HCI 2024 H2 Mathematics Preliminary Examinations Paper 1

Question	1	2	3	4	5	6	7	8	9	10	11
Marks	4	5	5	7	7	11	9	12	14	14	12

Question 1 [4]

The curve $y = f(x)$ cuts the axes at $(0, \frac{1-p}{p})$ and $(1 - p, 0)$, where p is a constant such that $0 < p < 1$. It is given that f^{-1} exists.

State, if possible to do so, the coordinates of the points where the following curves cut the axes.

(a) $y = f(x) + 1$

(b) $y = f(x - p)$

(c) $y = f(3x - p)$

(d) $y = f^{-1}(x)$

Question 2 [5]

Part (a) [1]

It is given that $f(x)$ and $g(x)$ are non-zero polynomials.

When solving an inequality $\frac{f(x)}{g(x)} \geq 1$, a student writes $f(x) > g(x)$.

Comment on the student's working. **[1]**

Part (b) [4]

Find the exact set of values of x for which $\frac{2x^2-x-9}{x^2-x-6} \geq 1$. **[4]**

Question 3 [5]

The region bounded by the curve with equation $x = \frac{y}{\sqrt{2y-y^2}}$, the lines $y = 1, y = 1.6$ and the y-axis is rotated through 2π radians about the y-axis to form a solid ornament.

Part (a) [4]

Find the exact volume V_1 of the ornament, giving your answer in terms of π.

Part (b) [1]

An ornament designer designs a different ornament by rotating the region bounded by another curve with equation $x = \frac{by}{\sqrt{2y-y^2}}$, where $b > 0$, the lines $y = 1, y = 1.6$ and the y-axis. The region is now rotated through 2π radians about the y-axis. The volume generated is now V_2. State the ratio of V_1 to V_2.

Question 4 [7]

Part (a) [4]

The 11th, 15th and 23rd terms of an arithmetic progression are three distinct consecutive terms of a geometric progression. Find the common ratio of the geometric progression.

Part (b) [3]

The sum, S_n, of the first n terms of a sequence v_1, v_2, v_3 is given by

$$S_n = \frac{3^{n+2}-(-2)^{n+2}-5}{6}.$$

Find an expression for v_n in terms of n, simplifying your answer.

Question 5 [7]

Part (a) [4]

By using the substitution $x = sec\,\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$, show that

$$\int_{\sqrt{2}}^{2} \frac{1}{\sqrt{x^2-1}}\,dx = \int_{\theta_1}^{\theta_2} g(\theta)\,d\theta,$$

Where θ_1 and θ_2 are exact constants to be stated,
and g is a single trigonometric function to be determined.

Part (b) [3]

Hence, find the exact value of $\int_{\sqrt{2}}^{2} \frac{1}{\sqrt{x^2-1}}\,dx$.

Question 6 [11]

The functions f and g are defined by

$$f: x \mapsto \ln\left[(x+4)^2 - 9\right], \qquad \text{for } x \in R,\ x > k,$$

$$g: x \mapsto \frac{3-2x}{1+2x}, \qquad \text{for } x \in R,\ x > \frac{1}{2}.$$

Part (a) [2]

Find the least value of k for which the function f^{-1} exists.

Use the value of k found in part (a) for the rest of this question.

Part (b) [2]

Without finding f^{-1}, find the exact value of α if $g(\frac{3}{2}) = f^{-1}(\alpha)$.

The function h is defined by

$$h: x \mapsto \frac{1}{\sqrt[4]{x(a-x)}}, \text{ for } 0 < x < a, \text{ where } a \text{ is a constant.}$$

Part (c) [3]

Sketch the graph of $y = h(x)$, stating the coordinates of the stationary point and the equations of any asymptotes.

Part (d) [2]

Given that the composite function gh exists, find the range of values of a.

Part (e) [2]

By considering $y = \frac{1}{h(x)}$ and its stationary point, or otherwise, find the value of a for which $[h(x)]^2 = 1$ only has one real root.

Question 7 [9]

It is given that $f(x) = ax^5 + bx^3 + cx$, where a, b, and c are non-zero real constants.

Part (a) [1]

Show that $f(-x) = -f(x)$.

Part (b) [3]

It is given that $f(x) = 0$ has only one real root and one of the non-real roots is $p + qi$, where p and q are non-zero real constants. Find, in terms of p and q, all the other non-real roots of $f(x) = 0$, justifying your answers.

Part (c) [2]

Given that $\int_{0}^{3} f(x)\,dx = -5$, state the values of $\int_{-3}^{3} f(x)\,dx$ and $\int_{-3}^{3} f(|x|)\,dx$.

Let $a = 1$ and $b = 3$.

Part (d) [3]

By considering $f'(x)$, find the range of values of c such that the curve with equation $y = f(x)$ has 2 stationary points, showing your working clearly.

Question 8 [12]

A curve C has parametric equations

$$x = t^2, \qquad y = lnt, \qquad \text{for } t > 0.$$

Part (a) [3]

Find the equation of the tangent to C at the point with parameter t.

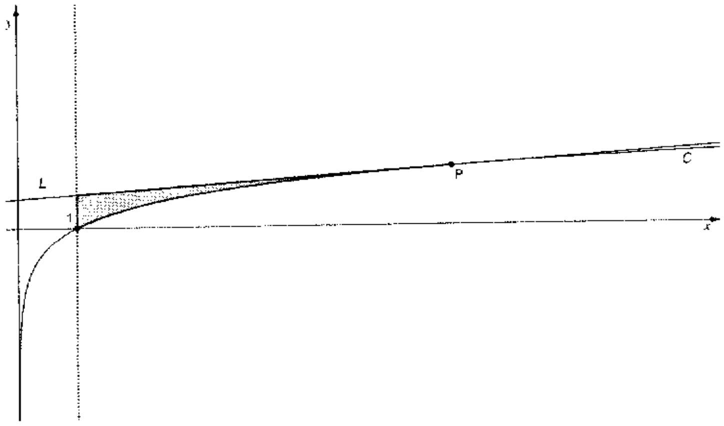
Part (b) [3]

The line L is the tangent to C at the point $P(p^2, ln\ p)$, where p is a positive constant.

Part (c) [2]

Find $\int ln\, x\, dx$.

The diagram below shows the parts of C and L for which $x > 0$.

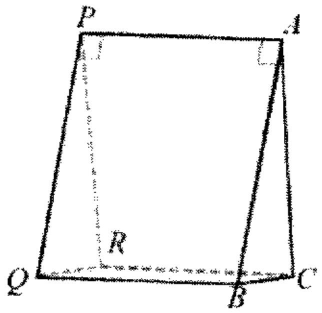


Part (d) [4]

Find the cartesian equation of C in the form of $y = f(x)$. **By using the results in parts (b) and (c)**, find the exact area of the shaded region bounded by C, L and the line $x = 1$.

Question 9 [14]

A right triangular prism has its 2 triangular faces ABC and PQR adjoined by 3 rectangles as shown in the diagram below.



The coordinates of points A, B and C are $(-5,-4,1)$, $(-3,6,2)$ and $(-3,-4,2)$ respectively.

Part (a) [3]

Find the area of triangle ABC.

Part (b) [2]

Find the cartesian equation of the plane which contains A, B and C.

Part (c) [3]

It is given that the plane which contains P, Q and R has equation

$$\underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2$$

Find the volume of the right triangular prism.

Part (d) [3]

Find the coordinates of P.

Part (e) [3]

A circle with centre at the origin O passes through A and another point D with coordinates $(1, 5, -4)$. Find the length of the minor arc AD, **giving your answer correct to 3 decimal places.**

(arc length = $r\theta$ where θ is in radians)

Question 10 [14]

In a particular chemical reaction, every 2 grams of compound Y and every 3 grams of compound Z react to form 1 gram of compound X. Let x, y and z denote the masses (in grams) of compounds X, Y and Z respectively at any time t (in minutes) after the start of the reaction. 24 grams of compound Y and 24 grams of compound Z are used at the start of the reaction, and there is none of compound X present initially.

Part (a) [1]

Express y as $\alpha + \beta x$, where α and β are constants to be determined.

Part (b) [2]

At any time t, the rate of change of x with respect to t is directly proportional to the product of y and z. Show that

$$\frac{dx}{dt} = k(x - 12)(x - 8), \text{ where } k \text{ is a positive constant.}$$

Part (c) [6]

By solving the differential equation in part **(b)**, obtain an expression for x in terms of t and k.

Part (d) [1]

State the theoretical mass of compound X formed in the long run.

Part (e) [2]

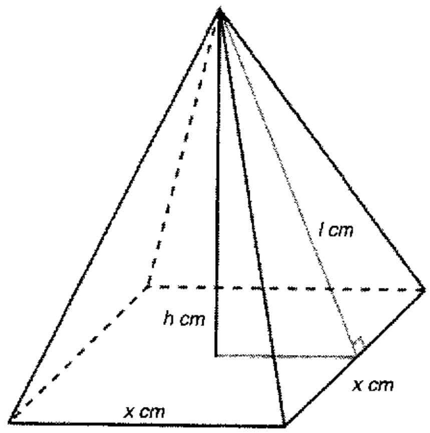
It is observed that there are 4 grams of compound X formed 5 minutes after the start of the reaction. Determine the exact value of k.

Part (f) [2]

Sketch the graph of x against t with the value of k found in part **(e)**.

Question 11 [12]

[The volume of a right square-based pyramid is $\frac{1}{3}$ × base area × height.]



Jane designs a model in the shape of a right square-based pyramid. The square base has sides x cm. Each of the four lateral faces is a triangle with base x cm and perpendicular height l cm. The four lateral faces converge at the top of the pyramid to form an apex directly above the centre of the square base. The vertical height of the pyramid is h cm. The model is assumed to be made of material of negligible thickness.

Part (a) [1]

Form an equation involving x, l and h.

In the design of the model, Jane hopes to fix the total surface area, A cm^2 of the model but maximise the volume, V cm^3 of the model.

Part (b) [1]

Using the result in part (a), show that

$$A = x^2 + 2x\sqrt{h^2 + \frac{x^2}{4}}.$$

Part (c) [2]

Hence show that

$$V^2 = \frac{Ax^2(A-2x^2)}{36}.$$

Part (d) [5]

Use differentiation to show that the maximum V occurs when $x = \frac{\sqrt{A}}{2}$ and find a simplified expression for the maximum V in terms of A.

