ and $\overrightarrow{AB}$ are all vectors on the required plane.

BP-789

5 **(a)** The complex number $z = x + \mathrm{i}y$ is represented by the point $P(x, y)$ in an Argand diagram and satisfies the equation

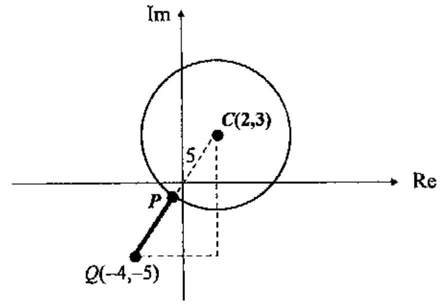
$$zz^* = (2+3\mathrm{i})z^* + (2-3\mathrm{i})z + 12 .$$

(i) Show that P is a point on a circle, and state the centre and radius of the circle. [3]

(ii) The point Q represents the complex number $-4-5\mathrm{i}$. Find the smallest possible length PQ. [2]

(b) **(i)** Show that $\dfrac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta} = \mathrm{i}\cot\dfrac{\theta}{2}$. [3]

(ii) It is given that $z = \mathrm{e}^{\mathrm{i}\theta}$. Find the set of integer values of n such that $\left(\dfrac{1+z}{1-z}\right)^n$ is always real. [2]

	[Solutions]	Remarks
5(a)(i)	Subst $z = x + y\mathrm{i}$ into $zz^* = (2+3\mathrm{i})z^* + (2-3\mathrm{i})z + 12$ $(x+\mathrm{i}y)(x-\mathrm{i}y) = (2+3\mathrm{i})(x-\mathrm{i}y) + (2-3\mathrm{i})(x+\mathrm{i}y) + 12$ $x^2 + y^2 = 2x - 2y\mathrm{i} + 3x\mathrm{i} + 3y + 2x + 2y\mathrm{i} - 3x\mathrm{i} + 3y + 12$ $= 4x + 6y + 12$ $x^2 - 4x + y^2 - 6y = 12$ $(x-2)^2 + (y-3)^2 = 12 + 4 + 9 = 25 = 5^2$ Thus P lies on a circle with centre (2, 3) and radius 5.	This is a **"Show"** qn. Need to give the equation of a circle: $(x-h)^2 + (y-k)^2 = r^2$ Use completing squares
5(a)(ii)	 Let C be the centre of the circle. Use Pythagoras Theorem, $CQ = \sqrt{(2-(-4))^2 + (3-(-5))^2} = \sqrt{6^2+8^2} = 10$ **Shortest length** $PQ = CQ$ **− radius of circle** $= 10 - 5 = 5$	**Sketch a diagram** to help you see the position of P for length PQ to be shortest. P and Q are points! *Poor presentation:* $Q \neq -4-5\mathrm{i}$ ✘ point ≠ complex no. $\lvert Q\rvert \neq \sqrt{4^2+5^2}$ ✘ Should be length OQ

