TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ____________________

CIVICS GROUP: ____________________

H2 MATHEMATICS — 9758/02

Paper 2

19 SEPTEMBER 2022

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and Civics Group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

For Examiners' Use	
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Total	

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of 6 printed pages and 0 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 The equation of a curve is given by $y^3 - xy = e^{2x} + 7$.

(i) Find $\frac{dy}{dx}$ in terms of x and y. [2]

(ii) Find the equation of the tangent to the curve at the point where $x = 0$. [2]

2 It is given that $z = -\sqrt{6} - i\sqrt{2}$ and $w = 3\left(\cos\frac{5\pi}{7} - i\sin\frac{5\pi}{7}\right)$. Without the use of a calculator, find the modulus and argument of $\frac{izw^2}{w^*}$ in exact form. [6]

3 A curve C has equation $y = \frac{x}{4+x^2}$.

(i) Sketch the graph of C, stating the equations of any asymptotes and the coordinates of any turning points. [2]

(ii) The region bounded by the curve, the y-axis and the line $y = \frac{1}{4}$ is rotated about the x-axis through 360°. Use the substitution $x = 2\tan\theta$ to find the exact volume of the solid obtained. [5]

4 In the triangle ABC, angle $BAC = \frac{\pi}{4}$ radians and angle $ABC = \left(\frac{\pi}{4} + 2x\right)$ radians. Show that $\frac{AB}{AC} = \frac{\sqrt{2}\cos 2x}{\cos 2x + \sin 2x}$. [3]

Given that x is sufficiently small for which x^3 and higher powers of x can be ignored, find the series expansion of $\frac{AB}{AC}$ in ascending powers of x. [4]

5 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-zero non-parallel vectors. The point C is on AB such that $AC:CB = 3:2$ and the point D is such that A is the mid-point of OD. It is also given that $ODPB$ forms a parallelogram.

(i) By finding $\overrightarrow{OC}$ and $\overrightarrow{OP}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, show that the area of triangle OPC can be written as $k|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be determined. [5]

The lines AB and OP intersect at point E.

(ii) Find the position vector of E in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

(iii) It is given further that angle AOB is acute and $\mathbf{a}$ is a unit vector. Find the range of values of $|\mathbf{b}|$ such that E is the foot of perpendicular from D to the line OP, giving your answers in exact form. [7]

Section B: Probability and Statistics [60 marks]

6 The probability function of a discrete random variable, X, is given as follows:

$$\mathrm{P}(X=x)=\begin{cases} p & \text{if } x=0 \\ \frac{1}{3}\mathrm{P}(X=x-1) & \text{if } x=2,4,6 \\ q & \text{if } x=1,3,5 \end{cases}$$

(i) Given that the expected value of X is 2, find the probability distribution of X. [4]

(ii) Given that X_1 and X_2 are two independent observations of X, find $\mathrm{P}(X_1+X_2=4)$. [2]

7 **(a)** A team of 5 people from a family of 7 adults and 3 children is to be selected for a competition. Find the number of teams that can be selected if

(i) there are no restrictions, [1]

(ii) at most one child and the oldest adult must be in the team. [2]

(b) For events A and B, it is given that $\mathrm{P}(B)=\frac{17}{30}$, $\mathrm{P}(A'\cap B')=\frac{1}{3}$ and $\mathrm{P}(B\mid A)=\frac{4}{5}$.

(i) Find $\mathrm{P}(A\cap B)$. [3]

(ii) Find $P(A\mid B')$. [2]

[Turn Over

8 To improve the prediction accuracy of earthquakes, a group of seismologists gathered the following information about the crack density ρ, measured in millimetres^{-1}, at different distances away from the fault line d, measured in millimetres.

d	1	10	100	200	500	1000
ρ	49	27	16	11	10	9

(i) Sketch a scatter diagram of the data. [1]

(ii) Find the product moment correlation coefficient between

(a) ρ and d,

(b) $\ln\rho$ and $\ln d$. [2]

(iii) Using the answers to parts **(i)** and **(ii)**, explain why the relationship between ρ and d is better modelled by $\ln\rho = A + B\ln d$, as compared to $\rho = C + Dd$, where A, B, C and D are real constants. Hence, find the equation of a suitable regression line for this better model. [3]

(iv) Use the regression line found in part **(iii)** to estimate the distance away from the fault line, to the nearest integer, when the crack density is 8 mm^{-1} and comment on its reliability. [2]

(v) Without further calculations, explain whether the product moment correlation coefficient between $\ln\rho$ and $\ln d$ would be different if d was recorded in metres instead. [1]

9 A fruit seller claims that the apples he sells have a mean weight of 200g. A consumer believes the fruit seller is overstating his claim and decides to do a hypothesis test. He buys a random sample of 30 apples from the fruit seller and measures x, the weight of each apple with the following results:

$$\sum(x-200)=-30 \text{ and } \sum(x-200)^2=1800.$$

(i) Explain why, in this context, the given data is summarised in terms of $(x-200)$ rather than x. [1]

(ii) Find unbiased estimates for the population mean and variance. [2]

(iii) Test, at the 10% significance level, whether the fruit seller has overstated the mean weight of apples that he sells. [4]

The fruit seller wishes to test whether the weight of oranges he sells has a mean weight of 120g and it is given that the weights of oranges sold by the fruit seller are normally distributed with a standard deviation of 8g.

(iv) For a random sample of 30 oranges, find the set of possible values of the sample mean weight to conclude that the mean weight of oranges is not 120g at the 10% level of significance. (Answer obtained by trial and improvement from a calculator will obtain no marks.) [4]

10 A company produces large batches of key chains. It is known that, on average, 3% of the key chains are defective. The key chains are packed in boxes of n.

(i) State, in context, two assumptions needed for the number of defective key chains in a box to be well-modelled by a binomial distribution. [2]

Assume that the assumptions stated above hold.

(ii) The probability that a box contains fewer than 3 defective key chains is less than 0.95. Find the smallest possible integer value of n. [2]

As part of the company's quality control process, the company is considering 2 methods for inspection of a batch of key chains.

Method A

The key chains are packed in boxes of 20. A box is randomly selected. If there are 2 or fewer defective key chains, the batch will be accepted, otherwise the batch will be rejected.

Method B

The key chains are packed in boxes of 10. A box is randomly selected. The batch is accepted if there are no defective key chains and rejected if there are 2 or more defective key chains. Otherwise, randomly select another box for inspection. If there are fewer than 2 defective key chains in the second box, the batch will be accepted.

(iii) For each model, find the probability that the batch will be accepted. [3]

(iv) By calculating the expected number of keychains to be sampled for each method, give a possible reason why the company would choose method B. [2]

The key chains are packed in boxes of 20.

(v) A random sample of 30 boxes is taken. Given that there are 3 boxes with 3 or more defective key chains, find the probability that the third box with 3 or more defective key chains occurs on the 15$^{\text{th}}$ box. [3]

11 **In this question you should state clearly all the distributions that you use, together with the values of the appropriate parameters.**

In a factory, oil is stored in barrels of different sizes. It is given that the volumes of oil in the barrels can be modelled using normal distributions with means and standard deviations as shown in the table.

	Mean volume (in litres)	Standard deviation (in litres)
Light oil	110	2.5
Heavy oil	145	3.5

(i) Find the probability that the volume of light oil in a randomly chosen barrel is between 104 litres and 116 litres. [1]

(ii) Sketch the distribution for the volume of light oil, indicating clearly the probability found in part **(i)**. [2]

(iii) A random sample of seven barrels of heavy oil is chosen. Find the probability that exactly four barrels of heavy oil have volume between 142 and 150 litres each and exactly one barrel of heavy oil has volume more than 150 litres. [3]

(iv) A random sample of n barrels of heavy oil is chosen and it is given that the probability that the mean volume of these n barrels of heavy oil exceeding k litres is at least 0.3. Find an inequality, expressing k in terms of n. [3]

A lorry with a maximum laden mass of 6800 kilograms is used to transport 25 barrels of light oil and 30 barrels of heavy oil. It is given that the densities of light oil and heavy oil are 0.83 kilograms per litre and 0.94 kilograms per litre respectively, and the empty barrels for light oil weigh 5 kilograms each and the empty barrels for heavy oil weigh 8 kilograms each.

(v) Find the probability that the load of the lorry exceeds its maximum laden mass. [4]

[Density is defined as $\frac{\text{Mass}}{\text{Volume}}$.]