Part (e) [3]

Given that V is a maximum, find the angle made by a lateral face and the base of the model, **giving your answer to the nearest degree.**

HCI 2024 H2 Mathematics Preliminary Examinations Paper 2 Section A (Pure Maths)

Question	1	2	3	4	5	6
Marks	6	5	7	5	7	10

Question 1 [6]

Part (a) [3]

By considering $tan\ (A - B)$, show that

$$tan^{-1}\left(\frac{1}{x}\right) - tan^{-1}\left(\frac{1}{1+x}\right) = tan^{-1}\left(\frac{1}{x^2+x+1}\right).$$

Part (b) [3]

Hence show that

$$\sum_{r=1}^{1} tan^{-1}\left(\frac{1}{r^2+r+1}\right) = k - f(n),$$

Where $f(n)$ is an inverse trigonometric function and k is an exact constant to be found.

Question 2 [5]

A sequence is defined by the recurrence relation

$$u_{n+1} = \frac{1+u_n}{1-u_n}, \qquad \text{for } n \geq 1.$$

Part (a) [1]

State what happens to the sequence when $u_1 = 0$.

It is now given that $u_1 = 2$.

Part (b) [2]

Find u_2, u_3, u_4, u_5, and u_6.

Part (c) [2]

By observing the pattern in part **(b)**, find $\sum_{r=1}^{4n} u_r$ in terms of n.

Question 3 [7]

Part (a) [5]

A curve C has equation $2y^3 - y^2 = xe^x$.

Find the equations of the tangents which are parallel to the y-axis.

Part (b) [2]

It is given that the tangents found in part **(a)** make an acute angle of $\frac{\pi}{6}$ radians with the line $y = mx + 1$. Find the values of m.

Question 4 [5]

Part (a) [2]

For any non-parallel and non-zero vectors **m** and **n**, **explain clearly** and show that

$$(\mathbf{m}\cdot\mathbf{n})^2 + |\mathbf{m}\times\mathbf{n}|^2 = |\mathbf{m}|^2\,|\mathbf{n}|^2$$

P and Q are two distinct points, where $\vec{OP} = \mathbf{p}$ and $\vec{OQ} = \mathbf{q}$. It is also known that **p** and **q** are non-zero vectors.

Two parallel lines l_1 and l_2 have vector equations

$\mathbf{r} = \mathbf{p} + s\mathbf{u}$ and $\mathbf{r} = \mathbf{q} + t\mathbf{v}$ respectively, where $s, t \in \mathbf{R}$.

Part (b) [3]

If $\mathbf{v} \times (\mathbf{p} - \mathbf{q}) = \mathbf{0}$, what can be said about the relationship between the two lines?
Justify your answer.

Question 5 [7]

Part (a) [4]

Using standard series from the **List of Formulae (MF26),** find the Maclaurin expansion of $\frac{1}{(1+cosx)^2}$ in ascending powers of x up to and including the term in x^4.

Part (b) [3]

Find the set of values of x for which $\frac{1}{(1+cosx)^2}$ is within ± 0.5 of the polynomial found in part **(a)**, where $0 \leq x \leq \pi$.

Question 6 [10]

Part (a) [1]

Given that $u = x + iy$, where x and y are real numbers, show that $|u|^2 = u\,u^*$.

Two complex numbers z and w with non-zero real and imaginary parts satisfy

$$|z + w| = |z - w|, \text{ where } z \neq w.$$

Part (b) [3]

By considering part (a), show that $zw^* + z^*w = 0$.

Part (c) [2]

Hence show that zw^* is purely imaginary.

It is now given that $w = -1 + i\sqrt{3}$, and the argument of z is θ, where $-\pi < \theta \leq \pi$.

Part (d) [4]

Using the result in part (c), find the possible exact values of θ.

HCI 2024 H2 Mathematics Preliminary Examinations Paper 2 Section B (Statistics)

Question	7	8	9	10	11	12
Marks	6	7	9	12	12	14

Question 7 [6]

A factory produces a large number of monitor screens. It is known that, on average, $100p\%$ of the monitor screens are faulty. The number of faulty monitor screens produced each day is independent of that on other days. Each day, the quality control manager will produce a check on n randomly chosen monitor screens produced on that day.

Let M be the number of faulty monitor screens found. You may assume that M can be modelled by a binomial distribution.

Part (a) [2]

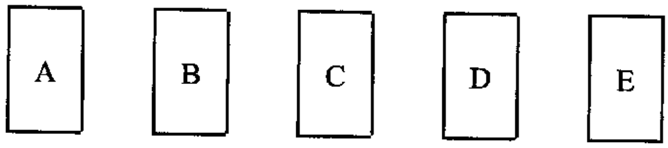
State the probability that on a particular day, there are at least 2 but no more than 3 faulty monitor screens found, giving your answer in terms of n and p.

Part (b) [4]

Each day, the quality control manager will perform a check on 10 randomly chosen monitor screens produced. Find the possible values of p such that there is a 25% chance that on a randomly chosen week with 5 working days, there are exactly 3 days with at least 2 but no more than 3 faulty monitor screens found.

Question 8 [7]

A conference hall has five doors, labelled A, B, C, D and E, which are located side by side as shown below. The doors are to be painted using four distinct colours, and each door will be painted with a single colour.



Part (a) [1]

By considering the number of colours available for each door, find the number of ways to paint the five doors such that there is no restriction to the colour of each door.

Part (b) [3]

Find the number of ways to paint the doors such that there are no consecutive doors which are of the same colour.

Part (c) [3]

Find the number of ways to paint the doors if all four colours are to be used.

Question 9 [9]

In this question, you should state the parameters of any distribution you use.

A ceramic shop sells handmade ceramic cups. The mouths of the cups are assumed to be circular in shape. The diameter of the outer circumferences of the top rim of the cups, S, are assumed to follow a normal distribution with mean μ mm and standard deviation σ mm.

Part (a) [3]

It is given that $P(S < 80.5) = P(S > 84.5)$ and that the probability of the diameter of the outer circumference of the top rim of a randomly chosen cup being more than 85mm is 1.15%. Find the value of μ, and show that $\sigma = 1.10$, when corrected to 3 significant figures.

Part (b) [3]

The shop also makes covers of circular shape that can be fitted over the mouths of the cups. The diameter of any randomly chosen cover, C, in mm, follows a normal distribution with mean 83mm and standard deviation 1.5mm. A cover would be considered to be well-fitted over the mouth of the cup if the diameter of the cover is not larger than that of the outer circumference of the top rim of the cup by not more than 2mm. Find the probability that a randomly chosen cover is well-fitted over the mouth of a randomly-chosen cup.

Part (c) [3]

A cover and a cup are randomly chosen. If the cover is well-fitted over the mouth of the cup, find the probability that the diameter of the cover is larger than that of the cup by more than 1.5mm.

Question 10 [12]

An online website, Star-Salary, which shares information on the salaries for fresh graduates in Singapore, claimed that the mean monthly salary of a fresh graduate with a Bachelor of Science (B.Sc) degree was $3600.

However, another website, First-Pay, stated a higher mean monthly salary for a fresh graduate

Question 10 [12]

An online website, Star-Salary, which shares information on the salaries for fresh graduates in Singapore, claimed that the mean monthly salary of a fresh graduate with a Bachelor of Science (B.Sc) degree was \$3600.

However, another website, First-Pay, stated a higher mean monthly salary for a fresh graduate with the same degree. A random sample of 80 fresh graduates with a B.Sc degree is surveyed and their monthly salaries, $\$x$, are summarised by

$$\Sigma(x - 3600) = 1000, \qquad \Sigma(x - 3600)^2 = 205000.$$

Part (a) [1]

Give a reason why it is challenging to obtain a random sample in this context.

Part (b) [2]

Calculate **exact** unbiased estimates of the population mean and variance for the monthly salaries of fresh graduates with a B.Sc degree.

Part (c) [4]

Test, at the 5% level of significance, whether First-Pay's claim is justified. You should state your hypotheses and define any parameters that you use.

Part (d) [1]

Explain, with justification, whether any assumption about the population is needed for the test in part **(c)** to be valid.

Part (e) [1]

State, in the context of the question, the meaning of "5% level of significance".

A second sample of 60 randomly chosen fresh graduates with B.Sc degree is surveyed and the sample mean and standard deviation of their monthly salaries are found to be $\$\bar{y}$ and \$355 respectively.

Part (f) [3]

Find the largest value of $\bar{y}$ such that this second sample would conclude the test in favour of Star-Salary's claim at 5% level of significance, giving your answer correct to the nearest dollar.

Question 11 [12]

An experiment was carried out to investigate the growth rate of a particular species of plant. The following table gives the height of the plant specimen, h centimetres, at the start of the nth month.

n	1	2	4	6	8	10
h	6.22	9.06	13.62	16.62	18.46	19.72

A possible model for the growth rate is given to be

$$h = \frac{an}{b+n}, \text{ where } a \text{ and } b \text{ are constants.}$$

Part (a) [4]

By writing the above equation in a form that is linear in $\frac{1}{h}$ and $\frac{1}{n}$, calculate the equation of the least squares regression line of $\frac{1}{h}$ on $\frac{1}{n}$. Hence, find estimates for the values of a and b, correct to 3 decimal places.

Part (b) [1]

Sketch a scatter diagram for $\frac{1}{h}$ on $\frac{1}{n}$ and include the least squares regression line found in part (a).

Part (c) [1]

Explain, in the context of the question, the significance of the value of a.

Part (d) [3]

Use the least squares regression line in part **(a)** to find the least integer value of n required for the plant to reach a height of 18 centimetres. Explain whether you would expect this estimate to be reliable.

For a line of best fit $y = f(x)$, the residual for a point (p, q) plotted on the scatter diagram is the vertical distance between $(p, f(p))$ and (p, q).

Part (e) [1]

Mark the residual for each point on the scatter diagram in part **(b)**.

Part (f) [1]

Find the sum of squares of the residuals for the least squares regression line of $\frac{1}{h}$ on $\frac{1}{n}$, giving your answer correct to 5 significant figures.

Question 12 [14]

The probability distribution function of a discrete random variable, X, is given as follows:

$$P(X = x) = \begin{cases} \frac{1}{2}P(X = |x| + 1) & , \text{ if } x = -2, -1 \\ = a & , \text{ if } x = 0, 1 \\ = b & , \text{ if } x = 2, 3 \\ = 0 & , \text{ otherwise} \end{cases}$$

Part (a) [3]

Show that $b = \frac{1-2a}{3}$.

Part (b)(i) [3]

Show that $E(X) = \frac{7}{6} - \frac{4a}{3}$.

Part (b)(ii) [3]

Find $Var\ (X)$ in terms of a.

Part (b)(iii) [2]

Find the range of values of a for which $Var\ (X)$ exists.

Let $a = \frac{7}{20}$.

Part (c) [3]

A random sample of 50 observations of X is taken. Find the probability that the sum of these observations differs from 36 by less than 5.