5(b)(i)	**Proof 1** $\dfrac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta} = \dfrac{1+\left(2\cos^2\frac{\theta}{2}-1\right)+\mathrm{i}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{1-\left(1-2\sin^2\frac{\theta}{2}\right)-\mathrm{i}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}$ $= \dfrac{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2}-\mathrm{i}\cos\frac{\theta}{2}\right)}$ $= \cot\dfrac{\theta}{2}\cdot\dfrac{\mathrm{i}\left(\sin\frac{\theta}{2}-\mathrm{i}\cos\frac{\theta}{2}\right)}{\left(\sin\frac{\theta}{2}-\mathrm{i}\cos\frac{\theta}{2}\right)}$ $= \mathrm{i}\cot\dfrac{\theta}{2}$ (shown)	Use double angle formula $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$ $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$ $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ *Factor out* i in the numerator (***see given answer***)
	Proof 2 $\dfrac{1+\cos\theta+\mathrm{i}\sin\theta}{1-\cos\theta-\mathrm{i}\sin\theta} = \dfrac{1+\mathrm{e}^{\mathrm{i}\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}}$ $= \dfrac{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}+\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right)}{\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\left(\mathrm{e}^{-\mathrm{i}\frac{\theta}{2}}-\mathrm{e}^{\mathrm{i}\frac{\theta}{2}}\right)}$ $= \dfrac{\cos\left(-\frac{\theta}{2}\right)+\mathrm{i}\sin\left(-\frac{\theta}{2}\right)+\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)}{\cos\left(-\frac{\theta}{2}\right)+\mathrm{i}\sin\left(-\frac{\theta}{2}\right)-\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)}$ $= \dfrac{2\cos\frac{\theta}{2}}{-2\mathrm{i}\sin\frac{\theta}{2}} \times \dfrac{\mathrm{i}}{\mathrm{i}}$ $= \mathrm{i}\cot\dfrac{\theta}{2}$ (shown)	$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ Note that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} = 2\cos\theta$ $\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = 2\sin\theta\,\mathrm{i}$ All steps should be clearly shown as the final answer is given
	Proof 3 $\dfrac{1+\cos\theta+\mathrm{i}\sin\theta}{(1-\cos\theta)-\mathrm{i}\sin\theta} \times \dfrac{(1-\cos\theta)+\mathrm{i}\sin\theta}{(1-\cos\theta)+\mathrm{i}\sin\theta}$ $= \dfrac{1-\cos^2\theta+(1+\cos\theta)\mathrm{i}\sin\theta+(1-\cos\theta)\mathrm{i}\sin\theta-\sin^2\theta}{(1-\cos\theta)^2-\sin^2\theta}$ $= \dfrac{1-\left(\cos^2\theta+\sin^2\theta\right)+2\mathrm{i}\sin\theta}{1-2\cos\theta+\left(\cos^2\theta+\sin^2\theta\right)}$ $= \dfrac{2\mathrm{i}\sin\theta}{2(1-\cos\theta)}$ $= \dfrac{\mathrm{i}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\left(1-\left(1-2\sin^2\frac{\theta}{2}\right)\right)}$ $= \mathrm{i}\cot\dfrac{\theta}{2}$ (shown)	The <u>conjugate</u> of $(1-\cos\theta) - \mathrm{i}\sin\theta$ is $(1-\cos\theta) + \mathrm{i}\sin\theta$ Use double angle formula

5(b)(ii)	**Method 1 (Preferred)** $z = e^{i\theta} = \cos\theta + i\sin\theta$ Using (i), $\left(\frac{1+z}{1-z}\right)^n = \left(i\cot\frac{\theta}{2}\right)^n = (i)^n \cot^n\frac{\theta}{2}$ See that $\cot^n\frac{\theta}{2}$ is real for all n. To be real, [redacted], [redacted] [redacted] of integer values of n = [redacted]	***Link to earlier result!*** *Distinguish the 2 sets* [redacted] = set of all integers [redacted] = set of all real nos *Give answer in set notation.*
	Method 2 (Not preferred) Using (i), $\left(\frac{1+z}{1-z}\right)^n = \left(i\cot\frac{\theta}{2}\right)^n$ $\arg\left(\frac{1+z}{1-z}\right)^n = n\arg\left(i\cot\frac{\theta}{2}\right)$ $= n\left(\arg i + \arg\left(\cot\frac{\theta}{2}\right)\right)$ $= n\left(\frac{\pi}{2} + m\pi\right),\ m \in \mathbb{Z}$ [redacted] For $n\left(\frac{\pi}{2} + m\pi\right) = k\pi \Rightarrow \frac{n\pi}{2} + mn\pi = k\pi$ [redacted] [redacted] of integer values of n = [redacted]	Note that Since $\cot\frac{\theta}{2}$ is real, $\arg\left(\cot\frac{\theta}{2}\right) = m\pi,\ m \in \mathbb{Z}$ z is real and positive $\Leftrightarrow \arg z = 2k\pi,\ k \in \mathbb{Z}$
	$\arg\left(\frac{1+z}{1-z}\right)^n = n(\arg(1+z) - \arg(1-z))$ Note that [redacted] ✗ Should be $\arg(zw) = \arg z + \arg w$	

Section B: Probability and Statistics [60 marks]

6 (a) The 11 letters of the word REFRESHMENT are arranged in a row.