(vi) State an assumption needed for your calculations in part **(v)**. [1]

End of Paper

[Turn Over

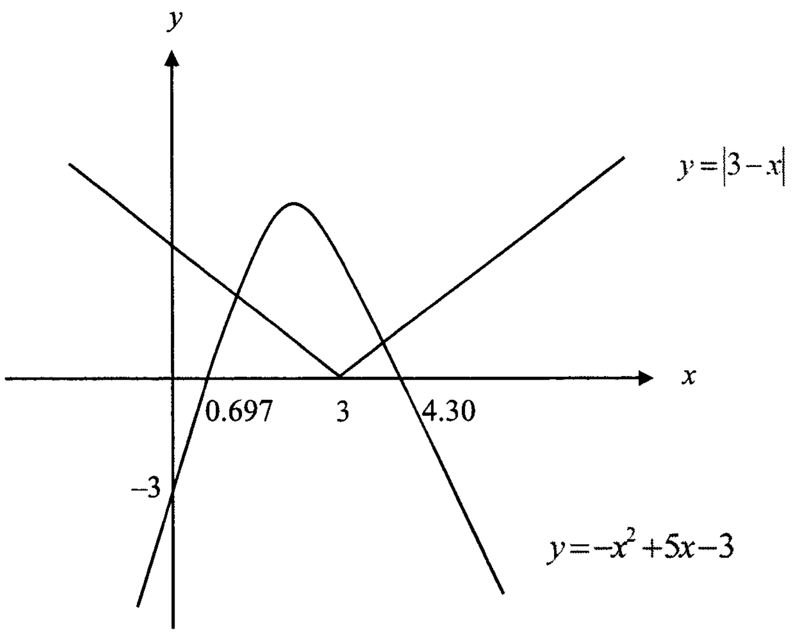
2022 H2 MATH (9758/01) JC 2 PRELIMINARY EXAMINATION SOLUTION

Qn	Solution
1	**Sequences and Series**
(i)	$\frac{3}{r}+\frac{2}{r+1}-\frac{5}{r+2}=\frac{3(r+1)(r+2)+2r(r+2)-5r(r+1)}{r(r+1)(r+2)}$ $=\frac{3r^2+9r+6+2r^2+4r-5r^2-5r}{r(r+1)(r+2)}$ $=\frac{8r+6}{r(r+1)(r+2)}$
(ii)	$\sum_{r=1}^{n}\frac{4r+3}{r(r+1)(r+2)}=\frac{1}{2}\sum_{r=1}^{n}\frac{8r+6}{r(r+1)(r+2)}$ $=\frac{1}{2}\sum_{r=1}^{n}\left(\frac{3}{r}+\frac{2}{r+1}-\frac{5}{r+2}\right)$ $=\frac{1}{2}\Big(\frac{3}{1}+\frac{2}{2}-\frac{5}{3}$ $+\frac{3}{2}+\frac{2}{3}-\frac{5}{4}$ $+\frac{3}{3}+\frac{2}{4}-\frac{5}{5}$ $+\frac{3}{4}+\frac{2}{5}-\frac{5}{6}$ $+$ $\vdots$ $+\frac{3}{n-2}+\frac{2}{n-1}-\frac{5}{n}$ $+\frac{3}{n-1}+\frac{2}{n}-\frac{5}{n+1}$ $+\frac{3}{n}+\frac{2}{n+1}-\frac{5}{n+2}\Big)$ $=\frac{1}{2}\left(\frac{3}{1}+\frac{2}{2}+\frac{3}{2}-\frac{5}{n+1}+\frac{2}{n+1}-\frac{5}{n+2}\right)$ $=\frac{11}{4}-\frac{3}{2(n+1)}-\frac{5}{2(n+2)}$
	$\sum_{r=3}^{n}\frac{4r-5}{r(r-1)(r-2)}=\sum_{r=1}^{n-2}\frac{4r+3}{r(r+1)(r+2)}$ $=\frac{11}{4}-\frac{3}{2(n-1)}-\frac{5}{2n}$

Qn	Solution
2	**Integration**
(a)	$\int \frac{1}{\sqrt{(1-x^2)\sin^{-1}x}}\,dx = \int \frac{1}{\sqrt{(1-x^2)}}(\sin^{-1}x)^{-\frac{1}{2}}\,dx$ $= \frac{(\sin^{-1}x)^{\frac{1}{2}}}{\frac{1}{2}} + C$ $= 2\sqrt{\sin^{-1}x} + C$
(b)	$\int \frac{x-3}{x^2-2x+4}\,dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x+4}\,dx - \int \frac{2}{(x-1)^2+3}\,dx$ $= \frac{1}{2}\ln(x^2-2x+4) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{x-1}{\sqrt{3}} + C$ **Note:** $x^2-2x+4=(x-1)^2+3>0$. Therefore, modulus is not required for ln(.)

Qn	Solution
3	**Chapter 11 Definite Integral**
	when $x=8, (at)^2=8 \Rightarrow t=\frac{2\sqrt{2}}{a}$ when $x=0, (at)^2=0 \Rightarrow t=0$ $x=a^2t^2$ $\frac{dx}{dt}=2a^2t$
	$\int_0^8 y\,dx$ $=\int_0^{\frac{2\sqrt{2}}{a}} e^{at}(2a^2t)\,dt$ $=2a^2\left\{\left[\frac{e^{at}}{a}(t)\right]_0^{\frac{2\sqrt{2}}{a}} - \int_0^{\frac{2\sqrt{2}}{a}} \frac{e^{at}}{a}(1)\,dt\right\}$ $=2a^2\left\{\left[\frac{e^{2\sqrt{2}}}{a}\left(\frac{2\sqrt{2}}{a}\right)-0\right]-\left[\frac{e^{at}}{a^2}\right]_0^{\frac{2\sqrt{2}}{a}}\right\}$ $=2a^2\left[\frac{e^{2\sqrt{2}}2\sqrt{2}}{a^2}-\left(\frac{e^{2\sqrt{2}}}{a^2}-\frac{1}{a^2}\right)\right]$ $=2e^{2\sqrt{2}}(2\sqrt{2}-1)+2$

Qn	Solution
4	**Graphing and Inequalities**



$-x^2+5x-3=|3-x|$

$-x^2+5x-3=3-x$ or $-x^2+5x-3=-(3-x)$

For $x<3$:

$-x^2+5x-3=3-x$

$x^2-6x+6=0$

$x=\dfrac{6\pm\sqrt{6^2-4(1)(6)}}{2}$

$=\dfrac{6\pm\sqrt{12}}{2}$

$=3\pm\sqrt{3}$

$=3-\sqrt{3}$ since $x<3$

For $x>4$:

$-x^2+5x-3=-(3-x)$

$x^2-4x=0$

$x(x-4)=0$

$x=0$ (reject since $x>3$) or $x=4$

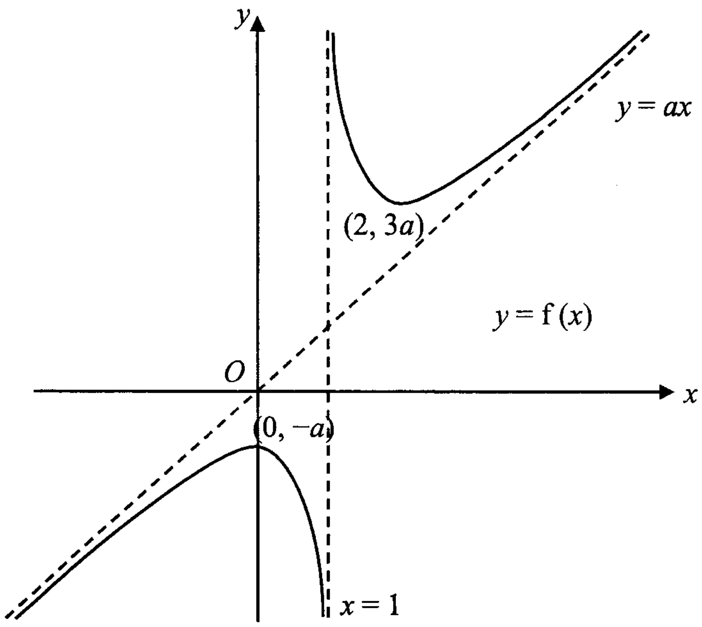
$-x^2+5x-3<|3-x|$

Using graph, $x<3-\sqrt{3}$ or $x>4$

Qn	Solution
5	**Maclaurin Series**
	$\frac{1}{y}\frac{dy}{dx}=k\cos kx$ $\frac{dy}{dx}=ky\cos kx$ $\frac{d^2y}{dx^2}=ky(-k\sin kx)+k\frac{dy}{dx}\cos kx$ $=-k^2y\sin kx+\frac{dy}{dx}(k\cos kx)$ $=-k^2y\ln y+\frac{dy}{dx}\left(\frac{1}{y}\frac{dy}{dx}\right)$ $\therefore \frac{d^2y}{dx^2}+k^2y\ln y-\frac{1}{y}\left(\frac{dy}{dx}\right)^2=0$ When $x=0, y=1$ $\frac{dy}{dx}=k$ $\frac{d^2y}{dx^2}=k^2$ $y=1+kx+\frac{k^2x^2}{2}+...$
	$y=e^{\sin kx}$ $=e^{\left(kx-\frac{(kx)^3}{3!}+...\right)}$ $=1+\left(kx-\frac{k^3x^3}{6}\right)+\frac{\left(kx-\frac{k^3x^3}{6}\right)^2}{2}+\frac{\left(kx-\frac{k^3x^3}{6}\right)^3}{6}+...$ $=1+kx-\frac{k^3x^3}{6}+\frac{1}{2}(k^2x^2)+\frac{1}{6}(k^3x^3)+...$ $=1+kx+\frac{k^2x^2}{2}+...$ (verified) Coefficient of $x^3=0$