HCI 2024 Prelims Paper 1 Solutions

Qn	Suggested Solutions
1 [4] (a)	Translation in the positive y direction by 1 unit $y=\mathrm{f}(x)\xrightarrow{\text{replace } y \text{ with } y-1} y=\mathrm{f}(x)+1$ $\left(0,\dfrac{1-p}{p}\right)\xrightarrow{\text{replace } y \text{ with } y-1}\left(0,\dfrac{1}{p}\right)$ **Note**: [only possible to state the y-intercept] $(1-p,0)\xrightarrow{\text{replace } y \text{ with } y-1}(1-p,1)$ doesn't cut the x-axis.
(b)	Translation in the positive x direction by p unit $y=\mathrm{f}(x)\xrightarrow{\text{replace } x \text{ with } x-p} y=\mathrm{f}(x-p)$ $(1-p,0)\xrightarrow{\text{replace } x \text{ with } x-p}(1,0)$ **Note**: [only possible to state the x-intercept] $\left(0,\dfrac{1-p}{p}\right)\xrightarrow{\text{replace } x \text{ with } x-p}\left(p,\dfrac{1-p}{p}\right)$ doesn't cut the y-axis.
(c)	**Step 1**: Translation in the positive x direction by p unit **Method 1**: From Part **(b)** **Step 2**: Scale parallel to the x axis by a factor of $\frac{1}{3}$ $(1-p,0)\xrightarrow{\text{replace } x \text{ with } x-p}(1,0)\xrightarrow{\text{replace } x \text{ with } 3x}\left(\dfrac{1}{3},0\right)$ **Note**: [only possible to state the x-intercept] $\left(0,\dfrac{1-p}{p}\right)\longrightarrow\left(\dfrac{p}{3},\dfrac{1-p}{p}\right)$ doesn't cut the y-axis.
	<u>Method 2:</u> **Step 1**: Scale parallel to the x axis by a factor of $\frac{1}{3}$ **Step 2**: Translation in the positive x direction by $\frac{p}{3}$ unit

$$y=\mathrm{f}(x)\xrightarrow[3x]{\text{replace } x \text{ with}}\mathrm{f}(3x)\xrightarrow[x-\frac{p}{3}]{\text{replace } x \text{ with}}\mathrm{f}(3x-p)$$

$$(1-p,0)\xrightarrow[3x]{\text{replace } x \text{ with}}\left(\frac{1-p}{3},0\right)\xrightarrow[x-\frac{p}{3}]{\text{replace } x \text{ with}}\left(\frac{1}{3},0\right)$$

Note:
[only possible to state the x-intercept]

$\left(0,\frac{1-p}{p}\right)\longrightarrow\left(\frac{p}{3},\frac{1-p}{p}\right)$ doesn't cut the y-axis.

(d)

$$y=\mathrm{f}(x)\xrightarrow{A}y=\mathrm{f}^{-1}(x)$$

A: Reflection about the line $y=x$

$$\left(0,\frac{1-p}{p}\right)\longrightarrow\left(\frac{1-p}{p},0\right)$$

$$(1-p,0)\longrightarrow(0,1-p)$$

2 [5] (a)

If $\mathrm{g}(x)>0$ for all $x\in\mathbb{R}$, then $\mathrm{f}(x)\geq\mathrm{g}(x)$.
Otherwise $\mathrm{f}(x)\geq\mathrm{g}(x)$ may not always be true.

(b)

$$\frac{2x^2-x-9}{x^2-x-6}\geq 1$$

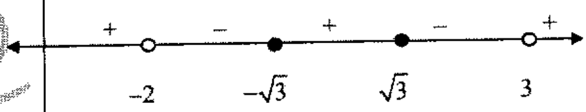
$$\frac{2x^2-x-9}{x^2-x-6}-1\geq 0$$

$$\frac{(2x^2-x-9)-(x^2-x-6)}{x^2-x-6}\geq 0$$

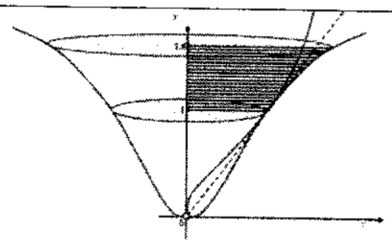
$$\frac{x^2-3}{x^2-x-6}\geq 0$$

$$\frac{(x-\sqrt{3})(x+\sqrt{3})}{(x-3)(x+2)}\geq 0$$

Critical values are: $-2,-\sqrt{3},\ \sqrt{3},3$



$$\{x\in\mathbb{R}:\ x<-2 \ \text{ or } \ -\sqrt{3}\leq x\leq\sqrt{3} \ \text{ or } \ x>3\}$$

3 [5] (a)	 Volume generated $V_1=\pi\int_1^{1.6}x^2\,\mathrm{d}y$ $=\pi\int_1^{1.6}\left[\dfrac{y}{\sqrt{2y-y^2}}\right]^2\mathrm{d}y$ $=\pi\int_1^{1.6}\left(\dfrac{y^2}{2y-y^2}\right)\mathrm{d}y$ $=\pi\int_1^{1.6}\left(\dfrac{y}{2-y}\right)\mathrm{d}y$
	$=\pi\int_1^{1.6}\left(-1+\dfrac{2}{2-y}\right)\mathrm{d}y$ $=\pi\left[-y-2\ln\lvert 2-y\rvert\right]_1^{1.6}$ $=\pi\left[-0.6-2\left(\ln\left\lvert\dfrac{2-1.6}{2-1}\right\rvert\right)\right]$ $=\pi\left[-0.6-2\ln\left(\dfrac{2}{5}\right)\right]$ $=2\pi\left[\ln\left(\dfrac{5}{2}\right)-0.3\right]$ unit^3
(b)	New volume generated $V_2=\pi\int_1^{1.6}x^2\,\mathrm{d}y$
	$=\pi\int_1^{1.6}\left[\dfrac{by}{\sqrt{2y-y^2}}\right]^2\mathrm{d}y$ $=b^2\pi\int_1^{1.6}\left[\dfrac{y}{\sqrt{2y-y^2}}\right]^2\mathrm{d}y$ Required ratio: $1:b^2$

4 [7] (a)	Let a be the first term of AP and d be the common difference. $\frac{a+22d}{a+14d}=\frac{a+14d}{a+10d}$ $(a+22d)(a+10d)=(a+14d)^2$ $a^2+32ad+220d^2=a^2+28ad+196d^2$ $4ad+24d^2=0$ $4d(a+6d)=0$ $d=0$ (reject) or $a=-6d$ Common ratio of GP = $\frac{a+14d}{a+10d}=\frac{-6d+14d}{-6d+10d}=\frac{8d}{4d}=2$
(b)	$v_n=S_n-S_{n-1}$ $=\frac{3^{n+2}-(-2)^{n+2}-5}{6}-\frac{3^{n+1}-(-2)^{n+1}-5}{6}$ $=\frac{1}{6}\left[3^{n+2}-(-2)^{n+2}-5-3^{n+1}+(-2)^{n+1}+5\right]$ $=\frac{1}{6}\left[9(3^n)-3(3^n)-(-2)^2(-2)^n+(-2)(-2)^n\right]$ $=\frac{1}{6}\left[6(3^n)-6(-2)^n\right]$ $=3^n-(-2)^n$

5 [7] (a)

$x = \sec\theta$

$\dfrac{dx}{d\theta} = \sec\theta\tan\theta$

When $x = \sqrt{2}$,

$\dfrac{1}{\cos\theta} = \sqrt{2} \;\Rightarrow\; \dfrac{1}{\cos\theta} = \dfrac{1}{\sqrt{2}} \;\Rightarrow\; \theta = \dfrac{\pi}{4}$

When $x = 2$,

$\dfrac{1}{\cos\theta} = 2 \;\Rightarrow\; \dfrac{1}{\cos\theta} = \dfrac{1}{2} \;\Rightarrow\; \theta = \dfrac{\pi}{3}$

$\int_{\sqrt{2}}^{2} \dfrac{1}{\sqrt{x^2 - 1}}\,dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{1}{\sqrt{\sec^2\theta - 1}}(\sec\theta\tan\theta)\,d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \dfrac{1}{\tan\theta}(\sec\theta\tan\theta)\,d\theta$,

Since $\sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$ where

$0 < \theta < \dfrac{\pi}{2}$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta\,d\theta$

$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec\theta\,d\theta$

$= \left[\ln|\sec\theta + \tan\theta|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$

$= \ln\left[2 + \sqrt{3}\right] - \ln\left[\sqrt{2} + 1\right]$

$= \ln\left[\dfrac{2 + \sqrt{3}}{\sqrt{2} + 1}\right]$

6(a)

$\text{f}: x \mapsto \ln\left[(x+4)^2 - 9\right]$

$(x+4)^2 - 9 > 0$

$(x+4)^2 - 3^2 > 0$

$[x+4-3][x+4+3] > 0$

$(x+1)(x+7) > 0$

$x < -7 \quad \text{or} \quad x > -1$

Minimum $k = -1$

6(b)

$\text{g}\left(\frac{3}{2}\right) = \text{f}^{-1}(\alpha)$

$\text{f}\left[\text{g}\left(\frac{3}{2}\right)\right] = \text{f}\left[\text{f}^{-1}(\alpha)\right]$

$\text{f}\left[\dfrac{3-2\left(\frac{3}{2}\right)}{1+2\left(\frac{3}{2}\right)}\right] = \text{f}\left[\text{f}^{-1}(\alpha)\right]$

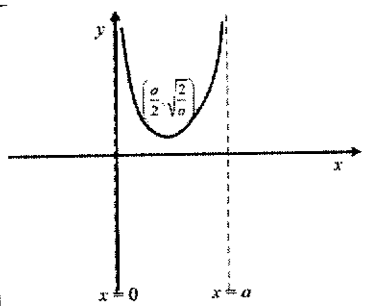
$\text{f}(0) = \text{ff}^{-1}(\alpha)$

$\text{f}(0) = \alpha$

$\alpha = \ln\left[(0+4)^2 - 9\right]$

$\alpha = \ln 7$

(c)



Method 1:

By observation, $x = \dfrac{a}{2}$, $y = \dfrac{1}{\sqrt[4]{\dfrac{a^2}{4}}} = \sqrt{\dfrac{2}{a}}$

Coordinates of stationary (minimum) point is

$\left(\dfrac{a}{2}, \sqrt{\dfrac{2}{a}}\right)$

Equations of 2 vertical asymptotes:

$x = 0, \quad x = a$

Method 2:

$$\frac{d}{dx}\left[x(a-x)\right]^{-\frac{1}{4}}$$

$$=-\frac{1}{4}\left[x(a-x)\right]^{-\frac{5}{4}}[a-2x]$$

$$=-\frac{a-2x}{4\sqrt[4]{\left[x(a-x)\right]^{5}}}$$

For stationary point, $\frac{dy}{dx}=0 \Rightarrow x=\frac{a}{2}$,

$$y=\frac{1}{\sqrt[4]{\frac{a^2}{4}}}=\sqrt{\frac{2}{a}}$$

Coordinates of stationary (minimum) point is

$$\left(\frac{a}{2},\sqrt{\frac{2}{a}}\right)$$

Equations of 2 vertical asymptotes:

$x=0$, $x=a$

(d)

$g: x \mapsto \frac{3-2x}{1+2x}$, for $x \geq \frac{1}{2}$,

$h: x \mapsto \frac{1}{\sqrt[4]{x(a-x)}}$, for $0<x<a$,

$R_h=\left[\sqrt{\frac{2}{a}},\infty\right)$, $D_g=\left[\frac{1}{2},\infty\right)$

For gh to exist, $R_h \subseteq D_g$

$$\sqrt{\frac{2}{a}} \geq \frac{1}{2}$$

$$\frac{2}{a} \geq \frac{1}{4}$$

$$\frac{a}{2} \leq 4$$

$$a \leq 8$$

Since $a>0$, $0<a\leq 8$

(e)

$\left[\mathrm{h}(x)\right]^2 = 1$

$\mathrm{h}(x) = \dfrac{1}{\mathrm{h}(x)}$

Method 1:

Consider minimum point $\left(\dfrac{a}{2}, \sqrt{\dfrac{2}{a}}\right)$ of $y = \mathrm{g}(x)$

intersecting

Maximum of point of $\left(\dfrac{a}{2}, \sqrt{\dfrac{a}{2}}\right)$ of $y = \dfrac{1}{\mathrm{g}(x)}$ at

exactly one point:

$\sqrt{\dfrac{2}{a}} = \sqrt{\dfrac{a}{2}}$

$\dfrac{2}{a} = \dfrac{a}{2}$

$a^2 = 4$

$a = \pm 2$

Since $a > 0$, $a = 2$

Method 2:

$\left[x(a-x)^{-\frac{1}{4}}\right]^2 = 1$

$\left[x(a-x)\right]^{\frac{1}{2}} = 1$

$x(a-x) = 1$

$x^2 - ax + 1 = 0$

For repeated roots,

$a^2 - 4 = 0$

$a = \pm 2$

Since $a > 0$, $a = 2$

7 (9) (a)

$$f(-x) = a(-x)^5 + b(-x)^3 + c(-x)$$
$$= -(ax^5 + bx^3 + cx)$$
$$= -f(x)$$

(b)

$f(x) = ax^5 + bx^3 + cx = 0$

Since all coefficients are real, by Conjugate Root Theorem, if $p+qi$ is a root, then $p-qi$ is also a root.

Also from part **(a)**,

$f(-x) = -f(x)$

We know that f is an odd function and

If $f(x) = 0$, $f(-x) = 0$

and hence $-p-qi$ and $-p+qi$ are also non-real roots.

Since $f(-x) = -f(x)$, So $-p-qi$ and $-p+qi$ are also the roots.

(c)

$$\int_{-3}^{3} f(x)\,dx$$
$$= \int_{-3}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx$$

Since f is an odd function and $\int_{0}^{3} f(x)\,dx = -5$

$$\int_{-3}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx$$
$$= 5 + (-5)$$
$$= 0$$

$$\int_{-3}^{3} f(|x|)\,dx$$
$$= \int_{-3}^{0} f(-x)\,dx + \int_{0}^{3} f(x)\,dx$$
$$= \int_{3}^{0} -f(x)\,dx + \int_{0}^{3} f(x)\,dx$$
$$= \int_{0}^{3} f(x)\,dx + \int_{0}^{3} f(x)\,dx$$
$$= -5 + (-5)$$
$$= -10$$

(d)

$\mathrm{f}(x)=x^5+3x^3+cx$

$\mathrm{f}'(x)=5x^4+9x^2+c$

At stationary points, $5x^4+9x^2+c=0$

$$x^2=\frac{-9\pm\sqrt{9^2-4(5)(c)}}{2(5)}$$

Note : $x^2=\dfrac{-9-\sqrt{9^2-4(5)(c)}}{2(5)}$ (rejected)

If they are 2 stat points, $x^2>0$

$$-9+\sqrt{9^2-4(5)(c)}>0$$

$$\sqrt{9^2-4(5)(c)}>9$$

$$81-20(c)>81$$

$$c<0$$

8[12] (a)

$$\frac{dx}{dt}=2t$$

$$\frac{dy}{dt}=\frac{1}{t}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=\frac{1}{2t^2}$$

At the point with parameter t, Equation of tangent to C at $\left(t^2,\ln t\right)$ is

$$y-\ln t=\frac{1}{2t^2}\left(x-t^2\right)$$

$$y=\frac{1}{2t^2}\left(x-t^2\right)+\ln t$$

$$y=\frac{1}{2t^2}x-\frac{1}{2}+\ln t$$

(b)

Equation of L, the tangent at P:

$$y=\frac{1}{2p^2}x-\frac{1}{2}+\ln p$$

Given that L passes through $\left(1,\dfrac{p^2+1}{2p^2}\right)$,

$$\frac{p^2+1}{2p^2}=\frac{1}{2p^2}(1)-\frac{1}{2}+\ln p$$

$$\ln p=1$$

$$p=\mathrm{e}$$

(c)

$\int \ln x \, dx$

$u = \ln x$, $v' = \int 1 \, dx$

$\dfrac{du}{dx} = \dfrac{1}{x}, v' = x$

$\therefore \int \ln x \, dx$

$= x \ln x - \int x\left(\dfrac{1}{x}\right) dx$

$= x \ln x - \int 1 \, dx$

$= x \ln x - x + C$

(d)

Cartesian equation of curve C_2:

Since $t > 0$,

$t = \sqrt{x}, \quad y = \ln t$

$\Rightarrow \quad y = \ln\left(\sqrt{x}\right)$

$\qquad = \dfrac{1}{2} \ln x$

Area bounded $= \int_1^{e^2} (y_1 - y_2) dx$

$= \int_1^{e^2} \left(\dfrac{1}{2e^2} x + \dfrac{1}{2} - \dfrac{1}{2} \ln x\right) dx$

$= \left[\dfrac{1}{4e^2} x^2 + \dfrac{1}{2} x\right]_1^{e^2} - \dfrac{1}{2} \int_1^{e^2} (\ln x) dx$

$= \left\{\dfrac{1}{4e^2}\left(e^2\right)^2 + \dfrac{1}{2} e^2 - \dfrac{1}{4e^2} - \dfrac{1}{2}\right\} \quad - \dfrac{1}{2}\left[x \ln x - x\right]_1^{e^2}$

$= \left\{\dfrac{3}{4} e^2 - \dfrac{1}{4e^2} - \dfrac{1}{2}\right\} - \dfrac{1}{2}\left\{2e^2 \ln e - 1^2 \ln 1 - e^2 + 1\right\}$

$= \left(\dfrac{3}{4} e^2 - \dfrac{1}{4e^2} - \dfrac{1}{2}\right) - \left(e^2 - \dfrac{1}{2} e^2 + \dfrac{1}{2}\right)$

$= \left(\dfrac{1}{4} e^2 - \dfrac{1}{4e^2} - 1\right)$ unit^2

9 [14] (a)

$$\overrightarrow{OA}=\begin{pmatrix}-5\\-4\\1\end{pmatrix},\ \overrightarrow{OB}=\begin{pmatrix}-3\\6\\2\end{pmatrix}\text{ and }\overrightarrow{OC}=\begin{pmatrix}-3\\-4\\2\end{pmatrix}$$

$$\overrightarrow{AB}=\begin{pmatrix}-3\\6\\2\end{pmatrix}-\begin{pmatrix}-5\\-4\\1\end{pmatrix}=\begin{pmatrix}2\\10\\1\end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix}-3\\-4\\2\end{pmatrix}-\begin{pmatrix}-5\\-4\\1\end{pmatrix}=\begin{pmatrix}2\\0\\1\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}2\\10\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}10\\0\\-20\end{pmatrix}$$

Area of triangle ABC

$$=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}10\\0\\-20\end{pmatrix}\right|$$

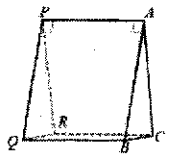
$$=\frac{1}{2}\sqrt{100+0+400}$$

$$=\frac{1}{2}\sqrt{500}\ \text{ or }\ =5\sqrt{5}\ \text{unit}^2$$

(b)

$$\underset{\sim}{n}_{ABC}=k\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$$

Plane ABC is parallel to Plane PQR



$$\underset{\sim}{n}_{ABC}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

Vector equation of π_{ABC} in **scalar product** form:

$$\underset{\sim}{r}\bullet\underset{\sim}{n}_{ABC}=\underset{\sim}{a}\bullet\underset{\sim}{n}_{ABC}$$

$$\underset{\sim}{r}\bullet\begin{pmatrix}1\\0\\-2\end{pmatrix}=\begin{pmatrix}-5\\-4\\1\end{pmatrix}\bullet\begin{pmatrix}1\\0\\-2\end{pmatrix}=-7$$

Vector equation of π_{ABC} in **cartesian** form:

$$x-2z=-7$$

(c) **Method 1**:

Let N be a point that lies on π_{PQR}

$$\pi_{PQR}: \quad \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2$$

By observation, $\overrightarrow{ON} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\overrightarrow{NA} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix}$$

Shortest distance from A to π_{PQR}

= Perpendicular height of the prism is

$$= \frac{\left| \overrightarrow{NA} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right|}{\sqrt{5}}$$

$$= \frac{\left| \begin{pmatrix} -5 \\ -4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \right|}{\sqrt{5}}$$

$$= \sqrt{5}$$

Volume of prism

$= \left(5\sqrt{5}\right)\left(\sqrt{5}\right)$

$= 25$ unit^3

Method 2:

$$\pi_{PQR}: \quad \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2,$$

$$\underset{\sim}{r} \cdot \frac{\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}{\sqrt{5}} = \frac{-2}{\sqrt{5}}$$

Perpendicular height of the prism is

$$\frac{\left|-7-(-2)\right|}{\sqrt{5}} = \sqrt{5} \text{ units}$$

Volume of prism

$= \left(5\sqrt{5}\right)\left(\sqrt{5}\right)$

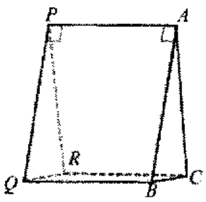
$= 25$ unit^3

(d) **Method 1**: Use intersection of l_{AP} and π_{PQR}

Let P be the foot of the perpendicular.