(i) Find the number of different arrangements that can be made. [2]

(ii) Find the number of different arrangements that can be made such that all the E's are together and all the R's are together but the E's and the R's are not together. [2]

(b) A 4-letter codeword is formed using the letters in the word REFRESHMENT. Find the number of different codewords that can be formed. [4]

	[Solutions]	Remarks
6(a)(i)	Required no. of arrangements = $\frac{11!}{2!3!} = 3\ 326\ 400$	R E F S H M N T R E E
6(a)(ii)	**M1: Slot in method** F S H M N T RR EEE Required no. of arrangements = $6! \times {}^7C_2 \times 2! = 30\ 240$ **M2: complement method** (EEE) (RR) _ _ _ _ _ _ $8! - 7! \times 2! = 30\ 240$ ((EEE) (RR)) _ _ _ _ _ _ 8! : Group Es as one unit, Rs as one unit. Total = 6 others + Es unit + Rs unit = 8 units arrange them. 7!: Group Es as one unit, Rs as one unit. Group the Es unit and Rs unit together. Total = 6 others + (E unit and R unit together)unit = 7 units arrange them 2! : Arrange the Es unit and the Rs unit	For objects not together means to **separate.** **Use slotting in method** Extra careful use of complement. Note the use of complement must have the no restriction with Es and Rs grouped respectively into unit first! It is **not (i) answer.**

BP-791

6(b)	**Case 1: All distinct letters** No. of ways $= {}^{8}C_4 \times 4! = 1680$ **Case 2: 1 pair of identical and 2 other different letters** No. of ways $= {}^{2}C_1 \times {}^{7}C_2 \times \frac{4!}{2!} = 504$ **Case 3: 2 pairs of identical letters (RR & EE)** No. of ways $= \frac{4!}{2!2!} = 6$ **Case 4: 3 E's and 1 other different letter** No. of ways $= {}^{7}C_1 \times \frac{4!}{3!} = 28$ Total no. of arrangements $= 1680 + 504 + 6 + 28$ $= 2218$	- Do not over split your cases. - Choosing number of items from identical objects is always 1 way

7 Two boys, Joseph and Elliot, play a game by each tossing a coin. Joseph tosses a 20-cent coin and Elliot tosses a 50-cent coin. The probability that the 20-cent coin and the 50-cent coin shows a head are $\frac{3}{5}$ and p respectively.

If both coins show heads, Joseph gets to keep Elliot's coin.
If both coins show tails, Joseph gives his coin to Elliot.

If one coin shows a head and the other coin shows a tail, both get to keep their own coins. Let W, in cents, be the amount of money Joseph wins in a game.

(a) Find, in terms of p, the probability distribution of W and E(W). [4]

(b) Find the value of p for the game to be fair. [1]

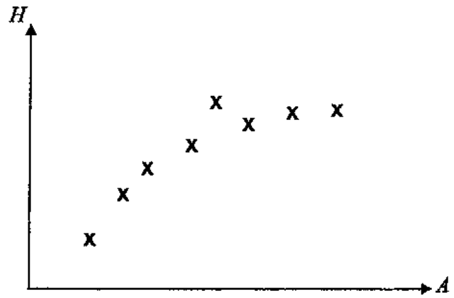
(c) Suppose Elliot's coin is fair i.e. $p = \frac{1}{2}$ and the boys played 40 games. Find the probability that Joseph wins an average of more than 15 cents per game. [3]