Qn	Solution
6	**Applications of Differentiation**
(a)	The water forms a smaller cone of radius r and height h. From the diagram, by similar triangles, $\frac{r}{h}=\frac{4}{8}$ $r=\frac{h}{2}$ Let the volume of water in the cone be V. $V=\frac{1}{3}\pi r^2 h$ $=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ $=\frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$ $=\frac{\pi}{12}h^3$ $V=\frac{\pi}{12}h^3$ $\frac{dV}{dh}=\frac{\pi}{4}h^2$ Given: $\frac{dV}{dt}=-1.5 \text{ cm}^3/\text{s}$ When $h=2$, $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$ $-1.5=\frac{\pi}{4}(2)^2\times\frac{dh}{dt}$ $\frac{dh}{dt}=-\frac{3}{2\pi}$ cm/s The rate at which the water level is **decreasing** at $\frac{3}{2\pi}$ cm/s.
(b)	$\tan\theta=\frac{DM}{5}$ $\quad\therefore DM=5\tan\theta$ $\cos\theta=\frac{5}{AD}$ $\quad\therefore AD=5\sec\theta$ $DA+AB+BC=20$ $5\sec\theta+AB+5\sec\theta=20$ $\therefore AB=20-10\sec\theta$ Area of trapezium $ABCD$ $=\frac{1}{2}(5)(AB+DC)$ $=\frac{5}{2}(2(20-10\sec\theta)+2(5\tan\theta))$ $=100-50\sec\theta+25\tan\theta$ (shown)

(ii)	$A = 100 - 50\sec\theta + 25\tan\theta$ $\frac{dA}{d\theta} = -50\sec\theta\tan\theta + 25\sec^2\theta = 0$ $25\sec\theta(-2\tan\theta + \sec\theta) = 0$ $25\sec\theta = 0$ or $2\tan\theta = \sec\theta$ (NA) $\sin\theta = \frac{1}{2}$ $\therefore \theta = \frac{\pi}{6}$ $A = 100 - 50\sec\left(\frac{\pi}{6}\right) + 25\tan\left(\frac{\pi}{6}\right)$ $= 100 - 50\left(\frac{2}{\sqrt{3}}\right) + 25\left(\frac{1}{\sqrt{3}}\right)$ $= 100 - \frac{75}{\sqrt{3}}$ $= \left(100 - 25\sqrt{3}\right)\text{cm}^2$

Qn	Solution
7	**Functions and Transformation of curves**
(a) (i)	 Let $y = a\left(x + \frac{1}{x-1}\right)$ $\frac{dy}{dx} = 0 \Rightarrow x = 0 \text{ or } 2$
(a) (ii)	Since the line $y = 4a$ cuts the graph of $y = f(x)$ more than once, f is not a one-one function, hence f does not have an inverse.
(a) (iii)	$R_f = (-\infty, -a] \cup [3a, \infty)$ $D_g = \mathbb{R} \setminus \{-1, 1\}$ Since $a > 1$, $-a < -1$ and $3a > 1$ Since $R_f \subseteq D_g$, $\therefore$gf exists.
(b)	$y = e^{\frac{x^2-4}{12}}$ $\downarrow$ **A** : Replace x by $\frac{x}{\left(\frac{1}{2}\right)} = 2x$ $y = e^{\frac{(2x)^2-4}{12}} = e^{\frac{x^2-1}{3}}$ $\downarrow$ **B** : Replace y by $y+1$ $y + 1 = e^{\frac{x^2-1}{3}}$ $\downarrow$ **C** : Replace y by x and x by y $x + 1 = e^{\frac{y^2-1}{3}}$ $\frac{y^2-1}{3} = \ln(x+1)$ $y^2 = 3\ln(x+1) + 1$ $\because y > 0, y = \sqrt{3\ln(x+1)+1}$

	$\therefore y = q(x) = \sqrt{3\ln(x+1)+1}$ Range of C: $\left(e^{-\frac{1}{3}}, \infty\right)$ $\left(e^{-\frac{1}{3}}, \infty\right) \xrightarrow{A} \left(e^{-\frac{1}{3}}, \infty\right) \xrightarrow{B} \left(e^{-\frac{1}{3}} - 1, \infty\right)$ $\therefore D_q = \left(e^{-\frac{1}{3}} - 1, \infty\right)$

Qn	Solution
8	
(i)	Equation of l_1 : $\mathbf{r}=\begin{pmatrix}-2\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\2\\-3\end{pmatrix}$ where $\lambda\in\mathbb{R}$ Equation of p_1 : $\mathbf{r}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}=3$ Normal of $p_2=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}5\\2\\-3\end{pmatrix}=\begin{pmatrix}-2\\8\\2\end{pmatrix}=2\begin{pmatrix}-1\\4\\1\end{pmatrix}$ Hence, equation of p_2 : $\mathbf{r}\cdot\begin{pmatrix}-1\\4\\1\end{pmatrix}=\begin{pmatrix}-2\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\1\end{pmatrix}=2-4+3=1$ Cartesian equation of p_2: $-x+4y+z=1$
(ii)	Solving $x+z=3---(1)$ $-x+4y+z=1---(2)$ From GC: $x=3-\mu$ $y=1-\frac{1}{2}\mu$ $z=\mu$ Hence, equation of l_2 : $\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+\alpha\begin{pmatrix}-2\\-1\\2\end{pmatrix}$ where $\alpha\in\mathbb{R}$
(iii)	Let F be the foot of perpendicular from B to p_2 **Method 1:** Let l_{BF} : $r=\begin{pmatrix}0\\4\\3\end{pmatrix}+t\begin{pmatrix}-1\\4\\1\end{pmatrix}$ where $t\in\mathbb{R}$ Sub into equation of p_2: $\begin{pmatrix}-t\\4+4t\\3+t\end{pmatrix}\cdot\begin{pmatrix}-1\\4\\1\end{pmatrix}=1$ $t+16+16t+3+t=1$ $t=-1$ Hence, $\overline{OF}=\begin{pmatrix}0-(-1)\\4-4\\3-1\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$ Perpendicular distance from B to $p_2=k=\lvert\overline{BF}\rvert=\left\lvert\begin{pmatrix}1\\-4\\-1\end{pmatrix}\right\rvert=3\sqrt{2}$

	Method 2: Since F is on l_2 as p_1 and p_2 are perpendicular planes, $\overrightarrow{OF} = \mathbf{r} = \begin{pmatrix}3\\1\\0\end{pmatrix} + \alpha\begin{pmatrix}-2\\-1\\2\end{pmatrix}$ $\overrightarrow{BF} = \overrightarrow{OF} - \overrightarrow{OB} = \begin{pmatrix}3-2\alpha\\1-\alpha\\2\alpha\end{pmatrix} - \begin{pmatrix}0\\4\\3\end{pmatrix} = \begin{pmatrix}3-2\alpha\\-3-\alpha\\-3+2\alpha\end{pmatrix}$ $\overrightarrow{BF}\cdot\begin{pmatrix}-2\\-1\\2\end{pmatrix} = 0$ $\begin{pmatrix}3-2\alpha\\-3-\alpha\\-3+2\alpha\end{pmatrix}\cdot\begin{pmatrix}-2\\-1\\2\end{pmatrix} = 0$ $-6+4\alpha+3+\alpha-6+4\alpha=0$ $\alpha=1$ $\overrightarrow{OF} = \begin{pmatrix}3-(1)\\1-1\\2(1)\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$ Perpendicular distance from B to $p_2 = k = \left\lvert\overrightarrow{BF}\right\rvert = \left\lVert\begin{pmatrix}1\\-4\\-1\end{pmatrix}\right\rVert = 3\sqrt{2}$
(iv)	Since the perpendicular distance from B to l_2 is $3\sqrt{2}$, the lines on p_1 with a distance of $3\sqrt{2}$ must be parallel to l_2. Using ratio theorem, find the reflected point B' in the line l_2 $\overrightarrow{OB'} = 2\overrightarrow{OF} - \overrightarrow{OB} = 2\begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}0\\4\\3\end{pmatrix} = \begin{pmatrix}2\\-4\\1\end{pmatrix}$ Hence, equation of lines: $\mathbf{r} = \begin{pmatrix}0\\4\\3\end{pmatrix} + \beta\begin{pmatrix}-2\\-1\\2\end{pmatrix}$ where $\beta\in\mathbb{R}$ or $\mathbf{r} = \begin{pmatrix}2\\-4\\1\end{pmatrix} + \gamma\begin{pmatrix}-2\\-1\\2\end{pmatrix}$ $\gamma\in\mathbb{R}$