$$l_{AP}: \quad \underset{\sim}{r} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

$$\overrightarrow{OP} = \begin{pmatrix} -5+\lambda \\ -4 \\ 1-2\lambda \end{pmatrix}, \qquad \text{for some } \lambda \in \mathbb{R}$$



$$\begin{pmatrix} -5+\lambda \\ -4 \\ 1-2\lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = -2$$

$$-5+\lambda-2(1-2\lambda) = -2$$

$$-7+5\lambda = -2$$

$$\lambda = 1$$

$$\overrightarrow{OP} = \begin{pmatrix} -5+1 \\ -4 \\ 1-2 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix}$$

$P(-4,-4,-1)$

(d) **Method 2:** Use $\overrightarrow{AP} // \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Let P be the foot of the perpendicular.

$$l_{AP}: \quad \underset{\sim}{r} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

$$\overrightarrow{OP} = \begin{pmatrix} -5+\lambda \\ -4 \\ 1-2\lambda \end{pmatrix}, \qquad \text{for some } \lambda \in \mathbb{R}$$

$$\overrightarrow{AP} = \begin{pmatrix} -\lambda \\ 0 \\ -2\lambda \end{pmatrix}$$

From part **(c)**, $\left|\overrightarrow{AP}\right| = \sqrt{5}$

$$\left|\begin{pmatrix} -\lambda \\ 0 \\ -2\lambda \end{pmatrix}\right| = \sqrt{5}$$

$$|\lambda| = 1$$

$$\lambda = \pm 1$$

When $\lambda = 1, \overrightarrow{OP} = \begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = -2$

$\therefore P(-4,-4,-1)$ lies on π_{PQR}

When $\lambda = -1, \overrightarrow{OP} = \begin{pmatrix} -6 \\ -4 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0$

$\therefore P(-6,-4,3)$ does not lie on π_{PQR}

$\therefore P(-4,-4,-1)$

(d)

<u>Method 3</u>: Using projection vector, $\overrightarrow{NA}$ projected onto normal vector

Let N be a point that lies on the plane $\underset{\sim}{r}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}=-2$

$$\overrightarrow{ON}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\overrightarrow{NA}=\begin{pmatrix}-5\\-4\\1\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}-5\\-4\\0\end{pmatrix}$$

$$\overrightarrow{PA}=\frac{\begin{pmatrix}-5\\-4\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}}{\sqrt{5}}\frac{\begin{pmatrix}1\\0\\-2\end{pmatrix}}{\sqrt{5}}$$

$$\overrightarrow{PA}=-\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$\overrightarrow{OA}-\overrightarrow{OP}=-\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

$$\overrightarrow{OP}=\begin{pmatrix}-5\\-4\\1\end{pmatrix}-\begin{pmatrix}-1\\0\\2\end{pmatrix}=\begin{pmatrix}-4\\-4\\-1\end{pmatrix}$$

$P(-4,-4,-1)$

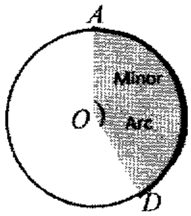
(e)

$\overrightarrow{OA} = \begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix}$, $\overrightarrow{OD} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$

Let the angle between $\overrightarrow{OA}$ and $\overrightarrow{OD} = \theta$

$$\begin{pmatrix} -5 \\ -4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} = \left(\sqrt{42}\right)^2 \cos\theta$$

$$\cos\theta = \frac{-5-20-4}{42} = -\frac{29}{42}$$



Since, $\theta = \cos^{-1}\left(-\frac{29}{42}\right)$

$\theta = 133.6678153°$

(Minor) Arc length

$= r\theta$, where θ is in radians.

$= \sqrt{42}\left[\cos^{-1}\left(-\frac{29}{42}\right)\right]$

$= 15.1192$

$= 15.119$ (3 d.p.)

OR

(minor) Arc length

$= \frac{\theta}{360}[2\pi r]$, where θ is in degrees.

$=15.119$ (3 d.p.)

10 [14] (a)

$y = 24 - 2x$

(b)

$z = 24 - 3x$

$\frac{dx}{dt} \alpha\ (yz)$ or $\frac{dx}{dt} = k_1(yz)$ where k_1 is a positive constant

$$\frac{dx}{dt} = k_1(24-2x)(24-3x)$$
$$= 6k_1(12-x)(8-x)$$
$$= k(x-12)(x-8)$$

$\therefore \quad \frac{dx}{dt} = k(x-12)(x-8)$, where k is a positive constant

(c)	**Method 1**:

$$\frac{dx}{dt} = k(x-12)(x-8)$$

$$\frac{dt}{dx} = \frac{1}{k(x-12)(x-8)}$$

$$t = \frac{1}{k}\int \frac{1}{(x-8)(x-12)}\,dx$$

$$t = \frac{1}{k}\left[\int \frac{1}{4(x-12)}\,dx - \int \frac{1}{4(x-8)}\,dx\right]$$

$$t = \frac{1}{4k}\ln\left|\frac{x-12}{x-8}\right| + C$$

$$4kt - 4C = \ln\left|\frac{x-12}{x-8}\right|$$

$$\frac{x-12}{x-8} = \pm e^{4kt}e^{-4C}$$

$$= Ae^{4kt}$$

where $A = \pm e^{-4C}$ is an arbitrary constant

When $t = 0$, $x = 0$:

$$\frac{0-12}{0-8} = Ae^{4k(0)}$$

$$A = \frac{3}{2}$$

$$\frac{x-12}{x-8} = \frac{3}{2}e^{4kt}$$

$$2x - 24 = (3x-24)e^{4kt}$$

$$x = \left[\frac{24(1-e^{4kt})}{2-3e^{4kt}}\right]$$

(c) **Method 2**:

$$\frac{dx}{dt} = k(x-12)(x-8)$$

$$\frac{1}{(x-12)(x-8)}\frac{dx}{dt} = k$$

$$\int \frac{1}{(x-8)(x-12)}\,dx = kt + C$$

$$\int \frac{1}{(x-10)^2 - 2^2}\,dx = kt + C$$

$$\frac{1}{2(2)}\ln\left|\frac{x-10-2}{x-10+2}\right| = kt + C$$

$$\ln\left|\frac{x-12}{x-8}\right| = 4kt + 4C$$

$$\frac{x-12}{x-8} = \pm e^{4kt}e^{4C}$$

$$= Ae^{4kt}$$

where $A = \pm e^{4C}$ is an arbitrary constant

When $t = 0$, $x = 0$:

$$\frac{0-12}{0-8} = Ae^{4k(0)}$$

$$A = \tfrac{3}{2}$$

$$\frac{x-12}{x-8} = \frac{3}{2}e^{4kt}$$

$$2x - 24 = (3x-24)e^{4kt}$$

$$x = \left[\frac{24(1-e^{4kt})}{2-3e^{4kt}}\right]$$

d

$x \to 8$ as $t \to \infty$

Theoretical Mass = 8g

e

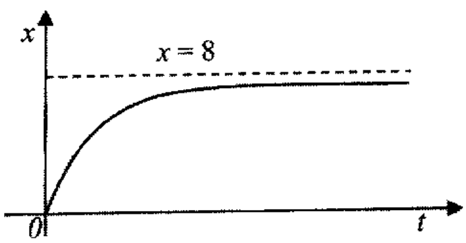
When $t = 5$, $x = 4$:

$$\frac{4-12}{4-8} = \frac{3}{2}e^{4k(5)}$$

$$\frac{-8}{-4} = \frac{3}{2}e^{4k(5)}$$

$$k = \frac{1}{20}\ln\left(\frac{4}{3}\right)$$

f



11 [12] (a)

By Pythagoras Theorem,

$$l^2 = h^2 + \left(\frac{x}{2}\right)^2$$

(b)

$A =$ Area of Square + Area of 4 Triangles

$$A = x^2 + 4\left[\frac{1}{2}(x)\sqrt{h^2 + \frac{x^2}{4}}\right]$$

$$\therefore A = x^2 + 2x\sqrt{h^2 + \frac{x^2}{4}} \text{ (shown)}$$

(c)

From part **(b)**,

$$A - x^2 = 2x\sqrt{h^2 + \frac{x^2}{4}}$$

$$\frac{A - x^2}{2x} = \sqrt{h^2 + \frac{x^2}{4}}$$

$$\left(\frac{A - x^2}{2x}\right)^2 = h^2 + \frac{x^2}{4}$$

$$h^2 = \left(\frac{A - x^2}{2x}\right)^2 - \frac{x^2}{4}$$

Volume of a right pyramid $= \frac{1}{3} \times$ base area $\times$ height

$$V = \frac{1}{3}x^2 h$$

$$V^2 = \frac{1}{9}x^4\left[\left(\frac{A - x^2}{2x}\right)^2 - \frac{x^2}{4}\right]$$

$$= \frac{x^2\left(A - x^2\right)^2 - x^6}{36}$$

$$= \frac{x^2\left(A^2 - 2Ax^2 + x^4\right) - x^6}{36}$$

$$V^2 = \frac{A^2x^2 - 2Ax^4}{36}$$

$$V^2 = \frac{Ax^2\left(A - 2x^2\right)}{36}$$

(d)

$$V^2 = \frac{Ax^2\left(A - 2x^2\right)}{36}$$

Method 1:

$$2V\frac{dV}{dx} = \frac{A\left(2Ax - 8x^3\right)}{36}$$

For stationary values, $\frac{dV}{dx} = 0$

$$2Ax - 8x^3 = 0$$

$$2x\left(A - 4x^2\right) = 0$$

$$x \neq 0 \ , \ x \neq -\sqrt{\frac{A}{4}} \ , \therefore \ x = \frac{1}{2}\sqrt{A}$$