<table>
<tr><th></th><th>[Solutions]</th><th>Remarks</th></tr>
<tr><td>7(a)</td><td>
<table>
<tr><td>outcome</td><td>HH</td><td>HT or TH</td><td>TT</td></tr>
<tr><td>w</td><td>50</td><td>0</td><td>−20</td></tr>
<tr><td>P($W=w$)</td><td>$\frac{3}{5}p$</td><td>$\frac{3}{5}(1-p)+\frac{2}{5}p$
$=\frac{3}{5}-\frac{1}{5}p$</td><td>$\frac{2}{5}(1-p)$</td></tr>
</table>
$\mathrm{E}(W) = 50\times\left(\frac{3}{5}p\right) - 20\times\left(\frac{2}{5}(1-p)\right)$
$= 38p - 8$
</td><td>Read the question carefully: W represents Joseph's winnings, so his initial 20-cent coin should not be included in the value of W.</td></tr>
<tr><td>[redacted]</td><td>For the game to be fair, E(W) = 0
$38p - 8 = 0 \Rightarrow p = \frac{4}{19}$</td><td><u>Important Understanding</u>
A "fair game" is one in which the expected winnings is 0.
It is not one where P(J wins) = P(E wins)</td></tr>
<tr><td>[redacted]</td><td>Given $p=\frac{1}{2}$, $\mathrm{E}(W) = 38\left(\frac{1}{2}\right) - 8 = 11$
Using GC, Var(W) = 709
$\overline{W} = \frac{W_1 + W_2 + W_3 + \ldots + W_{40}}{40}$
Since $n = 40$ is large, by Central Limit Theorem,
$\overline{W} \sim \mathrm{N}\left(11, \frac{709}{40}\right)$ approximately
Required probability $= \mathrm{P}\left(\overline{W} > 15\right) = 0.171$ (3 s.f.)</td><td>The question asks about Joseph's <u>average</u> winnings.
This is a big hint for the application of CLT. Remember to justify the use of CLT, and that CLT does not let the original r.v. (i.e. W) become normal. It only applies to $\overline{W}$.</td></tr>
</table>

8 A trainee nurse Angeline is investigating how the head circumferences of young children vary with age. The age, A months, and the head circumference, H cm, of a random sample of 8 young children are given in the table.

A	2	5	7	11	13	16	20	24
H	34.5	38.5	41	43	47	45	46.3	46.5

(a) The value of the product moment correlation coefficient between H and A is 0.880, correct to 3 decimal places, and a scatter diagram for the data is shown below.



(i) Explain whether a linear model is a good model for the relationship between H and A. [1]

(ii) Identify one of the data that Angeline may have recorded wrongly and justify your answer. [1]

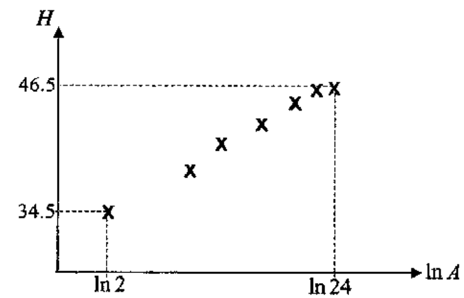
(b) **For the rest of this question, you should not include the wrongly recorded data.**

Based on the scatter diagram in **(a)**, Angeline thinks that a model with equation $H = a + b\ln A$ is an appropriate model.

(i) Sketch a scatter diagram for H against $\ln A$. [1]

(ii) Use your calculator to find the equation of the least square regression line of H on $\ln A$ and the value of the corresponding product moment correlation coefficient. [2]

(iii) Use your equation to estimate the head circumference of a 13-month-old child. Give two reasons why you would expect this estimate to be reliable. [3]