Qn	Solution
9	**Complex Numbers**
(a)	Given $z = x + \mathrm{i}y, \quad x, y \in \mathbb{R}$ $\frac{(z^2)^*}{z} = \frac{(z^*)^2}{z}$ $= \frac{(x-y\mathrm{i})^2}{x+y\mathrm{i}}$ $= \frac{(x-y\mathrm{i})^2(x-y\mathrm{i})}{(x+y\mathrm{i})(x-y\mathrm{i})}$ $= \frac{(x-y\mathrm{i})^3}{x^2-(y\mathrm{i})^2}$ $= \frac{x^3-3x^2(y\mathrm{i})+3x(y\mathrm{i})^2-(y\mathrm{i})^3}{x^2+y^2}$ $= \frac{x^3-3x^2y\mathrm{i}-3xy^2+y^3\mathrm{i}}{x^2+y^2}$ $= \frac{x^3-3xy^2}{x^2+y^2}+\frac{-3x^2y+y^3}{x^2+y^2}\mathrm{i}$ Given $\frac{(z^2)^*}{z}$ is real $\Rightarrow \mathrm{Im}\left[\frac{(z^2)^*}{z}\right]=0$ $\Rightarrow \frac{-3x^2y+y^3}{x^2+y^2}=0$ $-3x^2y+y^3=0$ $y(y^2-3x^2)=0$ $y(y-\sqrt{3}x)(y+\sqrt{3}x)=0$ $y=0$ (rejected as y is non-zero), $\quad y=\sqrt{3}x \quad$ or $\quad y=-\sqrt{3}x$ Since $\lvert z\rvert=1 \Rightarrow x^2+y^2=1$, For $y=\pm\sqrt{3}x$,, $\quad x^2+(\pm\sqrt{3}x)^2=1$ $x^2=\frac{1}{4}$ $x=\pm\frac{1}{2} \quad \Rightarrow y=\pm\frac{\sqrt{3}}{2}$ $\therefore$ Possible values of z are $\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i},\ \frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ or $-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$.

(b)	Let $z = x + iy, \quad x, y \in \mathbb{R}$ $(x+iy)^2 = 33 + 56i$ $x^2 + 2xyi - y^2 = 33 + 56i$ Comparing real and imaginary parts, $x^2 - y^2 = 33$ --- (1) $2xy = 56$ $y = \frac{28}{x}$ --- (2) Substitute (2) into (1): $x^2 - \left(\frac{28}{x}\right)^2 = 33$ $x^4 - 33x^2 - 784 = 0$ $(x^2 - 49)(x^2 + 16) = 0$ Since x is real, $x = -7$ or $x = 7$ When $x = -7, y = -4$ $\quad \therefore z = -7 - 4i$ When $x = 7, y = 4$ $\quad \therefore z = 7 + 4i$ $\therefore$ the roots of the equation are $-7 - 4i$ and $7 + 4i$. $w^2 = -33 + 56i$ $-w^2 = 33 - 56i$ $(-w^2)^* = 33 + 56i$ (conjugate both sides) $-(w^*)^2 = 33 + 56i$ $i^2 (w^*)^2 = 33 + 56i$ $(iw^*)^2 = 33 + 56i$ Replace z by iw^*, $iw^* = -7 - 4i$ or $iw^* = 7 + 4i$ $-w^* = -7i + 4$ $\quad -w^* = 7i - 4$ $w^* = 7i - 4$ $\quad w^* = -7i + 4$ $w = -4 - 7i$ $\quad w = 4 + 7i$

Qn	Solution
10	**Differential Equations**
(i)	$\frac{d^2x}{dt^2}+0.1\left(\frac{dx}{dt}\right)^2=10$ Since $y=\frac{dx}{dt}\Rightarrow\frac{dy}{dt}=\frac{d^2x}{dt^2}$ $\therefore\frac{dy}{dt}+0.1y^2=10$ $\frac{dy}{dt}=10-0.1y^2$ (shown)
(ii)	$\frac{dy}{dt}=10-0.1y^2=0.1\left(100-y^2\right)$ $\frac{1}{100-y^2}\frac{dy}{dt}=0.1$ $\int\frac{1}{10^2-y^2}\ dy=\int 0.1\ dt$ $\frac{1}{2(10)}\ln\left\|\frac{10+y}{10-y}\right\|=0.1t+C$ $\ln\left\|\frac{10+y}{10-y}\right\|=2t+20C$ $\frac{10+y}{10-y}=\pm e^{2t+20C}$ $\frac{10+y}{10-y}=Ae^{2t}$, where $A=\pm e^{20C}$ When $t=0, y=0, \therefore A=1$ $10+y=e^{2t}(10-y)$ $y\left(1+e^{2t}\right)=10\left(e^{2t}-1\right)$ $y=\frac{10\left(e^{2t}-1\right)}{e^{2t}+1}$ $\frac{dx}{dt}=\frac{10\left(e^{2t}-1\right)}{e^{2t}+1}$ $=10\left(\frac{e^{2t}}{e^{2t}+1}-\frac{1}{e^{2t}+1}\right)$ $=10\left(\frac{e^{2t}}{e^{2t}+1}-\frac{e^{-2t}}{1+e^{-2t}}\right)$ $x=5\int\left(\frac{2e^{2t}}{e^{2t}+1}-\frac{2e^{-2t}}{1+e^{-2t}}\right)dt$ $=5\left(\ln\left\|e^{2t}+1\right\|+\ln\left\|1+e^{-2t}\right\|\right)+D$ $=5\left(\ln\left(e^{2t}+1\right)+\ln\left(1+e^{-2t}\right)\right)+D$, $\because e^{2t}+1>0$ and $1+e^{-2t}>0$ $=5\ln\left(2+e^{2t}+e^{-2t}\right)+d$

	When $t=0,\ x=0,\ d=-5\ln 4$ $x=5\ln\left(2+e^{2t}+e^{-2t}\right)-5\ln 4$ $x=5\ln\left(\dfrac{2+e^{2t}+e^{-2t}}{4}\right)$
(iii)	When $t=2, x=13.3$ m (3 s.f.). The squirrel has fallen 13.3 metres.
(iv)	Note that $y=\dfrac{dx}{dt}$ is the velocity of the falling squirrel at any time t. $y=\dfrac{10\left(e^{2t}-1\right)}{e^{2t}+1}$ $=\dfrac{10\left(1-e^{-2t}\right)}{1+e^{-2t}}$ As $t\to\infty,\ e^{-2t}\to 0,\ y\to 10$ Hence, the terminal velocity of the falling squirrel is 10 m/s.

Qn	Solution
11	**APGP**
(i)	(see table below)

n (no. of years)	Total amount in account A at the end of nth year
1	$1000(1.035)$
2	$\left[1000+1000(1.035)\right]1.035$ $=1000\left(1.035+1.035^2\right)$
3	$\left[1000\left(1.035+1.035^2\right)\right]1.035$ $=1000\left(1.035+1.035^2+1.035^3\right)$
	....
n	$1000\left(1.035+1.035^2+1.035^3+...+1.035^n\right)$

Total amount in account A at the end of n years

$=1000\left(1.035+1.035^2+1.035^3+...+1.035^n\right)$

$=1000\left(\dfrac{1.035\left(1-1.035^n\right)}{1-1.035}\right)$

$=1000\left(\dfrac{1.035\left(1.035^n-1\right)}{1.035-1}\right)$

$=\dfrac{207000}{7}\left(1.035^n-1\right)$

Therefore, $p=\dfrac{207000}{7}$ and $q=1.035$.