Method 2:

$$V = \frac{\sqrt{A}x\sqrt{\left(A - 2x^2\right)}}{6}$$

$$\frac{dV}{dx} = \frac{1}{6}\sqrt{A}\sqrt{A - 2x^2} + \frac{1}{6}\sqrt{A}x\left[\frac{1}{2}\left(A - 2x^2\right)^{-\frac{1}{2}}(-4x)\right]$$

$$\frac{dV}{dx} = \frac{\sqrt{A}\left(A - 2x^2\right) - 2\sqrt{A}x^2}{6\sqrt{A - 2x^2}}$$

$$= \frac{\sqrt{A}\left(A - 4x^2\right)}{6\sqrt{A - 2x^2}}$$

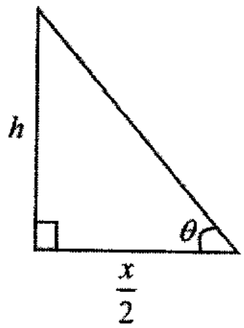
For stationary values, $\frac{dV}{dx} = 0$

$$A - 4x^2 = 0$$

$$x \neq 0 \ , \ x \neq -\sqrt{\frac{A}{4}} \ , \therefore \ x = \frac{1}{2}\sqrt{A}$$

$$\text{Maximum } V^2 = \frac{A^2\left(\frac{A}{4}\right) - 2A\left(\frac{A^2}{16}\right)}{36}$$

$$\text{Maximum } V = \sqrt{\frac{A^3}{288}} = \frac{\sqrt{A^3}}{12\sqrt{2}} = \frac{\sqrt{2}\sqrt{A^3}}{24}$$

(e)	$\frac{1}{3}(x)^2 h = \frac{\sqrt{A^3}}{12\sqrt{2}}$ $\frac{h}{x} = \frac{3\sqrt{A^3}}{12\sqrt{2}} \div \left(\frac{\sqrt{A^3}}{2^3}\right)$ $\frac{h}{x} = \frac{2^3}{4\sqrt{2}}$ $\therefore \frac{h}{x} = \sqrt{2}$ Let the angle the lateral face make with the horizontal be θ.  $\tan\theta = \frac{h}{\frac{x}{2}}$ $\tan\theta = \frac{2h}{x}$ $\tan\theta = 2\sqrt{2}$ $\theta = \tan^{-1}\left(2\sqrt{2}\right)$ $\theta = 70.529°$ $\theta = 71°$ (nearest degree)

HCI 2024 Prelims Paper 2 Solutions

Section A: Pure Mathematics

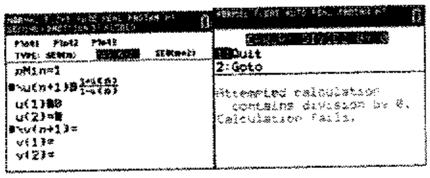
Qn	Suggested Solutions
1 (a)	$\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$ $\therefore\ A-B=\tan^{-1}\left(\dfrac{\tan A-\tan B}{1+\tan A\tan B}\right)$ ...(*) Let $A=\tan^{-1}\left(\dfrac{1}{x}\right)$. $\therefore\ \tan A=\dfrac{1}{x}$ Let $B=\tan^{-1}\left(\dfrac{1}{1+x}\right)$. $\therefore\ \tan B=\dfrac{1}{1+x}$ From (*): $\therefore\ A-B=\tan^{-1}\left(\dfrac{1}{x}\right)-\tan^{-1}\left(\dfrac{1}{1+x}\right)$ $=\tan^{-1}\left(\dfrac{\frac{1}{x}-\frac{1}{1+x}}{1+\frac{1}{x(1+x)}}\right)$ $=\tan^{-1}\left(\dfrac{1}{x(1+x)+1}\right)$ $=\tan^{-1}\left(\dfrac{1}{x^2+x+1}\right)$ (shown)
1(b)	$\sum_{r=1}^{n}\tan^{-1}\left(\dfrac{1}{r^2+r+1}\right)$ $=\sum_{r=1}^{n}\left[\tan^{-1}\left(\dfrac{1}{r}\right)-\tan^{-1}\left(\dfrac{1}{1+r}\right)\right]$ $=\begin{cases}\tan^{-1}(1) & -\tan^{-1}\left(\frac{1}{2}\right)\\ +\tan^{-1}\left(\frac{1}{2}\right) & -\tan^{-1}\left(\frac{1}{3}\right)\\ \vdots & \vdots\\ +\tan^{-1}\left(\frac{1}{n-1}\right) & -\tan^{-1}\left(\frac{1}{n}\right)\\ +\tan^{-1}\left(\frac{1}{n}\right) & -\tan^{-1}\left(\frac{1}{n+1}\right)\end{cases}$ $=\tan^{-1}(1)-\tan^{-1}\left(\dfrac{1}{n+1}\right)$ $=\dfrac{\pi}{4}-\tan^{-1}\left(\dfrac{1}{n+1}\right)$ where $k=\dfrac{\pi}{4}$, $\mathrm{f}(n)=\tan^{-1}\left(\dfrac{1}{n+1}\right)$. (shown)

2 [5] (a)

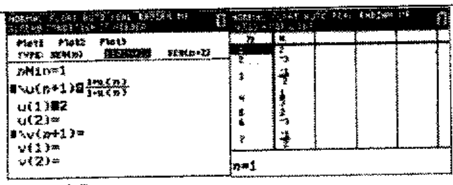
$u_2 = \dfrac{1+u_1}{1-u_1} = \dfrac{1+0}{1-0} = 1$

$u_3 = \dfrac{1+u_2}{1-u_2} = \dfrac{1+1}{1-1}$, which is undefined

- The sequence ends at $u_2 = 1$ as the subsequent terms are undefined. OR
- There are only two terms and terminate(ends) at the 2nd term.



(b)



From GC,

$u_2 = -3$, $u_3 = -\dfrac{1}{2}$, $u_4 = \dfrac{1}{3}$, $u_5 = 2$, $u_6 = -3$

(c)

From observation, the sequence repeats with a period of 4.

$$\sum_{r=1}^{4n} u_r = (u_1 + u_2 + u_3 + u_4) + (u_5 + \dots + u_8) + \dots + (u_{4n-3} + u_{4n})$$

$$= \left[2 - 3 + \left(-\frac{1}{2}\right) + \frac{1}{3}\right] \times \frac{4n}{4}$$

$$= -\frac{7}{6}n$$

3
[7]
(a)

$2y^3 - y^2 = xe^x$

Differentiate implicitly throughout with respect to x:

$6y^2\frac{dy}{dx} - 2y\frac{dy}{dx} = xe^x + e^x$

$\frac{dy}{dx} = \frac{e^x(x+1)}{2y(3y-1)}$

When tangent // to y-axis,

$2y(3y-1) = 0$

$y = 0,\ \frac{1}{3}$

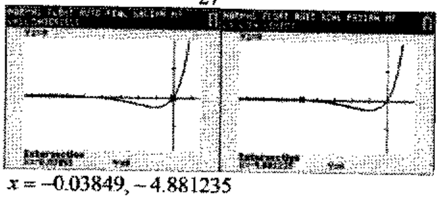
When $y = 0, x = 0$ is equation of the tangent // to y-axis

When $y = \frac{1}{3}$

$2\left(\frac{1}{3}\right)^3 - \left(\frac{1}{3}\right)^2 = xe^x$

$xe^x + \frac{1}{27} = 0$

From GC, $Y_1 = xe^x + \frac{1}{27}$, $Y_2 = 0$



$x = -0.03849, -4.881235$

Note: The more accurate answer is $x = -0.038490398$

Therefore, the equations of 3 tangents that are // to y-axis are:

$x = 0, \quad x = -0.0385\,(3\text{ s.f.}), \quad x = -4.88(3\text{ s.f.})$

(b)

Gradient $= \pm\tan\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \pm\sqrt{3}$ or ± 1.73 (3 s.f.)

Alternative Solution

$\frac{1}{m} = \pm\tan\frac{\pi}{6} \Longrightarrow m = \pm\sqrt{3}$

4
[5]
(a)

Let the acute angle between the vectors $\underset{\sim}{m}$ and $\underset{\sim}{n}$ be θ and the unit vector perpendicular to $\underset{\sim}{m}$ and $\underset{\sim}{n}$ be $\hat{\underset{\sim}{p}}$.

Applying definition of dot (scalar) product,

$\underset{\sim}{m}\bullet\underset{\sim}{n}=|\underset{\sim}{m}||\underset{\sim}{n}|\cos\theta$

$(\underset{\sim}{m}\bullet\underset{\sim}{n})^2=|\underset{\sim}{m}|^2|\underset{\sim}{n}|^2\cos^2\theta$

Applying definition of cross (vector) product,

$\underset{\sim}{m}\times\underset{\sim}{n}=|\underset{\sim}{m}||\underset{\sim}{n}|\sin\theta\,\hat{\underset{\sim}{p}}$

$|\underset{\sim}{m}\times\underset{\sim}{n}|=|\underset{\sim}{m}||\underset{\sim}{n}|\sin\theta\,|\hat{\underset{\sim}{p}}|$

$|\underset{\sim}{m}\times\underset{\sim}{n}|=|\underset{\sim}{m}||\underset{\sim}{n}|\sin\theta$ (since $|\hat{\underset{\sim}{p}}|=1$)

$|\underset{\sim}{m}\times\underset{\sim}{n}|^2=|\underset{\sim}{m}|^2|\underset{\sim}{n}|^2\sin^2\theta$

$\therefore(\underset{\sim}{m}\bullet\underset{\sim}{n})^2+|\underset{\sim}{m}\times\underset{\sim}{n}|^2$

$=|\underset{\sim}{m}|^2|\underset{\sim}{n}|^2\cos^2\theta+|\underset{\sim}{m}|^2|\underset{\sim}{n}|^2\sin^2\theta$

$=|\underset{\sim}{m}|^2|\underset{\sim}{n}|^2(\cos^2\theta+\sin^2\theta)$ since $\sin^2\theta+\cos^2\theta=1$

$=|\underset{\sim}{m}|^2|\underset{\sim}{n}|^2$ (Shown)

(b)

$\underset{\sim}{v}\times(\underset{\sim}{p}-\underset{\sim}{q})=\underset{\sim}{0}$

$\Rightarrow\underset{\sim}{v}=\underset{\sim}{0}$ (rej.) or $\underset{\sim}{p}-\underset{\sim}{q}=\underset{\sim}{0}$ (rej. $\because$ P and Q are distinct points)

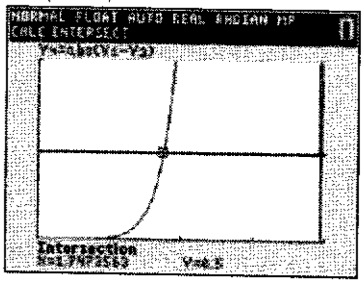
$\therefore\underset{\sim}{v}\,//\,(\underset{\sim}{p}-\underset{\sim}{q})$

$\Rightarrow\underset{\sim}{p}-\underset{\sim}{q}=m\underset{\sim}{v}$

$\Rightarrow\underset{\sim}{p}=\underset{\sim}{q}+m\underset{\sim}{v}$, where $m\in\mathbb{R}\setminus\{0\}$

Since $\underset{\sim}{p}$ satisfies the equation of l_2

Therefore the lines l_1 and l_2 intersect at the point P.