	[Solutions]
8(a) (i)	Although $r_{A,H} = 0.880$ is relatively close to 1, the scatter [redacted] the general trend that H increases at a decreasing rate **rather than a constant rate**. Hence a linear model is **not** appropriate. **Comments:** - r-value of 0.88 is not low and is not indicating weak linear. - [redacted] **is always a much accurate way than r-value** to see the suitability of a linear model. - **Question has given all data and is not asking to explain in context. So students need to learn to analyse questions when to use which.** - To explain WHY LINEAR cannot, hence the need to explain what about LINEAR model ie "constant rate" that cause it to be not suitable. **MUST do some comparative.**
8(a) (ii)	(13, 47) as it does not follow the increase at a decreasing rate **trend set by the rest of the data.** Or (13, 47) as it is **FAR away** from the increase at a decreasing rate graph formed **by the rest of the data.** **Comments:** - Important concept (***) **DATA points can be above or below the trend set by the rest of the data.** - Use of keywords "rest of the data", "follow", "far away", "trend/graph" - **CANNOT** use instantaneous point i.e. relative position of points to argue why point is outlier. (see ***) **MUST** argue is about the point **with respect to the trend/ graph.** - Must state the trend as one could pick another point eg A = 2 to say that it does not follow the linear trend set by the rest of the points!
8(b) (i)	 4 things to note when drawing scatter diagram 1) Max and min of respective axis 2) Relative position of points/ Shape form 3) Total number of points 4) Axis

BP-793

8(b) (ii)	From GC, regression line of H on $\ln A$ is $H = 5.0349 \ln A + 30.899$ $H = 5.03 \ln A + 30.9$ (3 s.f.) and $r = 0.997$ (3 s.f.)
8(b) (iii)	When $A = 13$, $H = 5.0349 \ln 13 + 30.899$ $= 43.8$ (3 s.f.) Thus the head circumference of a 13-month-old child is 43.8 cm. It is a reliable estimate as [redacted] is within the data range [2, 24] **and** $r = 0.997$ being **close to 1** indicates a **strong linear** correlation **between ln A and H.** Reliability must remember 2 'R's 1) Range. In particular is the [redacted] and NOT range of estimate obtained. 2) R-value. Pay attention to **all the keywords needed.** -Be clear in your answer, DO NOT USE 'it' as too ambiguous to know what you are referring to. - Final answer is non- exact, please remember to give 3 s.f.

9 In this question, you should state the parameters of any normal distributions you use.

In golf, a player's driving distance refers to the distance a ball travels when it is hit from the tee using a golf club known as a driver.

Records from past competitions show the following statistics.

- The proportion of male players with driving distance greater than 261 metres is equal to the proportion of male players with driving distance less than 170 metres.
- The proportion of male players with driving distance greater than 261 metres is equal to the proportion of female players with driving distance greater than 181 metres.
- The mean driving distance of a female player is 147.5 metres.

It may be assumed that the driving distances of male and female players follow normal distributions. The standard deviations of the driving distances of male and female players are denoted by σ_m and σ_f respectively.

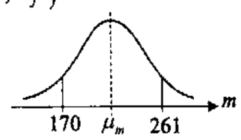
(a) State the mean driving distance of a male player. [1]

(b) Show that $\sigma_m = k\sigma_f$, where k is a constant to be determined. [2]

It is given that $\sigma_m = 24.14$ and $\sigma_f = 17.77$.

(c) Find the probability that the driving distance of a randomly chosen male player is more than 1.5 times the driving distance of a randomly chosen female player. [3]

(d) Find the probability that the difference in driving distances of two randomly chosen female players is less than 15 metres. [3]

	[Solutions]	Remarks
9(a)	Let M and F denote the driving distances (in metres) of **a** male and **a** female player respectively. $M \sim \mathrm{N}(\mu_m, \sigma_m^2)$ and $F \sim \mathrm{N}(147.5, \sigma_f^2)$. Given: $\mathrm{P}(M > 261) = \mathrm{P}(M < 170)$ **Then, by symmetry,** $\mu_m = \frac{170+261}{2} = 215.5$  Thus the mean driving distance of a male player is 215.5 metres.	Note that $\mu_m = \mathrm{E}(M)$ (population mean)
9(b)	Given: $\mathrm{P}(M > 261) = \mathrm{P}(F > 181)$ $\mathrm{P}\left(Z > \frac{261-215.5}{\sigma_m}\right) = \mathrm{P}\left(Z > \frac{181-147.5}{\sigma_f}\right)$ $\frac{45.5}{\sigma_m} = \frac{33.5}{\sigma_f}$ $\frac{\sigma_m}{\sigma_f} = \frac{45.5}{33.5}$ $\sigma_m = \frac{91}{67}\sigma_f$ or $1.36\sigma_f$ (3 s.f.)	**Need to show standardisation**