(ii)

<u>Method 1:</u>

Using result from part (i),

When $n=23$, $\dfrac{207000}{7}\left(1.035^{23}-1\right)=35666.53<36000$

When $n=24$, $\dfrac{207000}{7}\left(1.035^{24}-1\right)=37949.86>36000$

When $n=23$, total amount in account A at the end of 2045 = \$35666.53

Total amount in account A at the start of 2046

= \$35666.53 + \$1000 = \$36666.53

Hence, the total amount in account A first exceed \$36000 on **1 January 2046**.

<u>Method 2:</u>

n (no. of years)	Total amount in account A at the start of nth year
1	1000
2	$1000+1000(1.035)=1000(1+1.035)$
3	$\left[1000(1+1.035)\right]1.035+1000$ $=1000(1+1.035+1.035^2)$

	....
n	$1000\left(1+1.035+1.035^2+...+1.035^{n-1}\right)$

Total amount in account A at the start of the nth year

$= 1000\left(1+1.035+1.035^2+...+1.035^{n-1}\right)$

$= 1000\left(\dfrac{1\left(1-1.035^n\right)}{1-1.035}\right)$

$= \dfrac{200000}{7}\left(1.035^n-1\right)$

Using GC,

When $n=23$, $\dfrac{200000}{7}\left(1.035^{23}-1\right)=34460.41<36000$

When $n=24$, $\dfrac{200000}{7}\left(1.035^{24}-1\right)=36666.53>36000$

Since $n=24$, the total amount in account A first exceed \$36000 on **1 January 2046**.

(iii)

n (no. of years)	Total amount in account B at the end of nth year
1	$1000+40$
2	$(1000+40)+(1000+36)+40$ $=2(1000)+36+2(40)$
3	$\left[2(1000)+36+2(40)\right]+\left(1000+2(36)\right)+40$ $=3(1000)+\left(36+2(36)\right)+3(40)$
	....
n	$n(1000)+36\left[1+2+...+(n-1)\right]+n(40)$

Total amount in account B at the end of n years

$= n(1000)+36\left[1+2+...+(n-1)\right]+n(40)$

$= n(1040)+36\left[\dfrac{(n-1)n}{2}\right]$

$$\frac{207000}{7}\left(1.035^n-1\right)>n(1040)+36\left[\frac{(n-1)n}{2}\right]$$

$$\frac{207000}{7}\left(1.035^n-1\right)-n(1040)+36\left[\frac{(n-1)n}{2}\right]>0$$

Using GC,

When $n=6$,

$$\frac{207000}{7}\left(1.035^6-1\right)-6(1040)+36\left[\frac{(6-1)6}{2}\right]=-0.592<0$$

	When $n=7$, $\frac{207000}{7}\left(1.035^{7}-1\right)-7(1040)+36\left[\frac{(7-1)7}{2}\right]=15.687>0$ Least $n=7$ Therefore, the total amount in account A first exceeds the total amount in account B in 2029.
(iv)	At end of 31 Dec 2032, $n=10$ $10(1040)+k\left[\frac{(10-1)10}{2}\right]>\frac{207000}{7}\left(1.035^{10}-1\right)$ $k>\left[\frac{207000\left(1.035^{10}-1\right)}{7}-10(1040)\right]\frac{1}{45}$ $k>38.711$ Least $k=39$ (to the nearest whole number)

2022 H2 MATH (9758/02) JC 2 PRELIMINARY EXAMINATION SOLUTIONS

Qn	Solution
1	**Differentiation and applications**
(i)	$3y^2\frac{dy}{dx}-\left(y+x\frac{dy}{dx}\right)=2e^{2x}$ $\left(3y^2-x\right)\frac{dy}{dx}=2e^{2x}+y$ $\frac{dy}{dx}=\frac{2e^{2x}+y}{3y^2-x}$
(ii)	When $x=0$, $y^3-(0)y=e^{2(0)}+7$ $y^3=8$ $y=2$ $\frac{dy}{dx}=\frac{2e^{2(0)}+2}{3(2)^2-0}=\frac{4}{12}=\frac{1}{3}$ Equation of tangent to the curve at $x=0$: $y-2=\frac{1}{3}(x-0)$ $y=\frac{1}{3}x+2$

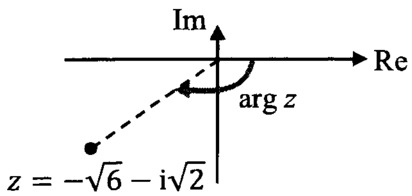
Qn	Solution
2	**Complex Numbers**

$$|z| = \sqrt{\left(-\sqrt{6}\right)^2 + \left(-\sqrt{2}\right)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\arg z = -\pi + \tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{6}}\right)$$

$$= -\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= -\pi + \frac{\pi}{6} = -\frac{5\pi}{6}$$



Method 1

$$\left|\frac{\mathrm{i}zw^2}{w^*}\right| = \frac{(1)\left(2\sqrt{2}\right)(3)^2}{(3)}$$

$$= 6\sqrt{2}$$

$$\arg\left(\frac{\mathrm{i}zw^2}{w^*}\right) = \arg(\mathrm{i}) + \arg z + 2\arg w - \arg\left(w^*\right)$$

$$= \frac{\pi}{2} - \frac{5\pi}{6} + 2\left(-\frac{5\pi}{7}\right) - \left(\frac{5\pi}{7}\right)$$

$$= -\frac{52\pi}{21}$$

$$\equiv -\frac{10\pi}{21}$$

Method 2: Using exponential form

$$z = 2\sqrt{2}\mathrm{e}^{-\frac{5\pi}{6}\mathrm{i}}$$

$$w = 3\left(\cos\frac{5\pi}{7} - \mathrm{i}\sin\frac{5\pi}{7}\right) = 3\mathrm{e}^{-\frac{5\pi}{7}\mathrm{i}}$$

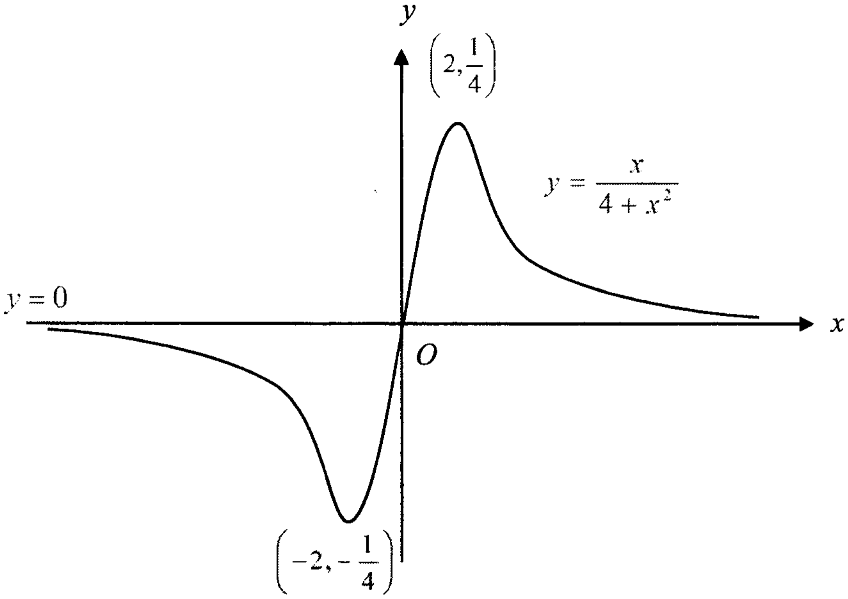
$$\frac{\mathrm{i}zw^2}{w^*} = \frac{\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}\left(2\sqrt{2}\mathrm{e}^{\frac{-5\pi}{6}\mathrm{i}}\right)\left(3\mathrm{e}^{\frac{-5\pi}{7}\mathrm{i}}\right)^2}{3\mathrm{e}^{\frac{5\pi}{7}\mathrm{i}}}$$

$$= 6\sqrt{2}\,\mathrm{e}^{\left(\frac{\pi}{2} - \frac{5\pi}{6} - \frac{10\pi}{7} - \left(\frac{5\pi}{7}\right)\right)\mathrm{i}}$$