5 **(a)**	$\mathrm{g}(x)$ $=\frac{1}{(1+\cos x)^2}$ $=(1+\cos x)^{-2}$ $\approx\left(1+1-\frac{x^2}{2}+\frac{x^4}{24}\right)^{-2}$, since $\cos x\approx 1-\frac{x^2}{2}+\frac{x^4}{24}$ $=\left(2-\frac{x^2}{2}+\frac{x^4}{24}\right)^{-2}$ $=2^{-2}\left(1-\frac{x^2}{4}+\frac{x^4}{48}\right)^{-2}$ $=\frac{1}{4}\left[1+\frac{(-2)}{1}\left(-\frac{x^2}{4}+\frac{x^4}{48}\right)+\frac{(-2)(-3)}{2!}\left(-\frac{x^2}{4}+\frac{x^4}{48}\right)^2+\ldots\right]$ $=\frac{1}{4}\left[1+\frac{1}{2}x^2-\frac{1}{24}x^4+\frac{3}{16}x^4+\ldots\right]$ $\approx\frac{1}{4}+\frac{1}{8}x^2+\frac{7}{192}x^4$
(b)	$Y_1=\frac{1}{(1+\cos x)^2}$ $Y_2=\frac{1}{4}+\frac{1}{8}x^2+\frac{7}{192}x^4$ $y=\lvert Y_1-Y_2\rvert$  $\lvert Y_1-Y_2\rvert\le 0.5$ From GC, $\{x\in\mathbb{R}:\ \ 0\le x\le 1.75\}$ (3 s.f.)

6 **[10]** **(a)**	$\lvert u \rvert^2$ $= \lvert x + \mathrm{i}y \rvert^2$ $= \left[\sqrt{x^2 + y^2}\right]^2$ $= x^2 + y^2$ $= (x + \mathrm{i}y)(x - \mathrm{i}y)$
(b)	**<u>Method 1</u>**: $\lvert z + w \rvert^2 = \lvert z - w \rvert^2$ $(z + w)(z + w)^* = (z - w)(z - w)^*$ $(z + w)(z^* + w^*) = (z - w)(z^* - w^*)$ $zz^* + zw^* + wz^* + ww^* = zz^* - zw^* - wz^* + ww^*$ $2(zw^* + wz^*) = 0$ $zw^* + wz^* = 0$

(method 2 omitted - avoid)

(c)	**<u>Method 1</u>**: $zw^* + wz^* = 0$ $zw^* + (w^*z)^* = 0$ $(w^*z) + (w^*z)^* = 0$ $2\,\mathrm{Re}(w^*z) = 0$ w^*z is purely imaginary.
	<u>Method 2</u>: zw^* $= (x + \mathrm{i}y)(a - \mathrm{i}b)$ $= (ax + by) + \mathrm{i}(ay - bx)$ From part (b), Since $ax + yb = 0$ zw^* $= 0 + \mathrm{i}(ay - bx)$ $\therefore \mathrm{Re}(zw^*) = 0$
(d)	$w = -1 + \mathrm{i}\sqrt{3} \quad \Rightarrow \arg(w) = \frac{2\pi}{3}$ $\arg(zw^*)$ $= \arg(z) + \arg(w^*)$ $= \arg(z) - \arg(w)$ $= \theta - \frac{2\pi}{3}$ Since zw^* is purely imaginary,
	$\theta - \frac{2\pi}{3} = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$ $\theta = \frac{7\pi}{6} + k\pi$ $\theta = \frac{\pi}{6}$ or $\theta = -\frac{5\pi}{6}$ since $-\pi < \theta \le \pi$.

Section B: Statistics

7 (a)

$M \sim \mathrm{B}(n,p)$

$\mathrm{P}(2 \le M \le 3) = \mathrm{P}(M=2) + \mathrm{P}(M=3)$

$= \binom{n}{2} p^2 (1-p)^{n-2} + \binom{n}{3} p^3 (1-p)^{n-3}$

$= \frac{n(n-1)}{2} p^2 (1-p)^{n-2} + \frac{n(n-1)(n-2)}{6} p^3 (1-p)^{n-3}$

7 (b)

$M \sim \mathrm{B}(10,p)$

$\mathrm{P}(2 \le M \le 3)$

$\mathrm{P}(2 \le M \le 3)$

$= \mathrm{P}(M=2) + \mathrm{P}(M=3)$

$= \binom{10}{2} p^2 (1-p)^8 + \binom{10}{3} p^3 (1-p)^7$

$= 45p^2(1-p)^8 + 120p^3(1-p)^7$

$= k$

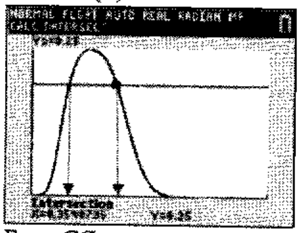
Let X be the number of days in which $2 \le M \le 3$ out of 5 working days

$X \sim \mathrm{B}(5,k)$

$\mathrm{P}(X=3) = 0.25$

$\binom{5}{3} k^3 (1-k)^2 = 0.25$

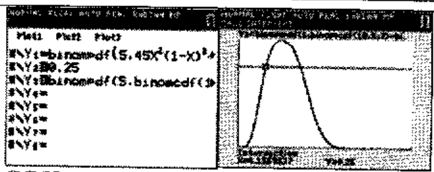
Let $Y_1 = \binom{5}{3} k^3 (1-k)^2$ and $Y_2 = 0.25$



From GC,

$p = 0.1559537$ or 0.3540735

$p = 0.156$ or 0.354 (3 s.f.)



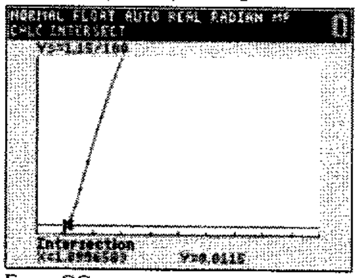
GC Keystrokes:

- $\mathrm{P}(2 \le M \le 3)$
  $= \mathrm{P}(M \le 3) - \mathrm{P}(M \le 1)$
  $= \mathrm{binomcdf}(10, X, 3) - \mathrm{binomcdf}(10, X, 1)$

- $\mathrm{P}(2 \le M \le 3)$
  $= \mathrm{P}(M=2) + \mathrm{P}(M=3)$
  $= \mathrm{binompdf}(10, X, 2) + \mathrm{binompdf}(10, X, 3)$

- $\mathrm{P}(X=3)$
  $= (5, \mathrm{binomcdf}(10, x, 3) - \mathrm{binomcdf}(10, x, 1), 3)$

8(a)	A, B, C, D, E: $^4C_1$ $^4C_1$ $^4C_1$ $^4C_1$ $^4C_1$ ← Each door can be painted in any of the 4 colours No. of ways $=(^4C_1)^5=1024$
8(b)	Consider each door A to E in sequence → A: $^4C_1$, B: $^3C_1$, C: $^3C_1$, D: $^3C_1$, E: $^3C_1$ $^4C_1$ ways to paint A $^3C_1$ ways to paint B different from A $^3C_1$ ways to paint C different from B $^3C_1$ ways to paint D different from C $^3C_1$ ways to paint E different from D No. of ways $=\,^4C_1\times(^3C_1)^4=324$
8(c)	If all 4 colours are used, there must be 2 doors out of 5 painted with the same colour. Method 1: (consider colours to be used) No. of ways $=\,^4C_1\times\frac{5!}{2!}=240$ $^4C_1$ ways to choose a colour to paint 2 doors $\frac{5!}{2!}$ ways to arrange 5 colours in a row, with 1 colour appearing twice Method 2: (consider doors to be painted) No. of ways $=\,^5C_2\times\,^4C_1\times 3!=240$ $^5C_2$ ways to choose 2 doors out of 5 to paint same colour $^4C_1$ ways to choose a colour to paint the 2 chosen doors 3! ways to paint remaining 3 doors with 3 colours

9 (a)	$S \sim \mathrm{N}\left(\mu, \sigma^2\right)$ Since $\mathrm{P}(S < 80.5) = \mathrm{P}(S > 84.5)$, By symmetry, $\mu = \frac{80.5 + 84.5}{2} = 82.5$.
	Method 1: Using GC $S \sim \mathrm{N}\left(82.5, \sigma^2\right)$ Let $Y_1 = \mathrm{P}(S > 85)$ and $Y_2 = 0.0115$  From GC, $\sigma = 1.0996583 = 1.10$ (3 s.f.) (Shown)
	Method 2: Using Standard Normal Distribution $\mathrm{P}(S > 85) = 0.0115$ $\mathrm{P}\left(Z > \frac{85 - \mu}{\sigma}\right) = 0.1155$ $\frac{85 - 82.5}{\sigma} = 2.27343$ $\sigma = 1.099657 = 1.10$ (3 s.f.) (Shown)
9 (b)	$C \sim \mathrm{N}\left(83, 1.5^2\right)$ Let $W = C - S$. $W \sim \mathrm{N}\left(83 - 82.5, 1.1099657^2 + 1.5^2\right)$ i.e. $W \sim \mathrm{N}(0.5, 3.45925)$ or $W \sim \mathrm{N}(0.5, 3.46)$ [if use σ=1.10] $\mathrm{P}(0 < C - S \leq 2)$ $= \mathrm{P}(0 \leq W \leq 2)$ $= 0.39599$ or 0.39595 [if use σ=1.10] $= 0.396$ (3 s.f.)
9 (c)	$\mathrm{P}(C - S > 1.5 \mid 0 < C - S \leq 2)$ $= \frac{\mathrm{P}(1.5 < C - S \leq 2)}{\mathrm{P}(0 < C - S \leq 2)}$ $= \frac{0.0854258}{0.39599}$ or $= \frac{0.0854208}{0.39595}$ [if use σ=1.10] $= 0.215727$ or $= 0.215734$ [if use σ=1.10] $= 0.216$ (3 s.f.)

10 (a)	The population consists of all fresh graduates with a B.Sc degree. While universities may have data on students before graduation, these graduates can work in various industries across Singapore after graduation. To obtain a truly random sample, every graduate must have an equal chance of being selected, and the selection process must be independent. However, several challenges make this difficult: • Not all graduates will be employed immediately after graduation, making it harder to gather salary data and select a truly random sample. • It may be difficult to track where fresh graduates are employed, as their contact details may have changed since leaving university. • Some graduates may be unwilling to respond to the survey, particularly if they are uncomfortable sharing salary information. • Graduates in different job sectors or industries (e.g., private vs. public) may have different levels of transparency regarding salary data. For example, starting salaries in the private sector may be confidential, further complicating data collection.
10 (b)	$\sum(x-3600)=1000$ $\sum x-\sum 3600=1000$ $\sum x=1000+\sum 3600$ $\sum x=1000+80(3600)$ An unbiased estimate of population mean is $=\bar{x}=\dfrac{\sum x}{80}$ $\bar{x}=\dfrac{1000}{80}+3600$ $=3612.5$ $=\dfrac{7225}{2}$ An unbiased estimate of population variance is $=s^2$ $s^2=\dfrac{1}{79}\left[205000-\dfrac{1000^2}{80}\right]$ $=\dfrac{192500}{79}$ $=2436\dfrac{56}{79}$ $=2436.708861$

10 (c)

Let μ and σ^2 be the **population mean and variance** of starting monthly salaries of fresh graduates with B.Sc.