9(c)	Given: $M \sim N(215.5, 24.14^2)$ and $F \sim N(147.5, 17.77^2)$ $E\left(M - \frac{3}{2}F\right) = 215.5 - \frac{3}{2}(147.5) = -5.75$ $Var\left(M - \frac{3}{2}F\right) = 24.14^2 + \left(\frac{3}{2}\right)^2 (17.77)^2 = 1293.228625$ (exact) Hence $M - \frac{3}{2}F \sim N(-5.75, 1293.228625)$ Required probability $= P\left(M > \frac{3}{2}F\right) = P\left(M - \frac{3}{2}F > 0\right) = 0.436$ (3 s.f.)	Give the final **values** of the mean and variance
9(d)	$E(F_1 - F_2) = E(F) - E(F) = 0$ $Var(F_1 - F_2) = 2Var(F) = 2(17.77)^2 = 631.5458$ (exact) Hence [redacted] Required probability [redacted] $= 0.449$ (3 s.f.) $P(\lvert X \rvert < a) = P(-a < X < a)$ $P(\lvert X \rvert > a) = 1 - P(-a \le X \le a)$	

10 A ministry spokesman reported that students spend an average of 6.5 hours per week on co-curricular activities (CCA) in school. Mr Gru believes that the average time spend on CCA per week in his school is less than this average. To test his belief, he tasks his student Kevin to take a random sample of 50 students in his school. The times, x hours, spent on CCA per week are summarised below.

$$\sum x = 306.68 \qquad \sum x^2 = 1916.22$$

(a) State what it means for a sample to be random in this context. [1]

(b) Calculate unbiased estimates of the population mean and variance of the times spent on CCA per week. [2]

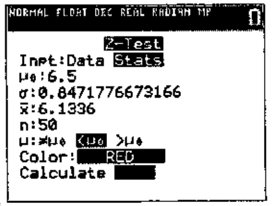
(c) Carry out a test and determine whether the p-value provides strong evidence to support Mr Gru's belief. [4]

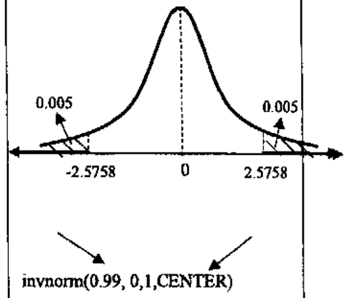
(d) Kevin suggests to Mr Gru that it is necessary to assume that the times spent on CCA per week is normally distributed in order to carry out the test. Explain whether this assumption is necessary. [1]

(e) Student Bob takes another random sample of 50 students and finds that the mean and standard deviation of their times spent on CCA per week are m hours and 0.9 hours respectively. The result of a test at the 1% significance level is that the average time spent on CCA per week by students in his school differs from the average time reported by the ministry spokesman. Find the range of values of m. [5]

	[Solutions]	**Remarks**
10(a)	A random sample is obtained when the students **are selected independently** <u>and</u> ***every*** student in the school has an ***equal probability*** of being ***selected.***	To indicate random sample, need to state 2 factors: 1. equal chance/probability of being chosen 2. **selection** is independent Note that it is the selection that is independent and not the time spend on CCA is independent. Do not write probability of a student chosen is independent of another student chosen.
10(b)	X denotes the time spent (in hours) by a student in Mr Gru's school on CCA per week. Unbiased estimate of population mean, $\bar{x} = \frac{306.68}{50} = 6.1336$ **(exact)** Unbiased estimate of population variance, $s^2 = \frac{1}{49}\left[1916.22 - \frac{306.68^2}{50}\right] \approx 0.71771 = 0.718$ (3 s.f.)	DO NOT ROUND OFF <u>EXACT DECIMALS</u> TO 3 s.f.