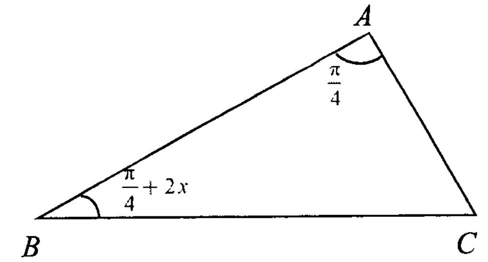
$$= 6\sqrt{2}\,\mathrm{e}^{\frac{-52\pi}{21}\mathrm{i}} \equiv 6\sqrt{2}\,\mathrm{e}^{\frac{-10\pi}{21}\mathrm{i}}$$

$$\left|\frac{\mathrm{i}zw^2}{w^*}\right| = 6\sqrt{2}$$

$$\arg\left(\frac{\mathrm{i}zw^2}{w^*}\right) = -\frac{10\pi}{21}$$

Qn	Solution
3	**Graphing (Rational Function), Definite Integral (Volume)**
(i)	
(ii)	$\text{Vol}=\pi\left(\frac{1}{4}\right)^2(2)-\pi\int_0^2\left(\frac{x}{4+x^2}\right)^2\,\mathrm{d}x$ $=\frac{\pi}{8}-\pi\int_0^2\frac{x^2}{\left(4+x^2\right)^2}\,\mathrm{d}x$ $=\frac{\pi}{8}-\pi\int_0^2\frac{x^2}{\left(4+x^2\right)^2}\,\mathrm{d}x$ $=\frac{\pi}{8}-\pi\int_0^{\frac{\pi}{4}}\frac{4\tan^2\theta}{\left(4+4\tan^2\theta^2\right)^2}\left(2\sec^2\theta\right)\,\mathrm{d}\theta$ $=\frac{\pi}{8}-\pi\int_0^{\frac{\pi}{4}}\frac{4\tan^2\theta}{16\sec^4\theta}\left(2\sec^2\theta\right)\,\mathrm{d}\theta$ $=\frac{\pi}{8}-\pi\int_0^{\frac{\pi}{4}}\frac{\tan^2\theta}{2\sec^2\theta}\,\mathrm{d}\theta$ $=\frac{\pi}{8}-\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sin^2\theta\,\mathrm{d}\theta$ $=\frac{\pi}{8}-\frac{\pi}{4}\int_0^{\frac{\pi}{4}}1-\cos2\theta\,\mathrm{d}\theta$ $=\frac{\pi}{8}-\frac{\pi}{4}\left[\theta-\frac{\sin2\theta}{2}\right]_0^{\frac{\pi}{4}}$ $=\frac{\pi}{8}-\frac{\pi}{4}\left[\left(\frac{\pi}{4}-\frac{1}{2}\right)-0\right]$ $=\frac{\pi}{4}-\frac{\pi^2}{16}$

Qn	Solution
4	**Maclaurin Series**



$$\angle ACB = \pi - \frac{\pi}{4} - \left(\frac{\pi}{4}+2x\right) = \frac{\pi}{4} - 2x$$

Using Sine Rule,

$$\frac{AB}{\sin\left(\frac{\pi}{2}-2x\right)} = \frac{AC}{\sin\left(\frac{\pi}{4}+2x\right)}$$

$$\frac{AB}{AC} = \frac{\sin\left(\frac{\pi}{2}-2x\right)}{\sin\left(\frac{\pi}{4}+2x\right)}$$

$$= \frac{\cos 2x}{\sin\frac{\pi}{4}\cos 2x + \cos\frac{\pi}{4}\sin 2x}$$

$$= \frac{\cos 2x}{\frac{1}{\sqrt{2}}\left(\cos 2x + \sin 2x\right)}$$

$$\frac{AB}{AC} = \frac{\sqrt{2}\cos 2x}{\cos 2x + \sin 2x} \text{ (shown)}$$

$$\frac{AB}{AC} = \frac{\sqrt{2}\cos 2x}{\cos 2x + \sin 2x}$$

$$\approx \frac{\sqrt{2}\left(1-\frac{(2x)^2}{2!}\right)}{\left(1-\frac{(2x)^2}{2!}\right)+(2x)}$$

$$= \frac{\sqrt{2}\left(1-2x^2\right)}{1+2x-2x^2}$$

$$= \sqrt{2}\left(1-2x^2\right)\left(1+2x-2x^2\right)^{-1}$$

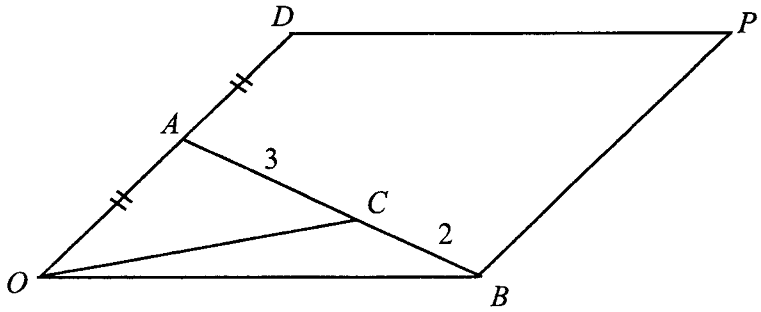
$$= \sqrt{2}\left(1-2x^2\right)\left(1+(-1)\left(2x-2x^2\right)+\frac{(-1)(-2)}{2!}\left(2x-2x^2\right)^2+\ldots\right)$$

$$= \sqrt{2}\left(1-2x^2\right)\left(1-2x+2x^2+4x^2+\ldots\right)$$

$$= \sqrt{2}\left(1-2x^2\right)\left(1-2x+6x^2+\ldots\right)$$

$$= \sqrt{2}\left(1-2x+6x^2-2x^2+\ldots\right)$$

$$= \sqrt{2}\left(1-2x+4x^2+\ldots\right)$$

Qn	Solution
5	**Vectors**
	
(i)	$\overrightarrow{OC}=\dfrac{2\overrightarrow{OA}+3\overrightarrow{OB}}{5}=\dfrac{2}{5}\mathbf{a}+\dfrac{3}{5}\mathbf{b}$ $\overrightarrow{OP}=\overrightarrow{OD}+\overrightarrow{OB}=2\overrightarrow{OA}+\overrightarrow{OB}=2\mathbf{a}+\mathbf{b}$ Area of triange $OPC=\dfrac{1}{2}\left\|\overrightarrow{OP}\times\overrightarrow{OC}\right\|$ $=\dfrac{1}{2}\left\|(2\mathbf{a}+\mathbf{b})\times\left(\dfrac{2}{5}\mathbf{a}+\dfrac{3}{5}\mathbf{b}\right)\right\|$ $=\dfrac{1}{10}\left\|2\mathbf{a}\times2\mathbf{a}+2\mathbf{a}\times3\mathbf{b}+\mathbf{b}\times2\mathbf{a}+\mathbf{b}\times3\mathbf{b}\right\|$ $=\dfrac{1}{10}\left\|6\mathbf{a}\times\mathbf{b}-2\mathbf{a}\times\mathbf{b}\right\|$ $=\dfrac{2}{5}\left\|\mathbf{a}\times\mathbf{b}\right\|$ $\therefore k=\dfrac{2}{5}$
(ii)	Line AB: $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a}),\ \lambda\in\mathbb{R}$ Line OP: $\mathbf{r}=\mu(2\mathbf{a}+\mathbf{b}),\ \mu\in\mathbb{R}$ To find intersection point: $\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})=\mu(2\mathbf{a}+\mathbf{b})$ $(1-\lambda)\mathbf{a}+\lambda\mathbf{b}=2\mu\mathbf{a}+\mu\mathbf{b}$ Since , $\mathbf{a}$ is not parallel to $\mathbf{b}$ and they are nonzero vectors, $1-\lambda=2\mu$ $\lambda=\mu$ Solving, $\lambda=\mu=\dfrac{1}{3}$ $\overrightarrow{OE}=\dfrac{1}{3}(2\mathbf{a}+\mathbf{b})$