Test $H_0: \mu = 3600$
Against $H_1: \mu > 3600$

Perform a 1-tailed test at 5% level of significance.
Under H_0, since $n = 80$ is large, by **Central Limit Theorem**, $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ approximately

Test Statistic: $Z = \dfrac{\bar{X} - \mu}{\frac{S}{\sqrt{n}}} \sim N(0,1)$ approximately

At 5% level of significance, p-value $= 0.01175874 \approx 0.0118$ (3 s.f)

Since p-value $= 0.0118 < 0.05$, we reject H_0 and conclude that there is **sufficient evidence** at **5% level of significance** that the **population mean monthly salary** is **higher than $3600**. Therefore **First-Pay's claim is justifiable**.

10 (d)

Since the sample size of 80 is large, by **Central Limit Theorem**, **sample mean monthly salary** of fresh graduates with B.Sc, $\bar{X}$ follows a **normal distribution approximately**. Thus, no assumption on the population, X is needed.

10 (e)

"5% level of significance" means that there is a 5% probability that we **wrongly conclude** that **population mean monthly salaries** of fresh graduates with B.Sc. is **higher than** $3600 **when it is in fact** $3600.

10 (f)

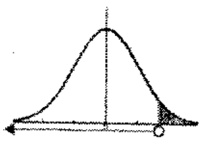
Test $H_0: \mu = 3600$
Against $H_1: \mu > 3600$

Perform a 1-tailed test at 5% level of significance.
$s^2 = \frac{60}{59} \times 355^2 = 128161.0169$

Under H_0, since $n = 60$ is large, by **Central Limit Theorem**, $\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ approximately.

Test Statistic: $Z = \dfrac{\bar{Y} - \mu}{\frac{S}{\sqrt{60}}} \sim N(0,1)$ approximately

At 5% level of significance,
reject H_0 when critical region is $z \geq 1.644853626$
Since H_0 is **not** rejected,

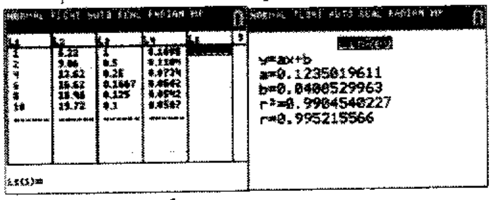


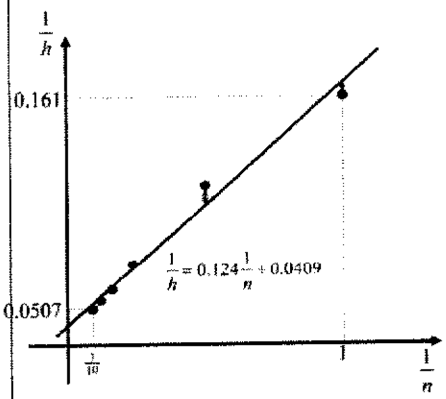
$\dfrac{\bar{y} - 3600}{\sqrt{\frac{128161.0169}{60}}} < 1.644853626$

$\bar{y} < 1.644853626 \times \sqrt{\frac{128161.0169}{60}} + 3600$

$0 < \bar{y} < 3676.020304$

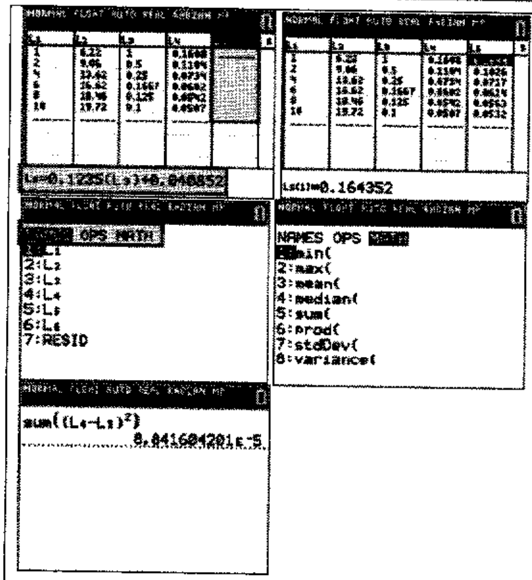
Largest $\bar{y} = 3676$ (nearest dollar)

11 (a)	$h=\dfrac{an}{b+n}$ $\dfrac{1}{h}=\dfrac{b+n}{an}$ $\dfrac{1}{h}=\left(\dfrac{b}{a}\right)\left(\dfrac{1}{n}\right)+\dfrac{1}{a}$ From GC, $L_3=\dfrac{1}{L_1}$, where $L_1=n$, $L_4=\dfrac{1}{L_2}$, where $L_2=h$  $\dfrac{1}{h}=0.1235019611\dfrac{1}{n}+0.0408529963$ $\dfrac{1}{h}=0.124\dfrac{1}{n}+0.0409$ (3 s.f.)
	$\dfrac{b}{a}=0.1235019611$ $\dfrac{1}{a}=0.0408529963 \Rightarrow a=24.47800893$ $b=0.1235019611\times 24.47800893=3.023082082$ $\therefore a=24.478$ and $b=3.023$ (3 d.p.)

11(b)	Red vertical line on the graph represents the residuals 
11 (c)	$h=\dfrac{an}{b+n}$ or $\dfrac{1}{h}=\left(\dfrac{b}{a}\right)\left(\dfrac{1}{n}\right)+\dfrac{1}{a}$ $h=a-\dfrac{ab}{b+n}$ As $n\to\infty$, $h\to a$ or $\dfrac{1}{h}\to\dfrac{1}{a}$ • The **theoretical maximum height** of the plant specimen. • The **maximum height** of the plant specimen in the **long run.**

11 (d)	Since $h \ge 18$, $\frac{1}{h} \le \frac{1}{18}$ $\frac{1}{h} = 0.1235019611\left(\frac{1}{n}\right) + 0.0408529963$ $0.1235019611\left(\frac{1}{n}\right) + 0.0408529963 \le \frac{1}{18}$ $n \ge 8.400031487$ Minimum number of months is 9. The estimate is **reliable** since • $h = 18$ cm is within the data range $[6.22, 19.72]$ and • scatter diagram in part **(b)** shows that there is a strong positive linear relationship between $\frac{1}{h}$ and $\frac{1}{n}$. • $r = 0.995215566 = 0.995$ (3 s.f.) is close to 1 indicating a strong positive linear relationship between $\frac{1}{h}$ and $\frac{1}{n}$.

11(f)	$L_4 = \frac{1}{h}$ From least squares regression line: $L_5 = \frac{1}{h} = 0.1235019611\left(\frac{1}{n}\right) + 0.0408529963$, or $L_5 = 0.12350L_3 + 0.040852$ (5 s.f.) From GC, sum of squares of residual $= \text{Sum } (L_4 - L_5)^2$ $= 8.8416 \times 10^{-5}$ (5 s.f.) $= 0.000088416$ (5 s.f.)



12 (a)

Probability Distribution of X

x	-2	-1	0	1	2	3
$P(X=x)$	$\frac{b}{2}$	$\frac{b}{2}$	a	a	b	b

$$\sum_{\text{all } x} P(X=x)=1$$

$$\frac{b}{2}+\frac{b}{2}+a+a+b+b=1$$

$$2a+3b=1$$

$$b=\frac{1-2a}{3}$$

12 (b) (i)

$$E(X)=\sum_{\text{all } x} xP(X=x)$$

$$E(X)=-\frac{2b}{2}-\frac{b}{2}+0+a+2b+3b$$

$$E(X)=a+\frac{7b}{2}$$

Since $b=\frac{1-2a}{3}$,

$E(X)$

$$=a+\frac{7\left(\frac{1-2a}{3}\right)}{2}$$

$$=\frac{7}{6}-\frac{4a}{3}$$

12 (b) (ii)

$$E(X^2)=\sum_{\text{all } x} x^2P(X=x)$$

$E(X^2)$

$$=\frac{4b}{2}+\frac{b}{2}+0+a+4b+9b$$

$$=a+\frac{31b}{2}$$

Since $b=\frac{1-2a}{3}$,

$E(X^2)$

$$=a+\frac{31\left(\frac{1-2a}{3}\right)}{2}$$

$$=\frac{31}{6}-\frac{28a}{3}$$

$$\text{Var}(X)=E(X^2)-[E(X)]^2$$

$$=\frac{31}{6}-\frac{28a}{3}-\left(\frac{7}{6}-\frac{4a}{3}\right)^2$$

$$=\frac{31}{6}-\frac{28a}{3}-\left(\frac{49}{36}-\frac{56a}{18}+\frac{16a^2}{9}\right)$$

$$=\frac{137}{36}-\frac{56a}{9}-\frac{16a^2}{9}$$

12 (b) (iii)	For X to be defined, $a, b > 0$ as stated in the question. From part **(a)**, $b = \frac{1-2a}{3}$ $\therefore 1-2a > 0$ $a \leq \frac{1}{2}$ When X is defined, $\text{Var}(X)$ is defined when $0 < a < \frac{1}{2}$
12 (c)	For $a = \frac{7}{20}$, $\text{E}(X) = \frac{7}{6} - \frac{4\left(\frac{7}{20}\right)}{3} = \frac{7}{10}$ $\text{Var}(X) = \frac{137}{36} - \frac{56\left(\frac{7}{20}\right)}{9} - \frac{16\left(\frac{7}{20}\right)^2}{9} = \frac{141}{100}$ Since $n = 50$ is large, by **Central Limit Theorem**, Let $T = X_1 + X_2 + ... + X_{50} \sim \text{N}\left(50\left(\frac{7}{10}\right), 50\left(\frac{141}{100}\right)\right)$ **approximately.** $\therefore T \sim \text{N}(35, 70.5)$ **approximately.**
	$\text{P}(\lvert T-36 \rvert < 5)$ $= \text{P}(-5 < T - 36 < 5)$ $= \text{P}(31 < T < 41)$ $= 0.445667$ $= 0.446$ (3 s.f.)