BP-795

10(c)	Let μ be the population mean time (in hours) spent on CCA by students in Mr Gru's school. $H_0 : \mu = 6.5$ $H_1 : \mu < 6.5$ Under H_0, since $n = 50$ is large, $\overline{X} \sim N\left(6.5, \frac{0.71771}{50}\right)$ approximately by **Central Limit Theorem**. Test statistic, $Z = \frac{\overline{X} - 6.5}{\sqrt{\frac{0.71771}{50}}} \sim N(0,1)$ approx Using GC, p-value = 0.00111 Since p-value of 0.00111 is very small, this indicates that H_0 will only be rejected if level of significance is at least 0.111% which is very **small**. This means that there is **very strong** evidence to support Mr Gru's belief.	Define μ Under H₀, μ=6.5 $\overline{X} \sim N\left(6.1336, \frac{0.71771}{50}\right)$ ✘ $Z = \frac{6.1336 - 6.5}{\sqrt{\frac{0.71771}{50}}} \sim N(0,1)$ ✘  Answer the question! Decide if p-value obtained provides strong evidence to support H_1. The smaller the p-value, the stronger the evidence!
10(d)	It is not necessary as the sample size of 50 is large and thus by Central Limit Theorem, the **sample mean** of the **times spent on CCA per week**, i.e. $\overline{X}$ would follow a normal distribution approximately.	State clearly and explicitly what exactly is normally distributed under CLT! Common wrong answers: 1. It is not necessary as the sample size of 50 is large and thus by CLT **it** is normally distributed. 2. It is not necessary as the sample size of 50 is large and thus by CLT, **the time** is normally distributed. 3. It is not necessary as the sample size of 50 is large and thus by CLT **the mean time** is normally distributed.

10(e)	Using Bob's sample, $s^2 = \frac{50}{49}(0.9^2)$ $H_0 : \mu = 6.5$ $H_1 : \mu \neq 6.5$ Under H_0, since $n = 50$ is large, by Central Limit Theorem, $\overline{X} \sim N\left(6.5, \frac{0.9^2}{49}\right)$ approximately. Test statistic, $Z = \frac{\overline{X} - 6.5}{\sqrt{\frac{0.9^2}{49}}} \sim N(0,1)$ Level of significance: 1% Critical region: $z \le -2.5758$ or $z \ge 2.5758$ To conclude that the mean time spent on CCA per week by students in Mr Gru's school differs from the average time reported by the ministry spokesman, H_0 is rejected, i.e., $\frac{m - 6.5}{\left(\frac{0.9}{7}\right)} \le -2.5758$ or $\frac{m - 6.5}{\left(\frac{0.9}{7}\right)} \ge 2.5758$ $0 \le m \le 6.17$ or $m \ge 6.83$ (3 s.f.)	m = sample mean 0.9 = sample standard deviation, NOT population s.d Since σ^2 is unknown, we need to estimate it using s^2. Recall: $s^2 = \frac{n}{n-1}$(sample variance) 

11 On average, Alex sleeps less than 6 hours on 75% of nights. The probability that he wakes up late on a school day is 0.625. On days where he wakes up late for school, there is a 96% chance that he has slept less than 6 hours the night before.

(a) Find the probability that he wakes up late when he has slept less than 6 hours the night before. [3]

(b) Determine, with justification, whether the event that he wakes up late for school is independent of the event that he has slept less than 6 hours. [1]

A school week has 5 school days. The number of days he wakes up late for school in a school week is denoted by X.

(c) State, in context, 2 assumptions needed for X to be well-modelled by a binomial distribution. [2]

Assume now that X can be modelled by a binomial distribution.

(d) Find the probability that, in a randomly chosen week, Alex wakes up late for school on at most 3 days. [1]

A school term has 10 weeks.