(iii)	**Method 1:** Since E is the foot of perpendicular from D to the line OP, $\overrightarrow{DE}\cdot\overrightarrow{OP}=0$ $\left(\frac{1}{3}(2\mathbf{a}+\mathbf{b})-2\mathbf{a}\right)\cdot(2\mathbf{a}+\mathbf{b})=0$ $\frac{1}{3}(\mathbf{b}-4\mathbf{a})\cdot(2\mathbf{a}+\mathbf{b})=0$ $\mathbf{b}\cdot 2\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-4\mathbf{a}\cdot 2\mathbf{a}-4\mathbf{a}\cdot\mathbf{b}=0$ $\lvert\mathbf{b}\rvert^2-8\lvert\mathbf{a}\rvert^2-2\mathbf{a}\cdot\mathbf{b}=0$ $2\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{b}\rvert^2-8\lvert\mathbf{a}\rvert^2$ $2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos A\hat{O}B=\lvert\mathbf{b}\rvert^2-8\lvert\mathbf{a}\rvert^2$ $\cos A\hat{O}B=\frac{\lvert\mathbf{b}\rvert^2-8\lvert\mathbf{a}\rvert^2}{2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ $=\frac{\lvert\mathbf{b}\rvert^2-8}{2\lvert\mathbf{b}\rvert}$ $(\because \mathbf{a}$ is a unit vector$)$ Since $A\hat{O}B$ is acute, $\frac{\lvert\mathbf{b}\rvert^2-8}{2\lvert\mathbf{b}\rvert}>0$ and since $\lvert\mathbf{b}\rvert>0$, $\lvert\mathbf{b}\rvert>2\sqrt{2}$ Also, $\frac{\lvert\mathbf{b}\rvert^2-8}{2\lvert\mathbf{b}\rvert}<1$ $(\because \mathbf{a}$ is not parallel to $\mathbf{b})$ $\lvert\mathbf{b}\rvert^2-8<2\lvert\mathbf{b}\rvert$ $(\because \lvert\mathbf{b}\rvert>0)$ $\lvert\mathbf{b}\rvert^2-2\lvert\mathbf{b}\rvert-8<0$ $(\lvert\mathbf{b}\rvert-4)(\lvert\mathbf{b}\rvert+2)<0$ we get $0<\lvert\mathbf{b}\rvert<4$. Thus, $2\sqrt{2}<\lvert\mathbf{b}\rvert<4$

	Method 2: Since E is the foot of perpendicular from D to the line OP, $\overrightarrow{DE}\cdot\overrightarrow{OP}=0$ $\left(\frac{1}{3}(2\mathbf{a}+\mathbf{b})-2\mathbf{a}\right)\cdot(2\mathbf{a}+\mathbf{b})=0$ $\frac{1}{3}(\mathbf{b}-4\mathbf{a})\cdot(2\mathbf{a}+\mathbf{b})=0$ $\mathbf{b}\cdot 2\mathbf{a}+\mathbf{b}\cdot\mathbf{b}-4\mathbf{a}\cdot 2\mathbf{a}-4\mathbf{a}\cdot\mathbf{b}=0$ $\lvert\mathbf{b}\rvert^2-8\lvert\mathbf{a}\rvert^2-2\mathbf{a}\cdot\mathbf{b}=0$ $2\mathbf{a}\cdot\mathbf{b}=\lvert\mathbf{b}\rvert^2-8\lvert\mathbf{a}\rvert^2$ $2\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos A\hat{O}B=\lvert\mathbf{b}\rvert^2-8\lvert\mathbf{a}\rvert^2$ $\lvert\mathbf{b}\rvert^2-2\lvert\mathbf{b}\rvert\cos A\hat{O}B-8=0 \qquad (\because \mathbf{a} \text{ is a unit vector})$ $\lvert\mathbf{b}\rvert=\frac{2\cos A\hat{O}B\pm\sqrt{4\cos^2 A\hat{O}B-4(1)(-8)}}{2}$ $=\cos A\hat{O}B\pm\sqrt{\cos^2 A\hat{O}B+8}$ $=\cos A\hat{O}B+\sqrt{\cos^2 A\hat{O}B+8}$ or $\cos A\hat{O}B-\sqrt{\cos^2 A\hat{O}B+8}$ Since $\lvert\mathbf{b}\rvert>0$, $\lvert\mathbf{b}\rvert=\cos A\hat{O}B+\sqrt{\cos^2 A\hat{O}B+8}$ Since $A\hat{O}B$ is acute, $0<\cos A\hat{O}B<1$ $0<\cos^2 A\hat{O}B<1$ And $\cos A\hat{O}B$ is strictly decreasing for the given domain, We have $2\sqrt{2}<\lvert\mathbf{b}\rvert<4$

Qn	Solution
6	**Discrete Random Variable**
(i)	Probability Distribution of X

x	0	1	2	3	4	5	6
$\mathrm{P}(X=x)$	p	q	$\frac{q}{3}$	q	$\frac{q}{3}$	q	$\frac{q}{3}$

$\mathrm{E}(X)=2$

$0+q+\frac{2q}{3}+3q+\frac{4q}{3}+5q+\frac{6q}{3}=2$

$p+q+\frac{q}{3}+q+\frac{q}{3}+q+\frac{q}{3}=1$

Solving, $p=\frac{5}{13}$, $q=\frac{2}{13}$

Probability Distribution of X

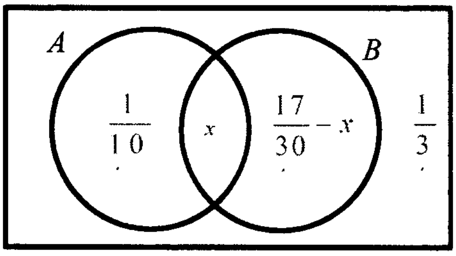
x	0	1	2	3	4	5	6
$\mathrm{P}(X=x)$	$\frac{5}{13}$	$\frac{2}{13}$	$\frac{2}{39}$	$\frac{2}{13}$	$\frac{2}{39}$	$\frac{2}{13}$	$\frac{2}{39}$

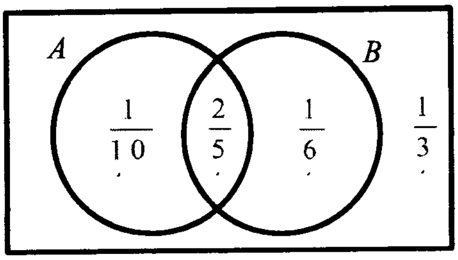
(ii)

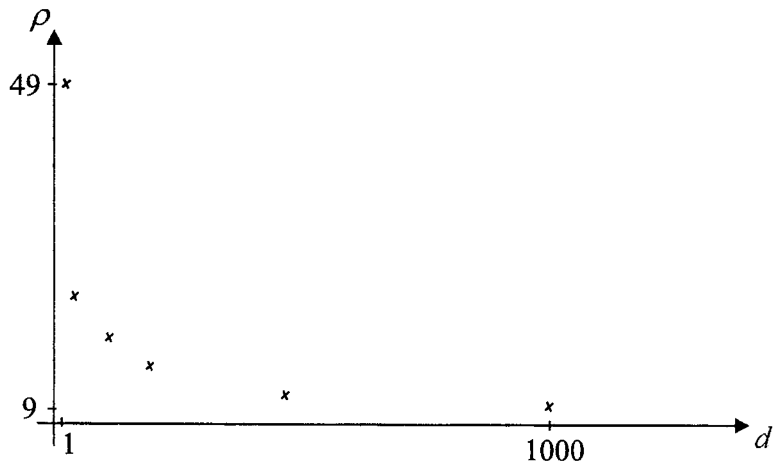
$$\begin{aligned}\mathrm{P}(X_1+X_2=4)&=\mathrm{P}(X_1=0,X_2=4)+\mathrm{P}(X_1=4,X_2=0)\\&\quad+\mathrm{P}(X_1=1,X_2=3)+\mathrm{P}(X_1=3,X_2=1)\\&\quad+\mathrm{P}(X_1=2,X_2=2)\end{aligned}$$

$$=\frac{5}{13}\left(\frac{2}{39}\right)\times 2+\frac{2}{13}\left(\frac{2}{13}\right)\times 2+\frac{2}{39}\left(\frac{2}{39}\right)$$

$$=\frac{136}{1521} \text{ or } 0.0894 \text{ (3 s.f.)}$$

Qn	Solution
7	**Permutations & Combinations and Probability**
(a)(i)	Number of teams $= {}^{10}C_5 = 252$
(a)(ii)	Number of teams $= {}^{6}C_4 + \left({}^{3}C_1\right)\left({}^{6}C_3\right) = 75$
(b)(i)	**Method 1:**  Let $\mathrm{P}(A \cap B) = x$ $\mathrm{P}(B \mid A) = \frac{4}{5}$ $\frac{x}{\frac{1}{10} + x} = \frac{4}{5}$ $x = \frac{2}{5}$
	Method 2: $\mathrm{P}(A \cup B) = 1 - \mathrm{P}(A' \cap B') = \frac{2}{3}$ $\mathrm{P}(B \mid A) = \frac{4}{5} = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} \Rightarrow \mathrm{P}(A \cap B) = \frac{4}{5}\mathrm{P}(A)$ $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ $= \mathrm{P}(A) + \frac{17}{30} - \frac{4}{5}\mathrm{P}(A)$ $= \frac{1}{5}\mathrm{P}(A) + \frac{17}{30}$ $\Rightarrow \mathrm{P}(A) = \frac{1}{2}$ $\Rightarrow \mathrm{P}(A \cap B) = \frac{2}{5}$