(e) Find the probability that Alex wakes up late for school on at most 3 days in a week for more than 4 weeks in a randomly chosen school term. State the distribution that you use. [3]

(f) Find the probability that Alex wakes up late for school on 32 days in a randomly chosen school term. State the distribution that you use. [2]

(g) Find the probability that Alex wakes up late for school on at most 3 days in a week for 4 weeks and wakes up late for school on 4 days in a week for the other weeks in a randomly chosen school term. [2]

	[Solutions]	Remarks
11(a)	Let S be the event that Alex sleeps less than 6 hours. Let L be the event that Alex wakes up late for school. Given: $\mathrm{P}(S)=0.75$, $\mathrm{P}(L)=0.625$ and $\mathrm{P}(S\mid L)=0.96 \Rightarrow \dfrac{\mathrm{P}(L\cap S)}{\mathrm{P}(L)}=0.96$ $\Rightarrow \mathrm{P}(L\cap S)=0.96(0.625)=0.6$ Required probability $=\mathrm{P}(L\mid S)=\dfrac{\mathrm{P}(L\cap S)}{\mathrm{P}(S)}=\dfrac{0.6}{0.75}=0.8$	A logical question to ask yourself when trying to determine whether the given scenario is a $P(A\mid B)$ scenario or $P(A\cap B)$ scenario is whether A and B are occurring simultaneously or A is occurring under the constraint that B has already occurred. Need to remember the conditional probability "formula" correctly.

11(b)	Since $\mathrm{P}(S\mid L)=0.96\neq 0.75=\mathrm{P}(S)$, S and L are **not independent** events.	More than often, determination of independence between two events is a mathematical question on whether either of the conditions $P(A\mid B)=P(A)$ or $P(A\cap B)=P(A)\times P(B)$ has been satisfied.
11(c)	The 2 assumptions are (1) Whether Alex wakes up late on **a** school day is **independent** of whether he wakes up late on **other** days. (2) The probability that Alex wakes up late is the **same constant** for **all school days**.	Always answer using the context of the questions. Do not generically specify the terms "trial" or "outcome", without mention of the scenario. Fixed no. of trials and two outcomes only are often already implied in the question. Hence they are usually not assumptions. When phrasing the assumption on the independence of events, it is recommended to have the structure "whether … **instance of event A** happens is independent of the occurrences of **other instances of event A** i.e. repeat the event when comparing. This will avoid misphrasing.
11(d)	Given: X be the number of days Alex wakes up late for school in a school week of 5 days. $X\sim \mathrm{B}(5, 0.625)$ Required probability $=\mathrm{P}(X\le 3)=0.61853$ (5 s.f.) $=0.619$ (3 s.f.)	Always give an intermediate value that is 2 more significant figures than the requirement in case a subsequent part requires the carry-over of the more accurate value.

BP-797

11(e)	Let Y be the number of weeks (in a school term of 10 weeks) where he wakes up late for school on at most 3 days in a week. $Y \sim \mathrm{B}(10, 0.61853)$ Required probability $= \mathrm{P}(Y > 4)$ $= 1 - \mathrm{P}(Y \leq 4) = 0.863$ (3 s.f.)	When defining the distribution, always use the more accurate value (with more significant figures) if a previous value is required.
11(f)	Let W be the number of days (in a school term of $10 \times 5 = 50$ days) where he wakes up late for school. $W \sim \mathrm{B}(50, 0.625)$ Required probability $= \mathrm{P}(W = 32) = 0.114$ (3 s.f.)	
11(g)	$X \sim \mathrm{B}(5, 0.625)$ Required probability $= \frac{10!}{4!\,6!}\left[\mathrm{P}(X \leq 3)\right]^4 \left[\mathrm{P}(X = 4)\right]^6$ $= 0.0169$ (3 s.f.)	Consider the following: You have 10 weeks. 4 of them needs Alex to wake up late for at most 3 times a week. 6 of them requires Alex to wake up 4 times a week. How many combinations of such a scenario are there?