(b)(ii)	 $P(A \mid B') = \dfrac{P(A \cap B')}{P(B')}$ $= \dfrac{P(A \cap B')}{1 - P(B)}$ $= \dfrac{\frac{1}{10}}{1 - \frac{17}{30}}$ $= \dfrac{3}{13}$

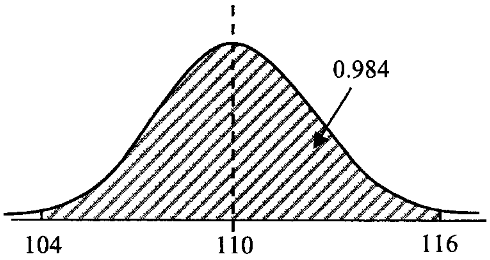
Qn	Solution
8	**Correlation and Regression**
(i)	
(ii)(a)	$r = -0.62016 \approx -0.620$
(ii)(b)	$r = -0.99371 = -0.994$
(iii)	Based on the scatter diagram, as d increases, ρ decreases at a decreasing rate. Also, $\lvert r\rvert = \lvert -0.99371\rvert = 0.994$ for $\ln \rho$ and $\ln d$ is closer to 1 as compared to $\lvert r\rvert = \lvert -0.62016\rvert = 0.620$ for ρ and d. Hence, the relationship between ρ and d is better modelled by $\ln \rho = A + B \ln d$. $\ln \rho = 3.8793 - 0.25338 \ln d \approx 3.88 - 0.253 \ln d$, where $A = 3.88, B = -0.253$
(iv)	$\ln \rho = 3.8793 - 0.25338 \ln d$ When $\rho = 8$, $\ln 8 = 3.8793 - 0.25338 \ln d$ $d = 1216.1 \approx 1216$ mm (to nearest integer) Even though $\lvert r\rvert = 0.994$ is close to 1, since $\rho = 8$ lies outside the data range of ρ, the linear relation may no longer hold, hence the estimate is not reliable.
(v)	The product moment correlation coefficient will be the **same**, as r is **independent of the scale of measurement**.

Qn	Solution
9	**Hypothesis Testing**
(i)	Since the weights of apples should be **close to the mean of 200g**, using $\sum(x-200)^2$ instead of $\sum x^2$ ensures the sum is manageable and **not too large**. **OR** By coding the summarised data of $(x-200)$ with reference to the mean of 200, it reduces the value of summarized data into a number that can be more easily handled when finding unbiased estimates.
(ii)	An unbiased estimate for the population mean is $\bar{x}=\frac{-30}{30}+200=199$ An unbiased estimate for the population variance is $s^2=\frac{1}{29}\left(1800-\frac{(-30)^2}{30}\right)=\frac{1770}{29}$
(iii)	Let μ be the population mean weight of apples, in g. $H_0:\mu=200$ $H_1:\mu<200$ Under H_0, Since $n=30$ is large, by Central Limit Theorem, $\bar{X}\sim N\left(200,\frac{1770}{(29)(30)}\right)$ approximately. Test Statistic: $Z=\frac{\bar{X}-200}{\sqrt{\frac{1770}{(29)(30)}}}$ Level of significance: 10% Reject H_0 if p-value < 0.1. Under H_0, using GC, $p\text{-value}=0.24162$ (5 s.f) $=0.242$ (3 s.f) Since $p\text{-value}=0.242>0.1$, we do not reject H_0 and conclude that there is insufficient evidence, at 10% level of significance, that the population mean weight of apples sold by the fruit seller is less than 200g. Thus the fruit's claim is valid.

(iv)	Let Y be the weight of a randomly chosen orange, in g. Let μ_y be the population mean weight of oranges, in g. $H_0 : \mu_y = 120$ $H_1 : \mu_y \neq 120$ Under H_0, $Y \sim N(120, 8^2)$ $\Rightarrow$ $\overline{Y} \sim N\left(120, \frac{8^2}{30}\right)$ Test Statistic: $Z = \frac{\overline{Y} - 120}{8/\sqrt{30}}$ Level of significance: 10% Reject H_0 if z-value < -1.6449 or z-value > 1.6449 Since H_0 is rejected, $\frac{\overline{y} - 120}{8/\sqrt{30}} < -1.6449$ or $\frac{\overline{y} - 120}{8/\sqrt{30}} > 1.6449$ $\overline{y} < 117.60$ or $\overline{y} > 122.40$ $\overline{y} < 117$ or $\overline{y} > 123$ $\{\overline{y} \in \mathbb{R}^+ : \overline{y} < 117 \text{ or } \overline{y} > 123\}$ or $\{\overline{y} \in \mathbb{R} : 0 < \overline{y} < 117 \text{ or } \overline{y} > 123\}$

Qn	Solution
10	**Binomial Distribution**
(i)	The probability that a randomly chosen key chain is defective remains constant at 0.03 for all key chains in a box. Whether a randomly chosen key chain is defective is independent of any other key chains in a box.
(ii)	Let X be the number of defective key chains out of n key chains in a box. $X \sim \mathrm{B}(n, 0.03)$ $\mathrm{P}(X \le 2) < 0.95$ when $n = 27$, $\mathrm{P}(X \le 2) = 0.9538 > 0.95$ when $n = 28$, $\mathrm{P}(X \le 2) = 0.9494 < 0.95$ Least value of $n = 28$
(iii)	**Method A:** Let Y be the number of defective key chains out of 20 key chains in a box. $Y \sim \mathrm{B}(20, 0.03)$ $\mathrm{P}(Y \le 2) = 0.97899 = 0.979$ (3 s.f.) **Method B:** Let W be the number of defective key chains out of 10 key chains in a box. $W \sim \mathrm{B}(10, 0.03)$ P(a batch is accepted) $= \mathrm{P}(W = 0) + P(W = 1)P(W \le 1)$ $= 0.95762$ $= 0.958$ (3 s.f.)
(iv)	Expected number for Method A = 20 Expected number for Method B $= 10 \times (1 - \mathrm{P}(W = 1)) + 20 \times \mathrm{P}(W = 1)$ $= 12.3$ (3 s.f.) Since the expected number of keychains to be sampled for method B is lower, the company might choose B instead of A as it saves time in checking (or any other valid reason).
(v)	$P(Y \ge 3) = 1 - P(Y \le 2) = 1 - 0.97899 = 0.02101$ Let S be the number of boxes with 3 or more defective key chains out of 30 boxes. $S \sim B(30, 0.02101)$ Let T be the number of boxes with 3 or more defective key chains out of 14 boxes. $T \sim B(14, 0.02101)$ Required probability = $\dfrac{P(T = 2) \times 0.02101 \times (0.97899)^{15}}{P(S = 3)} = 0.0224$

Qn	Solution
11	**Normal and Sampling Distribution**

(i)	Let X and Y be the volume of oil in a randomly chosen barrel of light and heavy oil respectively. $X \sim N(110,\ 2.5^2)$ $\quad Y \sim N(145,\ 3.5^2)$ $P(104 < X < 116) = 0.984$
(ii)	
(iii)	Required Probability $= P(142 < Y < 150)^4 \times P(Y > 150) \times P(Y < 142)^2 \times \frac{7!}{4!2!}$ $= 0.0863$
(iv)	Let $\bar{Y} = \frac{Y_1 + Y_2 + \ldots + Y_n}{n}$ and $\bar{Y} \sim N\left(145, \frac{3.5^2}{n}\right)$ $P(\bar{Y} > k) \geq 0.3$ $P\left(Z > \frac{k-145}{\frac{3.5}{\sqrt{n}}}\right) \geq 0.3$ $\frac{k-145}{\frac{3.5}{\sqrt{n}}} \leq 0.52440$ $k \leq 145 + \frac{1.84}{\sqrt{n}}$
(v)	Let $T = 0.83(X_1 + X_2 + \cdots X_{25}) + 0.94(Y_1 + Y_2 + \cdots Y_{30})$ $E(T) = 0.83(110 \times 25) + 0.94(145 \times 30) = 6371.5$ (exact) $Var(T) = 0.83^2(2.5^2 \times 25) + 0.94^2(3.5^2 \times 30) = 432.363625$ (exact) $T \sim N(6371.5, 432.363625)$ (exact) $P(T + 25(5) + 30(8) > 6800)$ $= P(T > 6435)$ $= 0.00113$
(vi)	The distributions of the volume of **all types of oil** are **independent** of one another.